Discrete Random Variables — Modules

20. Bernoulli, Binomial, Hypergeometric Distrns

21. Geometric, Negative Binomial Distrns

22. Poisson Distrn

3.20 Bernoulli, Binomial, Hypergeometric Distrns

Bernoulli Distrn

Binomial Distrn

Hypergeometric Distrn

Bernoulli(p) Distribution

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } q \end{cases}$$

Previous work showed that $\text{E}[X] = p$, $\text{Var}(X) = pq$, and $M_X(t) = pe^t + q$.

Further, $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Bern(p)} \Rightarrow \sum_{i=1}^n X_i \sim \text{Bin}(n, p)$.

Binomial(n, p) Distribution

Let $Y = \sum_{i=1}^{n} X_i$, where $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ Bern(p). Then $Y \sim \text{Bin}(n, p)$.

$$\Pr(Y = k) \; = \; \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, \ldots, n.$$

Example: Toss 2 dice 5 times. Let Y be the number of 7's you see. $Y \sim \text{Bin}(5, 1/6)$. Then, e.g.,

$$\Pr(Y = 4) \; = \; \binom{5}{4} (\frac{1}{6})^4 (\frac{5}{6})^{5-4}.$$

$X_1, \ldots X_n$ i.i.d. Bern(p) implies

$$\mathsf{E}[Y] = \mathsf{E}\Big[\sum_{i=1}^{n} X_i\Big] = \sum_{i=1}^{n} \mathsf{E}[X_i] = np$$

and, similarly,

$$\mathsf{Var}(Y) = npq.$$

We've already seen that $M_Y(t) = (pe^t + q)^n$.

If $Y_1, \ldots, Y_k$ are *indep* and $Y_i \sim \mathsf{Bin}(n_i, p)$, then

$$\sum_{i=1}^{k} Y_i \sim \mathsf{Bin}\Big(\sum_{i=1}^{k} n_i, p\Big).$$

Hypergeometric Distribution

You have a objects of type 1 and b objects of type 2.

Select n objects w/o replacement from the $a+b$.

Let X be the number of type 1's selected.

$$\Pr(X=k) \;=\; \frac{\binom{a}{k}\binom{b}{n-k}}{\binom{a+b}{n}}, \quad k=0,1,\ldots,n.$$

After some algebra, it turns out that

$$\mathsf{E}[X] = n\left(\frac{a}{a+b}\right) \quad \text{and}$$

$$\mathsf{Var}(X) = n\left(\frac{a}{a+b}\right)\left(1-\frac{a}{a+b}\right)\left(\frac{a+b-n}{a+b-1}\right).$$

$\frac{a}{a+b}$ here plays the role of p in the Binomial distrn.

Old Example: 25 sox in a box. 15 red, 10 blue. Pick 7 w/o replacement.

$$\Pr(\text{exactly 3 reds are picked}) = \frac{\binom{15}{3}\binom{10}{4}}{\binom{25}{7}}$$