

Basics of Probability — Modules (Updated 9/2/07)

1. Intro / Examples
2. Set Theory
3. Experiments and Sample Spaces
4. Definition of Probability
5. Finite Sample Spaces
6. Counting Techniques
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Mathematical Models for describing observable phenomena:

- Deterministic
- Probabilistic

Deterministic Models

- Ohm's Law ($I = E/R$)
- Drop an object from height h_0 . After t sec, height is $h(t) = h_0 - 16t^2$.
- Deposit \$1000 in a continuously compounding checking 3% account. At time t , it's worth $\$1000e^{.03t}$.

Probabilistic Models — Involve uncertainty

- How much snow will fall tomorrow?
- Will IBM make a profit this year?
- Should I buy a call or put option?
- Can I win in blackjack if I use a certain strategy?

Some Cool Examples

1. Birthday Problem — Assume all 365 days have equal probability of being a person's birthday (ignore Feb 29). Then...

If there are **23** people in the room, the odds are better than 50–50 that there will be a match.

If there are **50** people, the probability is about 97%!

2. Monopoly — In the long run, the property having the highest probability of being landed on is Illinois Ave.

3. Poker — Pick 5 cards from a standard deck. Then

$\Pr(\text{exactly 2 pairs}) \approx 0.0475$

$\Pr(\text{full house}) \approx 0.00144$

$\Pr(\text{flush}) \approx 0.00198$

4. Stock Market — Monkeys randomly selecting stocks could have outperformed most market analysts during the past year.

5. A couple has two kids and at least one is a boy. What's the probability that BOTH are boys?

Possibilities: GG, BG, GB, BB. Eliminate GG since we know that there's at least one boy. Then $\Pr(\text{BB}) = 1/3$.

6. Vietnam Lottery

7. Ask Marilyn. You are a contestant at a game show. Behind one of three doors is a car; behind the other two are goats. You pick door A. Monty Hall opens door B and reveals a goat. Monty offers you a chance to switch to door C. What should you do?

Working Definitions

Probability — Methodology that describes the random variation in systems. (We'll spend about 40% of our time on this.)

Statistics — Uses sample data to draw general conclusions about the population from which the sample was taken. (60% of our time.)

The Joy of Sets

Definition: A **set** is a collection of objects. Members of a set are called **elements**.

Notation:

A, B, C, \dots for sets; a, b, c, \dots for elements

\in for membership, e.g., $x \in A$

\notin for non-membership, e.g., $x \notin A$

U is the universal set (i.e., everything)

\emptyset is the empty set.

Examples:

$$A = \{1, 2, \dots, 10\}. \quad 2 \in A, \quad 49 \notin A.$$

$$B = \{\text{basketball, baseball}\}$$

$$C = \{x \mid 0 \leq x \leq 1\} \quad (\text{"|"} \text{ means "such that"})$$

$$D = \{x \mid x^2 = 9\} = \{\pm 3\} \quad (\text{either is fine})$$

$$E = \{x \mid x \in \mathfrak{R}, x^2 = -1\} = \emptyset \quad (\mathfrak{R} \text{ is the real line})$$

Definition: If every element of set A is an element of set B then A is a **subset** of B , i.e., $A \subseteq B$.

Definition: $A = B$ iff (if and only if) $A \subseteq B$ and $B \subseteq A$.

Properties:

$$\emptyset \subseteq A; A \subseteq U; A \subseteq A$$

$$A \subseteq B \text{ and } B \subseteq C \Rightarrow (\text{implies}) A \subseteq C$$

Remark: Order of element listing is immaterial, e.g,
 $\{a, b, c\} = \{b, c, a\}$.

Definitions: **Complement** of A with respect to U is $\bar{A} \equiv \{x|x \in U \text{ and } x \notin A\}$.

Intersection of A and B is $A \cap B \equiv \{x|x \in A \text{ and } x \in B\}$.

Union of A and B is $A \cup B \equiv \{x|x \in A \text{ or } x \in B \text{ (or both)}\}$.

(Remember Venn diagrams?)

Example:

Suppose $U = \{\text{letters of the alphabet}\}$, $A = \{\text{vowels}\}$,
and $B = \{a, b, c\}$. Then

$$\bar{A} = \{\text{consonants}\}$$

$$A \cap B = \{a\}$$

$$A \cup B = \{a, b, c, e, i, o, u\}$$

If $A \cap B = \emptyset$, then A and B are **disjoint** (or **mutually exclusive**).

Definitions:

Minus: $A - B \equiv A \cap \bar{B}$

Symmetric difference or XOR:

$$A \Delta B \equiv (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

The **cardinality** of A , $|A|$, is the number of elements in A . A is **finite** if $|A| < \infty$.

Examples:

$A = \{3, 4\}$ is finite since $|A| = 2$.

$B = \{1, 2, 3, \dots\}$ is **countably infinite**.

$C = \{x \mid x \in [0, 1]\}$ is **uncountably infinite**.

Laws of Operation:

1. Complement Law: $A \cup \bar{A} = U$, $A \cap \bar{A} = \emptyset$, $\bar{\bar{A}} = A$

2. Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$

3. DeMorgan's: $\overline{A \cup B} = \bar{A} \cap \bar{B}$, $\overline{A \cap B} = \bar{A} \cup \bar{B}$

4. Associative: $A \cup (B \cup C) = (A \cup B) \cup C$,

$A \cap (B \cap C) = (A \cap B) \cap C$

5. Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proofs: Easy. Could use Venn diagrams or many other ways.

Experiments and Sample Spaces

Consider a “random” experiment:

E_1 : Toss a die and observe the outcome.

Definition: A **sample space** associated with an experiment E is the set of *all* possible outcomes of E . It's usually denoted by S or Ω .

Examples:

E_1 has sample space $S_1 = \{1, 2, 3, 4, 5, 6\}$.

Another sample space for E_1 is $S'_1 = \{\text{odd, even}\}$.

So a sample space doesn't have to be unique!

E_2 : Toss a coin 3 times and observe the sequence of H 's and T 's.

$$S_2 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

E_3 : A new light bulb is tested to see how long it lasts.

$$S_3 = \{t | t \geq 0\}.$$

Definition: An **event** is a set of possible outcomes. Thus, any subset of S is an event.

Example (for E_1): If A_1 is the event “an even number occurs,” then $A_1 = \{2, 4, 6\}$, i.e., when the die is tossed, we get 2 or 4 or 6.

Remark: \emptyset is an event of S (“nothing happens”)
 S is an event of S (“something happens”)

Remark: If A is an event, then \bar{A} is the complementary (opposite) event.

Example (for E_1):

$A_1 = \{2, 4, 6\} \Rightarrow \bar{A}_1 = \{1, 3, 5\}$ (i.e., “an odd number occurs”)

Remark: If A and B are events, then $A \cup B$ and $A \cap B$ are events.

Example (for E_2): Let

$$\begin{aligned} A_2 &= \text{“exactly one } T \text{ was observed”} \\ &= \{HHT, HTH, THH\} \end{aligned}$$

$$B_2 = \text{“no } T\text{'s observed”} = \{HHH\}$$

$$\begin{aligned} C_2 &= \text{“first coin is } H\text{”} \\ &= \{HHH, HHT, HTH, HTT\} \end{aligned}$$

Then

$$\begin{aligned} A_2 \cup B_2 &= \text{“at most one } T \text{ observed”} \\ &= \{HHT, HTH, THH, HHH\} \end{aligned}$$

$$A_2 \cap C_2 = \{HHT, HTH\}$$

Probability Basics (see Meyer 1970)

Suppose A is some event for a sample space S . What's the prob that A will occur, i.e., $\Pr(A)$?

Example: Toss a fair coin. $S = \{H, T\}$. What's the prob that H will come up?

$$\Pr(\{H\}) = \Pr(H) = 1/2.$$

What does this mean?

Frequentist view: If the experiment were repeated n times, where n is very large, we'd expect about $1/2$ of the tosses to be H 's.

$$\frac{\text{Total \# of } H\text{'s out of } n \text{ tosses}}{n} \approx 1/2.$$

Example: Toss a fair die. $S = \{1, 2, 3, 4, 5, 6\}$, where each individual outcome has prob $1/6$. Then $\Pr(1, 2) = 1/3$.

Definition: With each event $A \subseteq S$, we associate a number $\Pr(A)$, called “the **probability** of A ,” satisfying the following *axioms*:

(1) $0 \leq \Pr(A) \leq 1$ (prob's are *always* betw. 0 and 1).

(2) $\Pr(S) = 1$ (prob of *some* outcome is 1). Example:

Die. $\Pr(S) = \Pr(1, 2, 3, 4, 5, 6) = 1$.

(3) If $A \cap B = \emptyset$, then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Example: $\Pr(1 \text{ or } 2) = \Pr(1) + \Pr(2) = 1/6 + 1/6 = 1/3$.

(4) Suppose A_1, A_2, \dots is a sequence of disjoint events (i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$). Then

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i).$$

Example: Toss a coin until the first H appears.

$$S = \{H, TH, TTH, TTTH, \dots\}.$$

Define the *disjoint* events

$$A_1 = \{H\}, A_2 = \{TH\}, A_3 = \{TTH\}, \dots$$

Then

$$1 = \Pr(S) = \Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i).$$

More Nifty Properties

Theorem 1: $\Pr(\emptyset) = 0$.

Proof: Since $A \cap \emptyset = \emptyset$, we have that A and \emptyset are disjoint. So Axiom (3) implies

$$\Pr(A) = \Pr(A \cup \emptyset) = \Pr(A) + \Pr(\emptyset).$$

Note: Converse is *false*: $\Pr(A) = 0$ does *not* imply $A = \emptyset$. Example: Pick a random number betw. 0 and 1.

Theorem 2: $\Pr(\bar{A}) = 1 - \Pr(A)$.

Proof:

$$\begin{aligned} 1 &= \Pr(S) \text{ (by Axiom (2))} \\ &= \Pr(A \cup \bar{A}) \\ &= \Pr(A) + \Pr(\bar{A}) \text{ (} A \cap \bar{A} = \emptyset \text{; Axiom (3)).} \end{aligned}$$

Example: The probability that it'll rain tomorrow is one minus the probability that it won't rain.

Theorem 3: For any two events A and B ,

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

Proof: First observe that $B = (A \cap B) \cup (\bar{A} \cap B)$ where $A \cap B$ and $\bar{A} \cap B$ are disjoint. Thus

$$\Pr(B) = \Pr(A \cap B) + \Pr(\bar{A} \cap B) \quad (*)$$

and so

$$\begin{aligned} \Pr(A \cup B) &= \Pr(A) + \Pr(\bar{A} \cap B) \quad (A, \bar{A} \cap B \text{ are disjoint}) \\ &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \quad (\text{by } (*)). \end{aligned}$$

Remark: Can also do an easy Venn diagram proof.
(Subtract $\Pr(A \cap B)$ to avoid double-counting.)

Remark: Axiom (3) is a “special case” of this theorem
in which $A \cap B = \emptyset$.

Example: Suppose there's...

40% chance of colder weather

10% chance of rain *and* colder weather

80% chance of rain *or* colder weather.

Find the chance of rain.

$$\begin{aligned}\Pr(R) &= \Pr(R \cup C) - \Pr(C) + \Pr(R \cap C) \\ &= 0.8 - 0.4 + 0.1 = 0.5.\end{aligned}$$

Theorem 4: For any three events A , B , and C ,

$$\begin{aligned} & \Pr(A \cup B \cup C) \\ &= \Pr(A) + \Pr(B) + \Pr(C) \\ &\quad - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) \\ &\quad + \Pr(A \cap B \cap C) \end{aligned}$$

Example: 75% of Atlantans jog (J), 20% like ice cream (I), and 40% enjoy music (M). Also, 15% J and I , 30% J and M , 10% I and M , and 5% do all three. Find the prob that a random resident will engage in at least one of the three activities.

$$\begin{aligned} &P(J \cup I \cup M) \\ &= P(J) + P(I) + P(M) \\ &\quad - P(J \cap I) - P(J \cap M) - P(I \cap M) \\ &\quad + P(J \cap I \cap M) \\ &= .75 + .20 + .40 - .15 - .30 - .10 + .05 = .85. \end{aligned}$$

Find the prob of precisely one activity.

$$\begin{aligned} P(J \cap \bar{I} \cap \bar{M}) + P(\bar{J} \cap I \cap \bar{M}) + P(\bar{J} \cap \bar{I} \cap M) \\ = .35 + 0 + .05 = .40. \end{aligned}$$

How'd we get those?? Use Venn diagram, starting from the center and building out.

Remark: Here is the general **principle of inclusion-exclusion**:

$$\begin{aligned} & \Pr(A_1 \cup A_2 \cup \cdots \cup A_n) \\ &= \sum_{i=1}^n \Pr(A_i) - \sum \sum_{i < j} \Pr(A_i \cap A_j) \\ & \quad + \sum \sum \sum_{i < j < k} \Pr(A_i \cap A_j \cap A_k) \\ & \quad + \cdots + (-1)^{n-1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n). \end{aligned}$$

(Bonus) Theorem 5: $A \subseteq B \Rightarrow \Pr(A) \leq \Pr(B)$.

Proof:

$$\begin{aligned}\Pr(B) &= \Pr(A \cup (\bar{A} \cap B)) \\ &= \Pr(A) + \Pr(\bar{A} \cap B) \\ &\geq \Pr(A).\end{aligned}$$

Remark: $A \subseteq B$ means that B occurs whenever A occurs; so the Theorem makes intuitive sense.

Finite Sample Spaces

Suppose S is finite, say $S = \{a_1, a_2, \dots, a_n\}$.

Let B be an event consisting of r ($\leq n$) outcomes, i.e., $B = \{a_{j_1}, a_{j_2}, \dots, a_{j_r}\}$, where the j_i 's represent r indices from $\{1, 2, \dots, n\}$. Then $\Pr(B) = \sum_{i=1}^r \Pr(a_{j_i})$.

Note: “Choosing an object **at random**” means that each object has the same prob of being chosen.

Example: You have 2 red cards, a blue card, and a yellow. Pick one card at random.

$$S = \{\text{red, blue, yellow}\} = \{a_1, a_2, a_3\}$$

$$\Pr(a_1) = 1/2, \Pr(a_2) = 1/4, \Pr(a_3) = 1/4.$$

$$\Pr(\text{red or yellow}) = \Pr(a_1) + \Pr(a_3) = 3/4.$$

Definition: A **simple sample space** (SSS) is a finite sample space in which all outcomes are *equally likely*.

Remark: In the above example, S is *not* simple since $\Pr(a_1) \neq \Pr(a_2)$.

Example: Toss 2 fair coins.

$S = \{HH, HT, TH, TT\}$ is a SSS (all prob's are $1/4$).

$S' = \{0, 1, 2\}$ (number of H 's) is *not* a SSS. Why?

Theorem: For any event A in a SSS S ,

$$\Pr(A) = \frac{|A|}{|S|} = \frac{\# \text{ elements in } A}{\# \text{ elements in } S}.$$

Example: Die. $A = \{1, 2, 4, 6\}$ (each with prob $1/6$).

$$\Pr(A) = 4/6.$$

Example: Roll a pair of dice. Possible results (each w.p. $1/36$):

$$\begin{array}{cccc} 1,1 & 1,2 & \cdots & 1,6 \\ 2,1 & 2,2 & \cdots & 2,6 \\ & \vdots & & \\ 6,1 & 6,2 & \cdots & 6,6 \end{array}$$

Sum	2	3	4	5	6	7	8	9	10	11	12
Prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Counting Techniques — count the elements in events from a SSS.

Multiplication Rule

Addition Rule

Permutations

Combinations

Multiplication Rule

Two operations are performed one after the other:

- (a) The first operation can be done in n_1 ways.
- (b) Regardless of the way in which the first operation was performed, the second can be performed in n_2 ways.

The $\#$ ways to perform the two operations together is $n_1 n_2$.

Example: 3 ways to go from City A to B, and 4 ways to go from B to C. Then the you can go from A to C (via B) in 12 ways.

Example: Roll 2 dice. How many outcomes? (Assume $(3, 2) \neq (2, 3)$.) Answer is 36.

Example: Select 2 cards from a deck without replacement and *care about order* (i.e., $(Q\spadesuit, 7\clubsuit) \neq (7\clubsuit, Q\spadesuit)$). How many ways can you do this? Answer: $52 \cdot 51 = 2652$.

Example: Box of 10 sox — 2 red and 8 black. Pick 2 w/o repl.

(a) Let A be the event that both are red.

$$\Pr(A) = \frac{\# \text{ ways to pick 2 reds}}{\# \text{ ways to pick 2 sox}} = \frac{2 \cdot 1}{10 \cdot 9} = \frac{1}{45}.$$

(b) Let B be the event that both are black.

$$\Pr(B) = \frac{8 \cdot 7}{10 \cdot 9} = \frac{28}{45}.$$

(c) Let C be one of each color.

$$\begin{aligned}\Pr(C) &= 1 - \Pr(\bar{C}) \\ &= 1 - \Pr(A \cup B) \\ &= 1 - \Pr(A) - \Pr(B) \quad (A \text{ and } B \text{ disjoint}) \\ &= 16/45.\end{aligned}$$

Remark: The multiplication rule can be extended to more than 2 operations.

Example: Flip 3 coins. $2 \times 2 \times 2 = 8$ possible outcomes.

Example: Toss n dice. 6^n possible outcomes.

Addition Rule

Can use method A in n_A ways.

Can use method B in n_B ways.

If only one method can be used, you have $n_A + n_B$ ways of doing so.

Example: Go to Starbucks and have a muffin (blueberry or oatmeal) or a bagel (sesame, plain, salt), but not both. $2 + 3 = 5$ choices.

Permutations

Definition: An arrangement of n symbols in a *definite order* is a **permutation** of the n symbols.

Example: How many ways to arrange the numbers 1,2,3? Answer: 6 ways — 123, 132, 213, 231, 312, 321.

Example: How many ways to arrange $1, 2, \dots, n$?

(choose first)(choose second) \cdots (choose n th)

$$n(n - 1)(n - 2) \cdots 2 \cdot 1 = n!.$$

Example: Baseball manager has 9 players on his team.

Find the # of possible batting orders. Answer: $9! = 362880$.

Definition: The # of r -tuples we can make from n different symbols (each used at most once) is called the # of **permutations of n things taken r -at-a-time**,

$$P_{n,r} \equiv \frac{n!}{(n-r)!} \quad (*).$$

Note that $0! = 1$ and $P_{n,n} = n!$.

Example: How many ways can you take two symbols from a, b, c, d ? Ans: $P_{4,2} = 4!/2! = 12$ — $ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc$.

Proof (of (*)):

$$\begin{aligned}P_{n,r} &= (\text{choose first})(\text{second})\cdots(r\text{th}) \\ &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 2 \cdot 1}{(n-r)\cdots 2 \cdot 1} \\ &= \frac{n!}{(n-r)!}.\end{aligned}$$

Example: How many ways to fill the first 4 positions of a batting order?

$n = 9$ players, $r = 4$ positions.

$P_{9,4} = 9!/(9 - 4)! = 3024$ ways.

Example: How many of these 3024 ways has Smith batting first?

Method 1: First 4 positions: (Smith,?,?,?). This is equiv to choosing 3 players from the remaining 8.

$$P_{8,3} = 8!/(8 - 3)! = 336 \text{ ways.}$$

Method 2: It's clear that each of the 9 players is equally likely to bat first. Thus, $3024/9 = 336$.

Example: How many license plates of 6 digits can be made from the numbers 1,2,...,9...

(a) with no repetitions? (e.g., 123465) $P_{9,6} = 9!/3! = 60480$.

(b) allowing repetitions? (e.g., 123345 or 123465) $9 \times 9 \times \dots \times 9 = 9^6 = 531441$.

(c) containing repetitions? $531441 - 60480 = 470961$.

Combinations

Suppose we only want to count the number of ways to choose r out of n objects *without* regard to order, i.e., count the number of different subsets of these n objects that contain exactly r objects.

Example: How many subsets of $\{1, 2, 3\}$ contain exactly 2 elements? (Order isn't important.)

3 subsets — $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$

Definition: The $\#$ of subsets with r elements of a set with n elements is called the number of **combinations of n things taken r -at-a-time**.

Notation: $\binom{n}{r}$ or $C_{n,r}$ (read as “ n choose r ”). These are also called **binomial coefficients**.

Difference between permutations and combinations:

Combinations — not concerned w/order: $(a, b, c) = (b, a, c)$.

Permutations — concerned w/order: $(a, b, c) \neq (b, a, c)$.

The number of permutations of n things taken r -at-a-time is always at least as large as the number of combinations. In fact, . . .

Remark: Choosing a permutation is the same as first choosing a combination *and* then putting the elements in order, i.e.,

$$\frac{n!}{(n-r)!} = \binom{n}{r} r!$$

So

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

$$\binom{n}{r} = \binom{n}{n-r}, \quad \binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = \binom{n}{n-1} = n.$$

Binomial Theorem:

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}$$

This is where Pascal's \triangle comes from!

Corollary: Surprising fact:

$$\sum_{i=0}^n \binom{n}{i} = 2^n.$$

Proof:

$$2^n = (1 + 1)^n = \sum_{i=0}^n \binom{n}{i} 1^i 1^{n-i}.$$

Example: An NBA team has 12 players. How many ways can the coach choose the starting 5?

$$\binom{12}{5} = \frac{12!}{5!7!} = 792.$$

Example: Smith is one of the players on the team. How many of the 792 starting line-ups include him?

$$\binom{11}{4} = \frac{11!}{4!7!} = 330.$$

(Smith gets one of the five positions for free; there are now 4 left to be filled by the remaining 11 players.)

Example: 7 red shoes, 5 blues. Find the number of arrangements.

R B R R B B R R R B R B

I.e., how many ways to put 7 reds in 12 slots?

Answer: $\binom{12}{7}$.

Some applications of counting techniques.

Hypergeometric problems

Permutations vs. Combinations

Birthday problem

Envelope problem

Poker probabilities

Multinomial coefficients

Hypergeometric Distribution

You have a objects of type 1 and b objects of type 2.

Select n objects w/o replacement from the $a + b$.

$$\begin{aligned} & \text{Pr}(k \text{ type 1's were picked}) \\ &= \frac{(\# \text{ ways to choose } k \text{ 1's})(\text{choose } n - k \text{ 2's})}{\# \text{ ways to choose } n \text{ out of } a + b} \\ &= \frac{\binom{a}{k} \binom{b}{n - k}}{\binom{a + b}{n}} \text{ (the **hypergeometric distr'n**).} \end{aligned}$$

Example: 25 sox in a box. 15 red, 10 blue. Pick 7 w/o replacement.

$$\Pr(\text{exactly 3 reds are picked}) = \frac{\binom{15}{3} \binom{10}{4}}{\binom{25}{7}}$$

Permutations vs. Combinations — It's all how you approach the problem!

Example: 4 red marbles, 2 whites. Put them in a row in random order. Find...

- (a) $\Pr(2 \text{ end marbles are } W)$
- (b) $\Pr(2 \text{ end marbles aren't both } W)$
- (c) $\Pr(2 \text{ } W\text{'s aren't side by side})$

Method 1 (using permutations): Let the sample space

$$S = \{\text{every random ordering of the 6 marbles}\}.$$

(a) A : 2 end marbles are W — WRRRRW.

$$|A| = 2!4! = 48 \Rightarrow \Pr(A) = \frac{|A|}{|S|} = \frac{48}{720} = \frac{1}{15}.$$

(b) $\Pr(\bar{A}) = 1 - \Pr(A) = 14/15.$

(c) B : 2 W 's side by side — $WWRRRR$ or $RWWRRR$
or ... or $RRRRWW$

$$\begin{aligned} |B| &= (\# \text{ ways to select pair of slots for 2 } W\text{'s}) \\ &\quad \times (\# \text{ ways to insert } W\text{'s into pair of slots}) \\ &\quad \times (\# \text{ ways to insert } R\text{'s into remaining slots}) \\ &= 5 \times 2! \times 4! = 240. \end{aligned}$$

$$\Pr(B) = \frac{|B|}{|S|} = \frac{240}{720} = \frac{1}{3}.$$

But — The above method took too much time! Here's an easier way...

Method 2 (using combinations): Which 2 positions do the W 's occupy? Now let

$S = \{\text{possible pairs of slots that the } W\text{'s occupy}\}.$

Clearly, $|S| = \binom{6}{2} = 15.$

(a) Since the W 's must occupy the end slots in order for A to occur, $|A| = 1 \Rightarrow \Pr(A) = |A|/|S| = 1/15.$

(b) $\Pr(\bar{A}) = 14/15.$

(c) $|B| = 5 \Rightarrow \Pr(B) = 5/15 = 1/3.$

Birthday Problem

n people in a room. Find the prob that at least two have the same birthday. (Ignore Feb. 29, and assume that all 365 days have equal prob.)

A : All birthdays are different.

$S = \{(x_1, \dots, x_n) : x_i = 1, 2, \dots, 365\}$ (x_i is person i 's birthday), and note that $|S| = (365)^n$.

$$|A| = P_{365,n} = (365)(364) \cdots (365 - n + 1)$$

$$\begin{aligned} \Pr(A) &= \frac{(365)(364) \cdots (365 - n + 1)}{(365)^n} \\ &= 1 \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365 - n + 1}{365} \end{aligned}$$

We want

$$\Pr(\bar{A}) = 1 - \left(1 \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365 - n + 1}{365} \right)$$

Notes: When $n = 366$, $\Pr(\bar{A}) = 1$.

For $\Pr(\bar{A})$ to be $> 1/2$, n must be ≥ 23 . (surprising)

When $n = 50$, $\Pr(\bar{A}) = 0.97$.

Envelope Problem

A group of n people receives n envelopes with their names on them — but someone has completely mixed up the envelopes! Find the prob that at least one person will receive the proper envelope.

A_i : Person i receives his correct envelope.

We obviously want $\Pr(A_1 \cup A_2 \cup \cdots \cup A_n)$.

By the general principle of inclusion-exclusion, we have. . . .

$$\begin{aligned}
 & \Pr(A_1 \cup A_2 \cup \cdots \cup A_n) \\
 &= \sum_{i=1}^n \Pr(A_i) - \sum \sum_{i < j} \Pr(A_i \cap A_j) \\
 &\quad + \sum \sum \sum_{i < j < k} \Pr(A_i \cap A_j \cap A_k) \\
 &\quad + \cdots + (-1)^{n-1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n) \\
 &= n\Pr(A_1) - \binom{n}{2} \Pr(A_1 \cap A_2) + \binom{n}{3} \Pr(A_1 \cap A_2 \cap A_3) \\
 &\quad + \cdots + (-1)^{n-1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n) \\
 &= \frac{n}{n} - \binom{n}{2} \frac{1}{n} \cdot \frac{1}{n-1} + \binom{n}{3} \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} + \cdots + (-1)^{n-1} \frac{1}{n!} \\
 &= 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{n-1} \frac{1}{n!} \approx 1 - \frac{1}{e}.
 \end{aligned}$$

Poker Problems

Draw 5 cards at random from a standard deck.

of possible hands is $|S| = \binom{52}{5} = 2,598,960$.

Terminology:

rank = 2, 3, ..., Q, K, A,

suit = , , , 

(a) 2 pairs — e.g., $A♥, A♣, 3♥, 3♦, 10♠$

Select 2 ranks (e.g., $A, 3$). Can do this $\binom{13}{2}$ ways.

Select 2 suits for first pair (e.g., $♥, ♣$). $\binom{4}{2}$ ways.

Select 2 suits for second pair (e.g., $♥, ♦$). $\binom{4}{2}$ ways.

Select remaining card to complete the hand. 44 ways.

$$|2 \text{ pairs}| = \binom{13}{2} \binom{4}{2} \binom{4}{2} 44 = 123,552$$

$$\Pr(2 \text{ pairs}) = \frac{123,552}{2,598,960} \approx 0.0475.$$

(b) Full house (1 pair, 3-of-a-kind) —

e.g., $A\heartsuit, A\clubsuit, 3\heartsuit, 3\diamondsuit, 3\spadesuit$

Select 2 *ordered* ranks (e.g., $A, 3$). $P_{13,2}$ ways.

Select 2 suits for pair (e.g., \heartsuit, \clubsuit). $\binom{4}{2}$ ways.

Select 3 suits for 3-of-a-kind (e.g., $\heartsuit, \diamondsuit, \spadesuit$). $\binom{4}{3}$ ways.

$$|\text{full house}| = 13 \cdot 12 \binom{4}{2} \binom{4}{3} = 3744$$

$$\Pr(\text{full house}) = \frac{3744}{2,598,960} \approx 0.00144.$$

(c) Flush (all 5 cards from same suit)

Select a suit. $\binom{4}{1}$ ways.

Select 5 cards from that suit. $\binom{13}{5}$ ways.

$$\Pr(\text{flush}) = \frac{5148}{2,598,960} \approx 0.00198.$$

(d) Straight (5 ranks in a row)

Select a starting point for the straight ($A, 2, 3, \dots, 10$).

$\binom{10}{1}$ ways.

Select a suit for each card in the straight. 4^5 ways.

$$\Pr(\text{straight}) = \frac{10 \cdot 4^5}{2,598,960} \approx 0.00394.$$

(e) Straight flush

Select a starting point for the straight. 10 ways.

Select a suit. 4 ways.

$$\Pr(\text{straight flush}) = \frac{40}{2,598,960} \approx 0.0000154.$$

Multinomial Coefficients

Example: n_1 blue sox, n_2 reds. # of assortments is $\binom{n_1 + n_2}{n_1}$ (binomial coefficients).

Generalization (for k types of objects): $n = \sum_{i=1}^k n_i$

of arrangements is $n! / (n_1! n_2! \cdots n_k!)$.

Example: How many ways can “Mississippi” be arranged?

$$\frac{\# \text{ perm's of 11 letters}}{(\# m's)!(\# p's)!(\# i's)!(\# s's)!} = \frac{11!}{1!2!4!4!} = 34,650.$$

Conditional Probability

Definition

Properties

Independence

Conditional Probability

Example: Die. $A = \{2, 4, 6\}$, $B = \{1, 2, 3, 4, 5\}$. So $\Pr(A) = 1/2$, $\Pr(B) = 5/6$.

Suppose we *know* that B occurs. Then the prob of A “given” B is

$$\Pr(A|B) = \frac{2}{5} = \frac{|A \cap B|}{|B|}$$

So the prob of A depends on the info that you have! The info that B occurs allows us to regard B as a new, restricted sample space. And...

$$\Pr(A|B) = \frac{|A \cap B|}{|B|} = \frac{|A \cap B|/|S|}{|B|/|S|} = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Definition: If $\Pr(B) > 0$, the **conditional prob of A given B** is $\Pr(A|B) \equiv \Pr(A \cap B)/\Pr(B)$.

Remarks: If A and B are disjoint, then $\Pr(A|B) = 0$.
(If B occurs, there's no chance that A can also occur.)

What happens if $\Pr(B) = 0$? Don't worry! In this case, makes no sense to consider $\Pr(A|B)$.

Example: Toss 2 dice and take the sum.

A : odd toss = $\{3, 5, 7, 9, 11\}$

B : $\{2, 3\}$

$$\Pr(A) = \Pr(3) + \dots + \Pr(11) = \frac{2}{36} + \frac{4}{36} + \dots + \frac{2}{36} = \frac{1}{2}.$$

$$\Pr(B) = \frac{1}{36} + \frac{2}{36} = \frac{1}{12}.$$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(3)}{\Pr(B)} = \frac{2/36}{1/12} = 2/3.$$

Example: 4 white socks, 8 red. Select 2 w/o repl.

A : 1st sock W; B : 2nd W; C : Both W ($= A \cap B$).

$$\Pr(C) = \Pr(A \cap B) = \Pr(A)\Pr(B|A) = \frac{4}{12} \cdot \frac{3}{11} = \frac{1}{11}.$$

$$\begin{aligned}\Pr(B) &= \Pr(A \cap B) + \Pr(\bar{A} \cap B) \\ &= \Pr(A)\Pr(B|A) + \Pr(\bar{A})\Pr(B|\bar{A}) \\ &= \frac{4}{12} \cdot \frac{3}{11} + \frac{8}{12} \cdot \frac{4}{11} = \frac{1}{3}.\end{aligned}$$

Could you have gotten this result w/o thinking?

A couple has two kids and at least one is a boy.
What's the prob that BOTH are boys?

$S = \{GG, GB, BG, BB\}$, (' BG ' means 'boy then girl')

C : Both are boys = $\{BB\}$.

D : At least 1 is a boy. = $\{GB, BG, BB\}$.

$$\Pr(C|D) = \frac{\Pr(C \cap D)}{\Pr(D)} = \frac{\Pr(C)}{\Pr(D)} = 1/3.$$

(My intuition was $1/2$ — the *wrong* answer! The problem was that we didn't know whether D meant the first or second kid.)

Properties — analogous to Axioms of probability.

$$(1') \quad 0 \leq \Pr(A|B) \leq 1.$$

$$(2') \quad \Pr(S|B) = 1.$$

$$(3') \quad A_1 \cap A_2 = \emptyset \Rightarrow \Pr(A_1 \cup A_2|B) = \Pr(A_1|B) + \Pr(A_2|B).$$

(4') If A_1, A_2, \dots are all disjoint, then

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i \mid B\right) = \sum_{i=1}^{\infty} \Pr(A_i|B).$$

Independence Day — Any unrelated events are independent.

A : It rains on Mars tomorrow.

B : Coin lands on H .

Definition: A and B are **independent** iff $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

Example: If $\Pr(\text{rains on Mars}) = 0.2$ and $\Pr(H) = 0.5$, then $\Pr(\text{rains and } H) = 0.1$.

Note: If $\Pr(A) = 0$, then A is indep of any other event.

Remark: Events don't have to be physically unrelated to be indep.

Example: Die. $A = \{2, 4, 6\}$, $B = \{1, 2, 3, 4\}$, $A \cap B = \{2, 4\}$, so $\Pr(A) = 1/2$, $\Pr(B) = 2/3$, $\Pr(A \cap B) = 1/3$.

$\Pr(A)\Pr(B) = 1/3 = \Pr(A \cap B) \Rightarrow A, B$ indep.

More natural interpretation of independence...

Theorem: Suppose $\Pr(B) > 0$. Then A and B are indep iff $\Pr(A|B) = \Pr(A)$.

Proof: A, B indep $\Leftrightarrow \Pr(A \cap B) = \Pr(A)\Pr(B) \Leftrightarrow \Pr(A \cap B)/\Pr(B) = \Pr(A)$.

Remark: So if A and B are indep, the prob of A doesn't depend on whether or not B occurs.

(Bonus) Theorem: A, B indep $\Rightarrow A, \bar{B}$ indep.

Proof: $\Pr(A) = \Pr(A \cap \bar{B}) + \Pr(A \cap B)$, so that

$$\begin{aligned}\Pr(A \cap \bar{B}) &= \Pr(A) - \Pr(A \cap B) \\ &= \Pr(A) - \Pr(A)\Pr(B) \quad (A, B \text{ indep}) \\ &= \Pr(A)[1 - \Pr(B)] = \Pr(A)\Pr(\bar{B}).\end{aligned}$$

Don't confuse independence with disjointness!

Theorem: If $\Pr(A) > 0$ and $\Pr(B) > 0$, then A and B can't be indep and disjt at the same time.

Proof: A, B disjt ($A \cap B = \emptyset$) $\Rightarrow \Pr(A \cap B) = 0 < \Pr(A)\Pr(B)$. Thus A, B not indep. Similarly, indep implies not disjt.

Extension to more than two events.

Definition: A, B, C are indep iff

$$(a) \Pr(A \cap B \cap C) = \Pr(A)\Pr(B)\Pr(C)$$

and

(b) All *pairs* must be indep:

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

$$\Pr(A \cap C) = \Pr(A)\Pr(C)$$

$$\Pr(B \cap C) = \Pr(B)\Pr(C)$$

Note that condition (a) by itself isn't enough.

Example: $S = \{1, 2, \dots, 8\}$ (each element w.p. $1/8$).

$A = \{1, 2, 3, 4\}$, $B = \{1, 5, 6, 7\}$, $C = \{1, 2, 3, 8\}$.

(a) $A \cap B \cap C = \{1\}$. $\Pr(A \cap B \cap C) = \Pr(A)\Pr(B)\Pr(C) = 1/8$, so (a) is satisfied. However, (b) is *not*...

(b) $A \cap B = \{1\}$. $\Pr(A \cap B) = 1/8 \neq 1/4 = \Pr(A)\Pr(B)$.

Also note that (b) by itself isn't enough.

Example: $S = \{1, 2, 3, 4\}$ (each element w.p. $1/4$).

$A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 4\}$. $A \cap B = A \cap C = B \cap C = A \cap B \cap C = \{1\}$.

(b) $\Pr(A \cap B) = 1/4 = \Pr(A)\Pr(B)$. Same deal with A, C and B, C . So (b) is OK. But (a) *isn't*...

(a) $\Pr(A \cap B \cap C) = 1/4 \neq 1/8 = \Pr(A)\Pr(B)\Pr(C)$.

General Definition: A_1, \dots, A_k are indep iff $\Pr(A_1 \cap \dots \cap A_k) = \Pr(A_1) \dots \Pr(A_k)$ and all subsets of $\{A_1, \dots, A_k\}$ are indep.

Independent Trials: Perform n trials of an experiment such that the outcome of one trial is indep of the outcomes of the other trials.

Example: Flip 3 coins indep'ly.

(a) $\Pr(\text{1st coin is } H) = 1/2$. Don't worry about the other two coins since they're indep of the 1st.

(b) $\Pr(\text{1st coin } H, \text{ 3rd } T) = \Pr(\text{1st coin } H)\Pr(\text{3rd } T) = 1/4$.

Remark: For indep trials, you just multiply the individual probs.

Example: Flip a coin infinitely many times (each flip is indep of the others).

$$\begin{aligned} p_n &\equiv \Pr(\text{1st } H \text{ on } n\text{th trial}) \\ &= \Pr(\underbrace{TT \cdots T}_{n-1} H) \\ &= \underbrace{\Pr(T)\Pr(T) \cdots \Pr(T)}_{n-1} \Pr(H) = 1/2^n. \end{aligned}$$

$$\Pr(H \text{ eventually}) = \sum_{n=1}^{\infty} p_n = 1.$$

Bayes' Theorem

Partitions

Bayes' Theorem

Examples

Partition of a sample space — split the sample space into disjoint, yet all-encompassing subsets.

Definition: The events A_1, A_2, \dots, A_n form a **partition** of the sample space S if

- (1) A_1, A_2, \dots, A_n are disjoint.
- (2) $\bigcup_{i=1}^n A_i = S$.
- (3) $\Pr(A_i) > 0$ for all i .

Remark: When an experiment is performed, *exactly one* of the A_i 's occurs.

Example: A and \bar{A} form a partition.

Example: “vowels” and “consonants” form a partition of the letters.

Example: Suppose A_1, A_2, \dots, A_n form a partition of S , and B is some arbitrary event. Then

$$B = \bigcup_{i=1}^n (A_i \cap B).$$

So if A_1, A_2, \dots, A_n is a partition,

$$\begin{aligned}\Pr(B) &= \Pr\left(\bigcup_{i=1}^n (A_i \cap B)\right) \\ &= \sum_{i=1}^n \Pr(A_i \cap B) \quad (A_1, A_2, \dots, A_n \text{ are disjoint}) \\ &= \sum_{i=1}^n \Pr(A_i) \Pr(B|A_i) \quad (\text{by defn of cond'l prob}).\end{aligned}$$

This is the **Law of Total Probability**.

Example: $\Pr(B) = \Pr(A) \Pr(B|A) + \Pr(\bar{A}) \Pr(B|\bar{A})$,
which we saw in the last module.

Bayes Theorem: If A_1, A_2, \dots, A_n form a partition of S and B is any event, then

$$\Pr(A_i|B) = \frac{\Pr(A_i \cap B)}{\Pr(B)} = \frac{\Pr(A_i)\Pr(B|A_i)}{\sum_{j=1}^n \Pr(A_j)\Pr(B|A_j)}.$$

The $\Pr(A_i)$'s are **prior** probabilities (“before B ”).

The $\Pr(A_i|B)$'s are **posterior** probabilities (“after B ”).

The $\Pr(A_i|B)$ add up to 1. That's why the funny-looking denominator.

Example: In a certain city with good police,
 $\Pr(\text{Any defendent brought to trial is guilty}) = 0.99.$

In any trial,

$\Pr(\text{Jury acquits if defendent is innocent}) = 0.95.$

$\Pr(\text{Jury convicts if defendent is guilty}) = 0.95.$

Find $\Pr(\text{Defendent is innocent} | \text{Jury acquits}).$

Events: $I =$ “innocent”, $G =$ “guilty” $= \bar{I}$, $A =$ “acquittal”. Since the partition is $\{I, G\}$, Bayes' \Rightarrow

$$\begin{aligned}\Pr(I|A) &= \frac{\Pr(I)\Pr(A|I)}{\Pr(I)\Pr(A|I) + \Pr(G)\Pr(A|G)} \\ &= \frac{(0.01)(0.95)}{(0.01)(0.95) + (0.99)(0.05)} \\ &= 0.161.\end{aligned}$$

Notice how the posterior prob's depend strongly on the prior prob's.

Example: A store gets $1/2$ of its items from Factory 1, $1/4$ from Factory 2, and $1/4$ from Factory 3.

2% of Factory 1's items are defective.

2% of Factory 2's items are defective.

4% of Factory 3's items are defective.

An item from the store is found to be bad. Find the prob it comes from Factory 1. [Answer should be $< 1/2$ since bad items favor Factory 3.]

Events: $F_i =$ “Factory i ”, $D =$ “defective item”. Partition is $\{F_1, F_2, F_3\}$.

$$\begin{aligned}\Pr(F_1|D) &= \frac{\Pr(F_1)\Pr(D|F_1)}{\sum_{j=1}^3 \Pr(F_j)\Pr(D|F_j)} \\ &= \frac{(0.5)(0.02)}{(0.5)(0.02) + (0.25)(0.02) + (0.25)(0.04)} \\ &= 0.4.\end{aligned}$$

It turns out that $\Pr(F_2|D) = 0.2$ and $\Pr(F_3|D) = 0.4$.