Continuous Random Variables

Example

Probability Density Function

Exponential Distribution

Uniform Distribution

Yet Another Example

Example: Pick a point X randomly between 0 and 1.

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \Pr(x_1 < X < x_2) &= \text{ area under } f(x) \text{ from } x_1 \text{ to } x_2 \\ &= x_2 - x_1. \end{aligned}$$

Definition: Suppose X is a continuous RV. $f(x)$ is the **probability density function** (pdf) if

- $\int_{\Re} f(x)\,dx = 1$ (area under $f(x)$ is 1)

- $f(x) \geq 0, \quad \forall x$ (always non-negative)

- If $A \subseteq \Re$, then $\Pr(X \in A) = \int_A f(x)\,dx$ (probability that X is in a certain region A)

Remarks: If X is a continuous RV, then

$\Pr(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x)\,dx$.

An individual point has prob 0, i.e., $\Pr(X = x) = 0$.

Think of $f(x)\,dx \approx \Pr(x < X < x + dx)$.

Note that $f(x)$ denotes both pmf (**discrete** case) and pdf (**continuous** case) — but they are **different**:

$f(x) = \Pr(X = x)$ if X is **discrete**.
Must have $0 \leq f(x) \leq 1$.

$f(x)\,dx \approx \Pr(x < X < x + dx)$ if X is **continuous**.
Must have $f(x) \geq 0$ (and possibly > 1).

If X is cts, we calculate the prob of an event by integrating, $\Pr(X \in A) = \int_A f(x)\,dx$.

Example: X has the **exponential distriibution** with parameter $\lambda > 0$ if it has pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Notation: $X \sim \text{Exp}(\lambda)$

Note: $\int_{\Re} f(x)\,dx = \int_0^\infty \lambda e^{-\lambda x}\,dx = 1$ (as desired).

Example: Suppose $X \sim \text{Exp}(1)$. Then

$\Pr(X \le 3) = \int_0^3 e^{-x}\,dx = 1 - e^{-3}$.

$\Pr(X \ge 5) = \int_5^\infty e^{-x}\,dx = e^{-5}$.

$\Pr(2 \le X < 4) = \Pr(2 \le X \le 4) = \int_2^4 e^{-x}\,dx$
$= e^{-2} - e^{-4}$.

$\Pr(X = 3) = \int_3^3 e^{-x}\,dx = 0$.

Example: If X is "equally likely" to be anywhere between a and b, then X has the **uniform distribution** on (a, b).

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Notation: $X \sim \mathrm{U}(a, b)$

Note: $\int_{\Re} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = 1$ (as desired).

Example: Suppose X is a cts RV with pdf

$$f(x) = \begin{cases} cx^2 & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find c.

Answer: $1 = \int_{\Re} f(x)\,dx = \int_0^2 cx^2\,dx = \frac{8}{3}c$, so $c = 3/8$.

Find $\Pr(0 < X < 1)$.

Answer: $\Pr(0 < X < 1) = \int_0^1 \frac{3}{8}x^2\,dx = 1/8$.

Find $\Pr(0 < X < 1 | \frac{1}{2} < X < \frac{3}{2})$.

Answer:

$$\Pr\left(0 < X < 1 \middle| \frac{1}{2} < X < \frac{3}{2}\right)$$

$$= \frac{\Pr(0 < X < 1 \text{ and } \frac{1}{2} < X < \frac{3}{2})}{\Pr(\frac{1}{2} < X < \frac{3}{2})}$$

$$= \frac{\Pr(\frac{1}{2} < X < 1)}{\Pr(\frac{1}{2} < X < \frac{3}{2})}$$

$$= \frac{\int_{1/2}^{1} \frac{3}{8} x^2 \, dx}{\int_{1/2}^{3/2} \frac{3}{8} x^2 \, dx} = 7/26.$$