2.15 Functions of a Random Variable

Problem Statement

Discrete Case

Continuous Case

Inverse Transform Theorem

Problem: You have a RV X and you know its pmf/pdf $f(x)$.

Define $Y \equiv h(X)$ (some fn of X).

Find $g(y)$, the pmf/pdf of Y.

Discrete Case: X discrete implies Y discrete implies

$$\begin{aligned} g(y) &= \Pr(Y = y) \\ &= \Pr(h(X) = y) \\ &= \Pr(\{x | h(x) = y\}) \\ &= \sum_{x | h(x) = y} f(x) \end{aligned}$$

Example: X is the # of H's in 2 coin tosses. Want pmf for $Y = h(X) = X^2 - X$.

x	0	1	2
$\Pr(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$y = x^2 - x$	0	0	2

$g(0) = \Pr(Y = 0) = \Pr(X = 0 \text{ or } 1) = 3/4$ and $g(2) = \Pr(Y = 2) = \Pr(X = 2) = 1/4$.

$$g(y) = \begin{cases} 3/4 & \text{if } y = 0 \\ 1/4 & \text{if } y = 2 \end{cases}$$

Example: X is discrete with

$$f(x) = \begin{cases} 1/8 & \text{if } x = -1 \\ 3/8 & \text{if } x = 0 \\ 1/3 & \text{if } x = 1 \\ 1/6 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = X^2$ (Y can only equal 0,1,4).

$$g(y) = \begin{cases} \Pr(Y = 0) = f(0) = 3/8 \\ \Pr(Y = 1) = f(-1) + f(1) = 11/24 \\ \Pr(Y = 4) = f(2) = 1/6 \\ 0, \ \text{otherwise} \end{cases}$$

Continuous Case: X continuous implies Y can be continuous *or* discrete.

Example: $Y = X^2$ (clearly cts)

Example:

$$Y = \begin{cases} 0 & \text{if } X < 0 \\ 1 & \text{if } X \geq 0 \end{cases}$$

is *not* continuous.

Method:

Compute $G(y)$, the cdf of Y.

$$\begin{aligned} G(y) &= \mathsf{Pr}(Y \leq y) \\ &= \mathsf{Pr}(h(X) \leq y) \\ &= \int_{\{x|h(x)\leq y\}} f(x)\, dx. \end{aligned}$$

If $G(y)$ is cts, construct the pdf $g(y)$ by differentiating.

Example: $f(x) = |x|$, $-1 \le x \le 1$.

Find the pdf of the RV $Y = X^2$.

$$G(y) \;=\; \text{Pr}(Y \le y) \;=\; \text{Pr}(X^2 \le y) \;=\; \begin{cases} 0 & \text{if } y \le 0 \\ 1 & \text{if } y \ge 1 \\ (\star) & \text{if } 0 < y < 1 \end{cases},$$

where

$$(\star) \;=\; \text{Pr}(-\sqrt{y} \le X \le \sqrt{y}) \;=\; \int_{-\sqrt{y}}^{\sqrt{y}} |x|\, dx \;=\; y.$$

Thus,

$$G(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ 1 & \text{if } y \geq 1 \\ y & \text{if } 0 < y < 1 \end{cases}$$

This implies

$$g(y) = G'(y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or } y \geq 1 \\ 1 & \text{if } 0 < y < 1 \end{cases}$$

This is the U(0,1) distribution!

Example: Suppose $U \sim \mathsf{U}(0,1)$. Find the pdf of $V = -\ell\mathsf{n}(1-U)$.

$$\begin{aligned}
G(y) &= \Pr(V \le y) \\
&= \Pr(-\ell\mathsf{n}(1-U) \le y) \\
&= \Pr(1-U \ge e^{-y}) \\
&= \Pr(U \le 1-e^{-y}) \\
&= \int_0^{1-e^{-y}} f(u)\,du \\
&= 1-e^{-y} \quad (\text{since } f(u)=1)
\end{aligned}$$

Thus,

$$G(y) = \begin{cases} 0 & \text{if } y \le 0 \\ 1 - e^{-y} & \text{if } y > 0 \end{cases}$$

Taking the derivative, we have

$$g(y) = \begin{cases} 0 & \text{if } y \le 0 \\ e^{-y} & \text{if } y > 0 \end{cases}$$

Wow! This implies $V \sim \text{Exp}(\lambda)$.

We can generalize this result. . .

Inverse Transform Theorem: Suppose X is a RV having cdf $F(x)$. Then the *random variable* $F(X) \sim$ U$(0,1)$.

Proof (only do cts case): Let $Y = F(X)$. Then the cdf of Y is

$$\begin{aligned} G(y) &= \text{Pr}(Y \le y) \\ &= \text{Pr}(F(X) \le y) \\ &= \text{Pr}(X \le F^{-1}(y)) \text{ (the cdf is mono. increasing)} \\ &= F(F^{-1}(y)) \; (F(x) \text{ is the cdf of } X) \\ &= y. \text{ Uniform!} \end{aligned}$$

Remark: this is a great theorem, since it applies to all RV's X.

Corollary: $X = F^{-1}(U)$, so you can plug in a U(0,1) RV into the inverse cdf to generate a realization of a RV having X's distribution.

Remark: This is what we did in the example on the previous page. This trick has tremendous applications in simulation.