Discrete Random Variables

Some More Definitions

Discrete Uniform and Binomial Distributions

Poisson Distribution

Definition: If X is a discrete RV, its **probability mass function** (pmf) is $f(x) \equiv \Pr(X = x)$.

Note that $f(x) \geq 0$, $\sum_x f(x) = 1$.

Example: Flip 2 coins. Let X be the number of heads.

$$f(x) = \begin{cases} 1/4 & \text{if } x = 0 \text{ or } 2 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Example: **Uniform Distribution** on integers $1, 2, \ldots, n$. X can equal $1, 2, \ldots, n$, each with prob $1/n$. $f(i) = 1/n$, $i = 1, 2, \ldots, n$.

Example/Definition: Let X denote the number of "successes" from n independent trials such that the Pr(success) at each trial is p $(0 \leq p \leq 1)$. Then X has the **Binomial Distribution** with parameters n and p.

The trials are referred to as **Bernoulli trials**.

Notation: $X \sim \text{Bin}(n, p)$. "X has the Bin distribution"

Example: Roll a die 3 indep times. Find

$$\text{Pr(Get exactly two 6's)}.$$

"success" (6)
"failure" (1,2,3,4,5)

All 3 trials are indep, and Pr(success) $= 1/6$ doesn't change from trial to trial.

Let $X =$ # of 6's. Then $X \sim \text{Bin}(3, \frac{1}{6})$.

Theorem: If $X \sim \text{Bin}(n, p)$, then the prob of k successes in n trials is

$$\Pr(X = k) \ = \ \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, \ldots, n,$$

where $q = 1 - p$.

Proof: Particular sequence of successes and failures:

$$\underbrace{SS \cdots S}_{k \text{ successes}} \ \underbrace{FF \cdots F}_{n-k \text{ failures}} \quad (\text{prob} = p^k q^{n-k})$$

Number of ways to arrange the seq is $\binom{n}{k}$. Done.

Back to the dice example, where $X \sim \text{Bin}(3, \frac{1}{6})$, and we want Pr(Get exactly two 6's).

$n = 3$, $k = 2$, $p = 1/6$, $q = 5/6$.

$$\Pr(X = 2) \;=\; \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^1 \;=\; \frac{15}{216}$$

k	0	1	2	3
$\Pr(X = k)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

Example: Toss a die 5 times.

$A =$ "outcome is divisible by 3" $= \{3, 6\}$

Find Pr(A will occur exactly 4 times).

Let $X =$ the number of times A occurs $\sim$ Bin(5, 1/3).

$$\Pr(X = 4) \;=\; \binom{5}{4} (\frac{1}{3})^4 (\frac{2}{3})^{5-4} \;=\; \frac{10}{243}$$

Example: Roll 2 dice 12 times.

Find Pr(Result will be 7 or 11 exactly 3 times).

Let $X =$ the number of times get 7 or 11.

$\text{Pr}(7 \text{ or } 11) \;=\; \text{Pr}(7) + \text{Pr}(11) \;=\; \frac{6}{36} + \frac{2}{36} \;=\; \frac{2}{9}.$

So $X \sim \text{Bin}(12, 2/9)$.

$$\text{Pr}(X = 3) \;=\; \binom{12}{3} \left(\frac{2}{9}\right)^3 \left(\frac{7}{9}\right)^9.$$

Definition: If $\Pr(X = k) = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, 2, \ldots$, $\lambda > 0$, we say that X has the **Poisson distribution** with parameter λ.

Notation: $X \sim \text{Pois}(\lambda)$.

Example: Suppose the number of raisins in a cup of cookie dough is Pois(10). Find the prob that a cup of dough has at least 4 raisins.

$$\begin{aligned} \Pr(X \geq 4) &= 1 - \Pr(X = 0, 1, 2, 3) \\ &= 1 - e^{-10}\Big(\frac{10^0}{0!} + \frac{10^1}{1!} + \frac{10^2}{2!} + \frac{10^3}{3!}\Big) \\ &= 0.9897. \end{aligned}$$