Moment Generating Functions

Intro / Definition

The Big Theorem

Other Applications

Introduction

Recall that $\mathsf{E}[X^k]$ is the kth **moment** of X.

Definition: $M_X(t) \equiv \mathsf{E}[e^{tX}]$ is the **moment generating function** (mgf) of the RV X.

Remark: $M_X(t)$ is a function of t, *not* of X!

Example: $X \sim \text{Bern}(p)$.

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } q \end{cases}$$

Then

$$M_X(t) = \text{E}[e^{tX}] = \sum_x e^{tx} f(x) = e^{t \cdot 1} p + e^{t \cdot 0} q = pe^t + q.$$

Example: $X \sim \text{Exp}(\lambda)$.

$$f(x) \;=\; \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} M_X(t) &= \text{E}[e^{tX}] \\ &= \int_{\Re} e^{tx} f(x)\, dx \\ &= \lambda \int_0^{\infty} e^{(t-\lambda)x}\, dx \\ &= \frac{\lambda}{\lambda - t} \quad \text{if } \lambda > t. \end{aligned}$$

Big Theorem: Under certain technical conditions,

$$\mathsf{E}[X^k] = \frac{d^k}{dt^k} M_X(t)\bigg|_{t=0}, \quad k = 1, 2, \ldots.$$

Thus, you can *generate* the moments of X from the mgf. (Sometimes, it's easier to get moments this way than directly.)

"Proof:"

$$M_X(t) = \mathsf{E}[e^{tX}] = \mathsf{E}\left[\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right] = \sum_{k=0}^{\infty} \mathsf{E}\left[\frac{(tX)^k}{k!}\right]$$

$$= 1 + t\mathsf{E}[X] + \frac{t^2\mathsf{E}[X^2]}{2} + \cdots$$

This implies

$$\frac{d}{dt}M_X(t) = \mathsf{E}[X] + t\mathsf{E}[X^2] + \cdots$$

and so

$$\frac{d}{dt}M_X(t)\Big|_{t=0} = \mathsf{E}[X].$$

Same deal for higher-order moments.

Example: $X \sim \text{Exp}(\lambda)$. Then $M_X(t) = \frac{\lambda}{\lambda - t}$ for $\lambda > t$. So

$$\text{E}[X] = \frac{d}{dt} M_X(t)\Big|_{t=0} = \frac{\lambda}{(\lambda - t)^2}\Big|_{t=0} = 1/\lambda.$$

Further,

$$\text{E}[X^2] = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \frac{2\lambda}{(\lambda - t)^3}\Big|_{t=0} = 2/\lambda^2.$$

Thus,

$$\text{Var}(X) = \text{E}[X^2] - (\text{E}[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = 1/\lambda^2.$$

Other Applications

You can do lots of nice things with mgf's. . .

Find the mgf of a linear function of X.

Find the mgf of the sum of indep RV's.

Identify distributions.

Derive cool properties of distributions.

Theorem: Suppose X has mgf $M_X(t)$ and let $Y = aX + b$. Then $M_Y(t) = e^{tb} M_X(at)$.

Proof:

$$\begin{aligned} M_Y(t) &= \mathsf{E}[e^{tY}] = \mathsf{E}[e^{t(aX+b)}] = e^{tb}\mathsf{E}[e^{(at)X}] \\ &= e^{tb} M_X(at). \end{aligned}$$

Example: Let $X \sim \text{Exp}(\lambda)$ and $Y = 3X - 2$. Then

$$M_Y(t) = e^{2t} M_X(3t) = e^{2t} \frac{\lambda}{\lambda - 3t}, \quad \text{if } \lambda > 3t.$$

Theorem: Suppose $X_1, \ldots, X_n$ are *indep*. Let $Y = \sum_{i=1}^n X_i$. Then

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

Proof:

$$\begin{aligned} M_Y(t) &= \mathsf{E}[e^{tY}] = \mathsf{E}[e^{t \sum X_i}] = \mathsf{E}\Big[\prod_{i=1}^{n} e^{tX_i}\Big] \\ &= \prod_{i=1}^{n} \mathsf{E}[e^{tX_i}] \quad (X_i\text{'s indep}) \\ &= \prod_{i=1}^{n} M_{X_i}(t). \end{aligned}$$

Corollary: If $X_1, \ldots, X_n$ are i.i.d. and $Y = \sum_{i=1}^n X_i$, then

$$M_Y(t) = [M_{X_1}(t)]^n.$$

Example: Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ Bern(p). Then by a previous example,

$$M_Y(t) = [M_{X_1}(t)]^n = (pe^t + q)^n.$$

So what use is a result like this?

Theorem: If X and Y have the same mgf, then they have the *same distribution* (at least in this course)!

Proof: Too hard for this course.

Example: The sum Y of n i.i.d. Bern(p) RV's is the same as a Binomial(n, p) RV.

By the previous example, $M_Y(t) = (pe^t + q)^n$. So all we need to show is that the mgf of the matches this.

Meanwhile, let $Z \sim \text{Bin}(n, p)$.

$$
\begin{aligned}
M_Z(t) &= \text{E}[e^{tZ}] = \sum_z e^{tz}\text{Pr}(Z = z) \\
&= \sum_{z=0}^{n} e^{tz} \binom{n}{z} p^z q^{n-z} \\
&= \sum_{z=0}^{n} \binom{n}{z} (pe^t)^z q^{n-z} \\
&= (pe^t + q)^n \quad \text{(by the binomial theorem)},
\end{aligned}
$$

and this matches the mgf of Y from the last pg.

Example: You can identify a distribution by its mgf.

$$M_X(t) \; = \; \left(\frac{3}{4}e^t + \frac{1}{4}\right)^{15}$$

implies that $X \sim \text{Bin}(15, 0.75)$.

Example:

$$M_Y(t) \; = \; e^{-2t}\left(\frac{3}{4}e^{3t} + \frac{1}{4}\right)^{15}$$

implies that Y has the same distribution as $3X - 2$, where $X \sim \text{Bin}(15, 0.75)$.

Theorem (Additive property of Binomials): If $X_1, \ldots, X_k$ are indep, with $X_i \sim \text{Bin}(n_i, p)$ (where p is the same for all X_i's), then

$$Y \equiv \sum_{i=1}^{k} X_i \sim \text{Bin}\Big(\sum_{i=1}^{k} n_i, p\Big).$$

Proof:

$$
\begin{aligned}
M_Y(t) &= \prod_{i=1}^{k} M_{X_i}(t) \quad \text{(mgf of indep sum)} \\
&= \prod_{i=1}^{k} (pe^t + q)^{n_i} \quad \text{(Binomial}(n_i, p)\text{ mgf)} \\
&= (pe^t + q)^{\sum_{i=1}^{k} n_i}.
\end{aligned}
$$

This is the mgf of the Bin($\sum_{i=1}^{k} n_i, p$), so we're done.