Cumulative Distribution Functions

Definition

Continuous cdf's

Discrete cdf's

Properties

Definition: For any RV X, the **cumulative distribution function** (cdf) is defined for all x by $F(x) \equiv P(X \leq x)$.

X continuous implies

$$F(x) = \int_{-\infty}^{x} f(t)\, dt.$$

X discrete implies

$$F(x) = \sum_{\{y|y \leq x\}} f(y) = \sum_{\{y|y \leq x\}} \Pr(X = y).$$

Continuous cdf's

Theorem: If X is a **continuous** RV, then $f(x) = F'(x)$.

Proof: $F'(x) = \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x)$, by the fundamental theorem of calculus.

Remark: If X is continuous, you can get from the pdf $f(x)$ to the cdf $F(x)$ by integrating.

Example: $X \sim \mathsf{U}(0,1)$.

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Example: $X \sim \text{Exp}(\lambda)$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - e^{-\lambda x} & \text{if } x > 0 \end{cases}$$

Discrete cdf's

Example: Flip a coin twice. $X =$ number of H's.

$$X = \begin{cases} \text{0 or 2} & \text{w.p. } 1/4 \\ 1 & \text{w.p. } 1/2 \end{cases}$$

$$F(x) = \Pr(X = x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/4 & \text{if } 0 \le x < 1 \\ 3/4 & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$$

(You have to be careful about "$\geq$" vs. "$<$".)

Properties of all cdf's

$F(x)$ is *non-decreasing* in x, i.e., $x_1 < x_2$ implies that $F(x_1) \leq F(x_2)$.

$\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.

$F(x)$ is *right-cts* at every point x.

Theorem: $\text{Pr}(X > x) = 1 - F(x)$.

Proof:

$$1 \;=\; \text{Pr}(X \le x) + \text{Pr}(X > x) \;=\; F(x) + \text{Pr}(X > x).$$

Theorem:

$$x_1 < x_2 \Rightarrow \Pr(x_1 < X \leq x_2) = F(x_2) - F(x_1).$$

Proof:

$$\begin{aligned}
&\Pr(x_1 < X \leq x_2) \\
&= \Pr(X > x_1 \cap X \leq x_2) \\
&= \Pr(X > x_1) + \Pr(X \leq x_2) - \Pr(X > x_1 \cup X \leq x_2) \\
&= 1 - F(x_1) + F(x_2) - 1.
\end{aligned}$$