

2.16 Bivariate Random Variables

Discrete Case

Continuous Case

Bivariate cdf's

Marginal Distributions

Now let's look at what happens when you consider 2 RV's simultaneously.

Example: Choose a person at random. Look at height and weight (X, Y) . Obviously, X and Y will be related somehow.

Discrete Case

Definition: If X and Y are discrete RV's, then (X, Y) is called a **jointly discrete bivariate RV**.

The joint (or bivariate) pmf is

$$f(x, y) = \Pr(X = x, Y = y).$$

Properties:

$$(1) 0 \leq f(x, y) \leq 1.$$

$$(2) \sum_x \sum_y f(x, y) = 1.$$

$$(3) A \subseteq \mathfrak{R}^2 \Rightarrow \Pr((X, Y) \in A) = \sum \sum_{(x, y) \in A} f(x, y).$$

Example: 3 sox in a box (numbered 1,2,3). Draw 2 sox at random w/o replacement. $X = \#$ of first sock, $Y = \#$ of second sock. The joint pmf $f(x, y)$ is

	$X = 1$	$X = 2$	$X = 3$	$\Pr(Y = y)$
$Y = 1$	0	1/6	1/6	1/3
$Y = 2$	1/6	0	1/6	1/3
$Y = 3$	1/6	1/6	0	1/3
$\Pr(X = x)$	1/3	1/3	1/3	1

$\Pr(X = x)$ is the “marginal” distribution of X .

$\Pr(Y = y)$ is the “marginal” distribution of Y .

By the law of total probability,

$$\Pr(X = 1) = \sum_{y=1}^3 \Pr(X = 1, Y = y) = 1/3.$$

In addition,

$$\begin{aligned} \Pr(X \geq 2, Y \geq 2) &= \sum_{x \geq 2} \sum_{y \geq 2} f(x, y) \\ &= f(2, 2) + f(2, 3) + f(3, 2) + f(3, 3) \\ &= 0 + 1/6 + 1/6 + 0 = 1/3. \end{aligned}$$

Continuous Case

Definition: If X and Y are cts RV's, then (X, Y) is a **jointly cts RV** if there exists a function $f(x, y)$ such that

$$(1) f(x, y) \geq 0, \forall x, y.$$

$$(2) \int \int_{\mathbb{R}^2} f(x, y) dx dy = 1.$$

$$(3) \Pr(A) = \Pr((X, Y) \in A) = \int \int_A f(x, y) dx dy.$$

In this case, $f(x, y)$ is called the **joint pdf**.

If $A \subseteq \mathbb{R}^2$, then $\Pr(A)$ is the volume between $f(x, y)$ and A .

Think of

$$f(x, y) dx dy \approx \Pr(x < X < x + dx, y < Y < y + dy).$$

Easy to see how all of this generalizes the 1-dimensional pdf, $f(x)$.

Easy Example: Choose a point at random in the interior of the circle $x^2 + y^2 = 16$. Find the pdf of (X, Y) .

Since the area of the circle is 16π ,

$$f(x, y) = \begin{cases} \frac{1}{16\pi} & \text{if } x^2 + y^2 \leq 16 \\ 0 & \text{otherwise} \end{cases}$$

Example: Suppose that

$$f(x, y) = \begin{cases} 4xy & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the prob (volume) of the region $0 \leq y \leq 1 - x^2$.

$$\begin{aligned} V &= \int_0^1 \int_0^{1-x^2} 4xy \, dy \, dx \\ &= \int_0^1 \int_0^{\sqrt{1-y}} 4xy \, dx \, dy \\ &= 1/3. \end{aligned}$$

Moral: Be careful with limits!

Bivariate cdf's

Definition: The **joint (bivariate) cdf** of X and Y is

$F(x, y) \equiv P(X \leq x, Y \leq y)$, for all x, y .

$$F(x, y) = \begin{cases} \sum \sum_{s \leq x, t \leq y} f(s, t) & \text{discrete} \\ \int_{-\infty}^y \int_{-\infty}^x f(s, t) ds dt & \text{continuous} \end{cases}$$

Properties:

$F(x, y)$ is non-decreasing in both x and y .

$$\lim_{x \rightarrow -\infty} F(x, y) = \lim_{y \rightarrow -\infty} F(x, y) = 0.$$

$$\lim_{x \rightarrow \infty} F(x, y) = F_Y(y) = \Pr(Y \leq y)$$

$$\lim_{y \rightarrow \infty} F(x, y) = F_X(x) = \Pr(X \leq x).$$

$F(x, y)$ is cts from the right in both x and y .

2.16 Bivariate RV's

Going from cdf's to pdf's (continuous case).

$$\text{1-dimension: } f(x) = F'(x) = \frac{d}{dx} \int_{-\infty}^x f(t) dt.$$

$$\text{2-D: } f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) = \frac{\partial^2}{\partial x \partial y} \int_{-\infty}^x \int_{-\infty}^y f(s, t) dt ds.$$

Example: Suppose

$$F(x, y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{if } x < 0 \text{ or } y < 0 \end{cases}$$

The “marginal” cdf of X is

$$F_X(x) = \lim_{y \rightarrow \infty} = \begin{cases} 1 - e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

The joint pdf is

$$\begin{aligned} f(x, y) &= \frac{\partial^2}{\partial x \partial y} F(x, y) \\ &= \frac{\partial}{\partial y} (e^{-x} - e^{-y} e^{-x}) \\ &= \begin{cases} e^{-(x+y)} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{if } x < 0 \text{ or } y < 0 \end{cases} \end{aligned}$$

Marginal Distributions — Distrns of X and Y .

Example (discrete case): $f(x, y) = \Pr(X = x, Y = y)$.

	$X = 1$	$X = 2$	$X = 3$	$X = 4$	$\Pr(Y = y)$
$Y = 1$.01	.07	.09	.03	.2
$Y = 2$.20	.00	.05	.25	.5
$Y = 3$.09	.03	.06	.12	.3
$\Pr(X = x)$.3	.1	.2	.4	1

By total probability,

$$\Pr(X = 1) = \Pr(X = 1, Y = \text{any } \#) = 0.3.$$

Definition: If X and Y are jointly discrete, then the **marginal pmf's** of X and Y are, respectively,

$$f_X(x) = \sum_y f(x, y)$$

and

$$f_Y(y) = \sum_x f(x, y)$$

Example (discrete case): $f(x, y) = \Pr(X = x, Y = y)$.

	$X = 1$	$X = 2$	$X = 3$	$X = 4$	$\Pr(Y = y)$
$Y = 1$.06	.02	.04	.08	.2
$Y = 2$.15	.05	.10	.20	.5
$Y = 3$.09	.03	.06	.12	.3
$\Pr(X = x)$.3	.1	.2	.4	1

Hmmm.... Compared to the last example, this has the *same marginals* but *different joint* distribution!

Definition: If X and Y are jointly cts, then the **marginal pdf's** of X and Y are, respectively,

$$f_X(x) = \int_{\mathfrak{R}} f(x, y) dy$$

and

$$f_Y(y) = \int_{\mathfrak{R}} f(x, y) dx$$

Example:

$$f(x, y) = \begin{cases} e^{-(x+y)} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_{\mathcal{R}} f(x, y) dy$$

$$= \int_0^{\infty} e^{-(x+y)} dy$$

$$= \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Example:

$$f(x, y) = \begin{cases} \frac{21}{4}x^2y & \text{if } x^2 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Note funny limits where the pdf is positive, i.e., $x^2 \leq y \leq 1$.

2.16 Bivariate RV's

$$\begin{aligned} f_X(x) &= \int_{\mathfrak{R}} f(x, y) dy \\ &= \int_{x^2}^1 \frac{21}{4} x^2 y dy \\ &= \begin{cases} \frac{21}{8} x^2 (1 - x^4) & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_{\mathcal{R}} f(x, y) dx \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{21}{4} x^2 y dx \\ &= \begin{cases} \frac{7}{2} y^{5/2} & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$