2.17 Conditional Distributions

Intro / Definition

Examples

Conditional Expectation

More Examples

A Word From Our Sponsor

Intro / Definition

Recall conditional probability: $\Pr(A|B) = \Pr(A \cap B)/\Pr(B)$ if $\Pr(B) > 0$.

Suppose that X and Y are jointly discrete RV's. Then if $\Pr(Y = y) > 0$,

$$\Pr(X = x|Y = y) = \frac{\Pr(X = x \cap Y = y)}{\Pr(Y = y)} = \frac{f(x, y)}{f_Y(y)}$$

$\Pr(X = x|Y = 2)$ defines the probabilities on X given that $Y = 2$.

Definition: If $f_Y(y) > 0$, then $f_{X|Y}(x|y) \equiv \frac{f(x,y)}{f_Y(y)}$ is the **conditional pmf/pdf of** X **given** $Y = y$.

Remark: Usually just write $f(x|y)$ instead of $f_{X|Y}(x|y)$.

Remark: Of course, $f_{Y|X}(y|x) = f(y|x) = \frac{f(x,y)}{f_X(x)}$.

Old Discrete Example: $f(x, y) = \Pr(X = x, Y = y)$.

	$X = 1$	$X = 2$	$X = 3$	$X = 4$	$f_Y(y)$
$Y = 1$	.01	.07	.09	.03	.2
$Y = 2$	**.20**	**.00**	**.05**	**.25**	**.5**
$Y = 3$	.09	.03	.06	.12	.3
$f_X(x)$	.3	.1	.2	.4	1

Find $f(x|2)$.

Then

$$f(x|2) = \frac{f(x,2)}{f_Y(2)} = \frac{f(x,2)}{0.5} = \begin{cases} 0.4 & \text{if } x = 1 \\ 0 & \text{if } x = 2 \\ 0.1 & \text{if } x = 3 \\ 0.5 & \text{if } x = 4 \\ 0 & \text{otherwise} \end{cases}$$

Old Cts Example:

$$f(x,y) = \frac{21}{4}x^2y, \quad \text{if } x^2 \le y \le 1$$

$$f_X(x) = \frac{21}{8}x^2(1-x^4), \quad \text{if } -1 \le x \le 1$$

$$f_Y(y) = \frac{7}{2}y^{5/2}, \quad \text{if } 0 \le y \le 1$$

Find $f(y|X = 1/2)$.

$$
\begin{aligned}
f(y|\frac{1}{2}) &= \frac{f(\frac{1}{2}, y)}{f_X(\frac{1}{2})} \\
&= \frac{\frac{21}{4} \cdot \frac{1}{4} y}{\frac{21}{8} \cdot \frac{1}{4} \cdot (1 - \frac{1}{16})}, \quad \text{if } \frac{1}{4} \le y \le 1 \\
&= \frac{32}{15} y, \quad \text{if } \frac{1}{4} \le y \le 1
\end{aligned}
$$

More generally,

$$f(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{\frac{21}{4}x^2 y}{\frac{21}{8}x^2(1-x^4)}, \quad \text{if } x^2 \le y \le 1$$

$$= \frac{2y}{1-x^4} \quad \text{if } x^2 \le y \le 1.$$

Note: $2/(1-x^4)$ is a constant with respect to y, and we can check to see that $f(y|x)$ is a legit condl pdf:

$$\int_{x^2}^{1} \frac{2y}{1-x^4}\, dy = 1.$$

Typical Problem: Given $f_X(x)$ and $f(y|x)$, find $f_Y(y)$.

Steps: (1) $f(x,y) = f_X(x) f(y|x)$

(2) $f_Y(y) = \int_{\Re} f(x,y)\, dx$.

Example: $f_X(x) = 2x,\ \ 0 < x < 1$.

Given $X = x$, suppose that $Y|x \sim \mathrm{U}(0,x)$. Now find $f_Y(y)$.

Solution: $Y|x \sim \text{U}(0,x) \;\Rightarrow\; f(y|x) = 1/x, \;\; 0 < y < x$.

So

$$\begin{aligned} f(x,y) &= f_X(x)f(y|x) \\ &= 2x \cdot \frac{1}{x}, \;\; \text{if } 0 < x < 1 \text{ and } 0 < y < x \\ &= 2, \;\; \text{if } 0 < y < x < 1. \end{aligned}$$

Thus,

$$f_Y(y) = \int_{\Re} f(x,y)\,dx = \int_y^1 2\,dx = 2(1-y), \;\; 0 < y < 1.$$

Conditional Expectation

Usual definition of expectation:

$$\mathsf{E}[Y] = \begin{cases} \sum_y y f(y) & \text{discrete} \\ \int_{\Re} y f(y)\, dy & \text{continuous} \end{cases}$$

$f(y|x)$ is the conditional pdf/pmf of Y given $X = x$.

Definition: The **conditional expectation** of Y given $X = x$ is

$$\mathsf{E}[Y|X = x] \equiv \begin{cases} \sum_y y f(y|x) & \text{discrete} \\ \int_{\Re} y f(y|x)\, dy & \text{continuous} \end{cases}$$

Note that $\mathsf{E}[Y|X=x]$ is a function of x.

Example: Suppose that

$$f(y|X=2) \;=\; \begin{cases} 0.2 & \text{if } y=1 \\ 0.3 & \text{if } y=2 \\ 0.5 & \text{if } y=3 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mathsf{E}[Y|X=2] \;=\; \sum_y y f(y|2) \;=\; 1(.2)+2(.3)+3(.5) \;=\; 2.3.$$

Old Cts Example:

$$f(x,y) \;=\; \frac{21}{4}x^2 y, \quad \text{if } x^2 \le y \le 1.$$

Recall that

$$f(y|x) \;=\; \frac{2y}{1-x^4} \;\; \text{if } x^2 \le y \le 1.$$

Thus,

$$\mathsf{E}[Y|x] \;=\; \int_{\Re} y f(y|x)\,dy \;=\; \frac{2}{1-x^4}\int_{x^2}^{1} y^2\,dy \;=\; \frac{2}{3}\cdot\frac{1-x^6}{1-x^4}.$$

Theorem (double expectations): $\mathsf{E}[\mathsf{E}(Y|X)] = \mathsf{E}[Y]$.

Remarks: Yikes, what the heck is this!? The exp value (averaged over all X's) of the conditional exp value (of $Y|X$) is the plain old exp value (of Y).

Think of the outside exp value as the exp value of $h(X) = \mathsf{E}(Y|X)$. Then the Law of the Unconscious Statistician miraculously gives us $\mathsf{E}[Y]$.

Proof (cts case): By the Unconscious Statistician,

$$
\begin{aligned}
\mathsf{E}[\mathsf{E}(Y|X)] &= \int_{\Re} \mathsf{E}(Y|x) f_X(x)\,dx \\
&= \int_{\Re} \left(\int_{\Re} y f(y|x)\,dy \right) f_X(x)\,dx \\
&= \int_{\Re} \int_{\Re} y f(y|x) f_X(x)\,dx\,dy \\
&= \int_{\Re} y \int_{\Re} f(x,y)\,dx\,dy \\
&= \int_{\Re} y f_Y(y)\,dy \;=\; \mathsf{E}[Y].
\end{aligned}
$$

Old Example: Suppose $f(x,y) = \frac{21}{4}x^2y$, if $x^2 \le y \le 1$. Find $\mathsf{E}[Y]$ **two ways**.

By previous examples, we know that

$$f_X(x) = \frac{21}{8}x^2(1-x^4), \quad \text{if } -1 \le x \le 1$$

$$f_Y(y) = \frac{7}{2}y^{5/2}, \quad \text{if } 0 \le y \le 1$$

$$\mathsf{E}[Y|x] = \frac{2}{3} \cdot \frac{1-x^6}{1-x^4}.$$

Solution #1 (old, boring way):

$$\mathsf{E}[Y] = \int_{\Re} y f_Y(y)\,dy = \int_0^1 \frac{7}{2} y^{7/2}\,dy = \frac{7}{9}.$$

Solution #2 (new, exciting way):

$$\begin{aligned} \mathsf{E}[Y] &= \mathsf{E}[\mathsf{E}(Y|X)] \\ &= \int_{\Re} \mathsf{E}(Y|x) f_X(x)\,dx \\ &= \int_{-1}^{1} \left(\frac{2}{3} \cdot \frac{1-x^6}{1-x^4}\right)\left(\frac{21}{8} x^2 (1-x^4)\right) dx = \frac{7}{9}. \end{aligned}$$

Notice that both answers are the same (good)!

Believe it or not, sometimes it's easier to calculate E$[Y]$ indirectly by using our double expectation trick.

And Now, A Word From Our Sponsor. . .

Congratulations! You are now done with the most difficult module of the course!

Things will get easier from here (I hope)!