Selecting the Best System

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Outline

1. Introduction
2. Find the Normal Distribution with the Largest Mean
   - Motivation
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   - Sequential Procedure
3. Find the Bernoulli Distribution with the Largest Success Prob
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   - Sequential Procedure
4. Find the Most Probable Multinomial Cell
   - Motivation
   - Single-Stage Procedure
   - Curtained Procedure
   - Sequential Procedure
   - Nonparametric Applications
Statistics / Simulation experiments are typically performed to analyze or compare a “small” number of systems, say \( \leq 200 \).

The appropriate method depends on the type of comparison desired and properties of the output data.

If we analyze one system, we could use traditional confidence intervals (based on the normal or t-distributions) from our baby statistics class.

If we compare two systems, we could again use CI’s from baby stats — maybe even clever ones based on paired observations.

For > 2 systems, we could use methods such as ANOVA. But those methods don’t tell us anything except that “at least one of the systems is different than the others”, which is no surprise.
And what measures do you use to compare different systems?

* Which has the biggest mean?

* The smallest variance?

* The highest probability of yielding a success?

* The lowest risk?

* A combination of criteria?
This module: We present ranking & selection procedures to find the best system with respect to one parameter.

Examples:

Great Expectations: Which of 10 fertilizers produces the largest mean crop yield? (Normal)

Great Expectorants: Find the pain reliever that has the highest probability of giving relief for a cough. (Binomial)

Great Ex-Patriots: Who is the most-popular former New England football player? (Multinomial)
R&S selects the best system, or a subset of systems that includes the best.

- Guarantee a probability of a correct selection.
- Multiple Comparisons Procedures (MCPs) add in certain confidence intervals.

R&S is relevant in simulation:

- Normally distributed data by batching.
- Independence by controlling random numbers.
- Multiple-stage sampling by retaining the seeds.
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We present procedures for selecting the normal distribution that has the largest mean.

We use the *indifference-zone* approach.

Assumptions: Independent $Y_{i1}, Y_{i2}, \ldots (1 \leq i \leq k)$ are taken from $k \geq 2$ normal popns $\Pi_1, \ldots, \Pi_k$. Here $\Pi_i$ has *unknown* mean $\mu_i$ and *known* or *unknown* variance $\sigma_i^2$.

Denote the vector of means by $\mu = (\mu_1, \ldots, \mu_k)$ and the vector of variances by $\sigma^2 = (\sigma_1^2, \ldots, \sigma_k^2)$.

The ordered (but unknown) $\mu_i$’s are $\mu_{[1]} \leq \cdots \leq \mu_{[k]}$.

The system having the largest mean $\mu_{[k]}$ is the “best.”
Goal: To select the population associated with mean $\mu[k]$.

A *correct selection* (CS) is made if the Goal is achieved.

Indifference-Zone Probability Requirement: For specified constants $(P^*, \delta^*)$ with $\delta^* > 0$ and $1/k < P^* < 1$, we require

$$P\{\text{CS}\} \geq P^* \quad \text{whenever} \quad \mu[k] - \mu[k-1] \geq \delta^*.$$  

(1)

The probability in (1) depends on the differences $\mu_i - \mu_j$, the sample size $n$, and $\sigma^2$.

The constant $\delta^*$ can be thought of as the “smallest difference worth detecting.”
Parameter configurations $\mu$ satisfying $\mu[k] - \mu[k-1] \geq \delta^*$ are in the preference-zone for a correct selection.

Configurations satisfying $\mu[k] - \mu[k-1] < \delta^*$ are in the indifference-zone.

Any procedure that guarantees (1) is said to be employing the indifference-zone approach.

There are 100’s of such procedures. Highlights:

* Single-Stage Procedure (Bechhofer 1954)
* Two-Stage Procedure (Rinott 1979)
* Sequential Procedure (Nelson and friends, 2001)
**Single-Stage Procedure $\mathcal{N}_B$ (Bechhofer 1954)**

This procedure takes all necessary observations and makes the selection decision at once (in a single stage).

Assumes popns have *common known variance*.

For the given $k$ and specified $(P^*, \delta^*/\sigma)$, determine sample size $n$ (usually from a table).

Take a random sample of $n$ observations $Y_{ij}$ ($1 \leq j \leq n$) in a single stage from $\Pi_i$ ($1 \leq i \leq k$).

Calculate the $k$ sample means, $\bar{Y}_i = \sum_{j=1}^{n} Y_{ij} / n$ ($1 \leq i \leq k$).

Select the popn that yielded the largest sample mean, $\bar{Y}_{[k]} = \max\{\bar{Y}_1, \ldots, \bar{Y}_k\}$, as the one associated with $\mu_{[k]}$. 
Find the Normal Distribution with the Largest Mean

Single-Stage Procedure

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Common Sample Size $n$ per Popn Required by $\mathcal{N}_B$
Remark: Don’t really need the above table. You can directly calculate

\[
n = \left\lceil 2 \left( \frac{\sigma Z_{k-1,1/2}}{\delta^*} \right)^2 \right\rceil,
\]

where \( \lceil \cdot \rceil \) is the “ceiling” function and the constant \( Z_{k-1,1/2} \) is a special case of the upper equicoordinate point of a certain multivariate normal distribution.

The value of \( n \) satisfies (1) for any \( \mu \) such that

\[
\mu[1] = \mu[k-1] = \mu[k] - \delta^*.
\]

(2)

Configurations (2) are termed least-favorable (LF) because, for fixed \( n \), they minimize the \( P\{CS\} \) among all \( \mu \) in the preference-zone.
How to calculate $n$ yourself (without multivariate normal tables).

Denote the standard normal p.d.f. and c.d.f. by $\phi(\cdot)$ and $\Phi(\cdot)$ and set

\[
P^* = P\{\text{CS} \mid \text{LF}\} = P\{\bar{Y}_i \leq \bar{Y}_k, i = 1, \ldots, k - 1 \mid \text{LF}\} \\
= \int_{\mathbb{R}} P\left\{\frac{\bar{Y}_i - \mu_k}{\sqrt{\sigma^2/n}} \leq \frac{\bar{Y}_k - \mu_k}{\sqrt{\sigma^2/n}}, i = 1, \ldots, k - 1 \mid \text{LF}\right\} \phi(x) \, dx \\
= \int_{\mathbb{R}} P\left\{\frac{\bar{Y}_i - \mu_i}{\sqrt{\sigma^2/n}} \leq x + \frac{\sqrt{n}\delta^*}{\sigma}, i = 1, \ldots, k - 1\right\} \phi(x) \, dx \\
= \int_{\mathbb{R}} \Phi^{k-1}\left(x + \frac{\sqrt{n}\delta^*}{\sigma}\right) \phi(x) \, dx = \int_{\mathbb{R}} \Phi^{k-1}(x + h) \phi(x) \, dx.
\]

Now solve numerically for $h$, and then set $n = \lceil (h\sigma/\delta^*)^2 \rceil$. 
Example: Suppose $k = 4$ and we want to detect a difference in means as small as 0.2 standard deviations with $P\{CS\} \geq 0.99$. The table for $N_B$ calls for $n = 361$ observations per popn.

Increasing $\delta^*$ and/or decreasing $P^*$ requires a smaller $n$. For example, when $\delta^*/\sigma = 0.6$ and $P^* = 0.95$, $N_B$ requires only $n = 24$ observations per popn.  

Robustness of Procedure: How does $N_B$ do under different types of violations of the underlying assumptions on which it’s based?

- Lack of normality — not so bad.
- Different variances — sometimes a big problem.
- Dependent data — usually a nasty problem (e.g., in simulations).
Find the Normal Distribution with the Largest Mean

Two-Stage Procedure $\mathcal{N}_R$ (Rinott 1979)

Assumes popns have *unknown and unequal variances*. Takes a first stage of observations to estimate the variances of each system, and then uses those estimates to determine how many observations to take in the second stage — the higher the variance estimate, the more observations needed.

For the given $t$, specify $(P^*, \delta^*)$, and a common first-stage sample size $n_0 \geq 2$.

Look up the constant $g(P^*, n_0, k)$ in an appropriate table or (if you have the urge) solve the following equation for $g$:

$$\int_0^\infty \int_0^\infty \left[ \Phi \left( \frac{g}{(n_0 - 1)(\frac{1}{x} + \frac{1}{y})} \right) f(x) \right]^{k-1} f(y) \, dx \, dy = P^*,$$

where $f(\cdot)$ is the $\chi^2(n_0 - 1)$ p.d.f.
Take an i.i.d. sample $Y_{i1}, Y_{i2}, \ldots, Y_{in_0}$ from each of the $k$ scenarios simulated independently.

Calculate the first-stage sample means and variances,

\[
\bar{Y}_i^{(1)} = \frac{1}{n_0} \sum_{j=1}^{n_0} Y_{ij} \quad \text{and} \quad S_i^2 = \frac{\sum_{j=1}^{n_0} \left( Y_{ij} - \bar{Y}_i^{(1)} \right)^2}{n_0 - 1},
\]

and then the final sample sizes

\[
N_i = \max \left\{ n_0, \left\lceil (gS_i/\delta^*)^2 \right\rceil \right\}, \quad i = 1, 2, \ldots, k.
\]

Take $N_i - n_0$ additional i.i.d. observations from scenario $i$, independently of the first-stage sample and the other scenarios, for $i = 1, 2, \ldots, k$. 


Compute overall sample means $\bar{Y}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} Y_{ij}$, $\forall i$.

Select the scenario with the largest $\bar{Y}_i$ as best.

**Bonus:** Simultaneously form MCP confidence intervals

$$
\mu_i - \max_{j \neq i} \mu_j \in \left[ -\left( \bar{Y}_i - \max_{j \neq i} \bar{Y}_j - \delta^* \right)^-, \left( \bar{Y}_i - \max_{j \neq i} \bar{Y}_j + \delta^* \right)^+ \right]
$$

$\forall i$, where $(a)^+ \equiv \max\{0, a\}$ and $-(b)^- \equiv \min\{0, b\}$. 
Find the Normal Distribution with the Largest Mean

Two-Stage Procedure

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$g$ Constant Required by $N_R$
Example: A Simulation Study of Airline Reservation Systems

Consider $k = 4$ different airline reservation systems.

Objective: Find the system with the largest expected time to failure (E[TTF]). Let $\mu_i$ denote the E[TTF] for system $i$.

From past experience we know that the E[TTF]'s are roughly 100,000 minutes (about 70 days) for all four systems.

Goal: Select the best system with probability at least $P^* = 0.90$ if the difference in the expected failure times for the best and second best systems is $\geq \delta^* = 3000$ minutes (about two days).

The competing systems are sufficiently complicated that computer simulation is required to analyze their behavior.
Let $T_{ij}$ ($1 \leq i \leq 4$, $j \geq 1$) denote the observed time to failure from the $j$th independent simulation replication of system $i$.

Application of the Rinott procedure $\mathcal{N}_R$ requires i.i.d. normal observations from each system.

If each simulation replication is initialized from a particular system under the same operating conditions, but with independent random number seeds, the resulting $T_{i1}, T_{i2}, \ldots$ will be i.i.d. for each system.

However, the $T_{ij}$ aren’t normal — in fact, they’re skewed right.
Instead of using the raw $T_{ij}$ in $\mathcal{N}_R$, apply the procedure to the so-called macroreplication estimators of the $\mu_i$.

These estimators group the $\{T_{ij} : j \geq 1\}$ into disjoint batches and use the batch averages as the “data” to which $\mathcal{N}_R$ is applied.

Fix a number $m$ of simulation replications that comprise each macroreplication (that is, $m$ is the batch size) and let

$$Y_{ij} \equiv \frac{1}{m} \sum_{k=1}^{m} T_{i,(j-1)m+k}, \quad 1 \leq i \leq 4, \ 1 \leq j \leq b_i,$$

where $b_i$ is the number of macroreplications to be taken from system $i$. 
The macroreplication estimators from the $i$th system, $Y_{i1}, Y_{i2}, \ldots, Y_{ib}$, are i.i.d. with expectation $\mu_i$.

If $m$ is sufficiently large, say at least 20, then the CLT yields approximate normality for each $Y_{ij}$.

No assumptions on the variances of the macroreplications.

To apply $\mathcal{N}_R$, first conduct a pilot study to serve as the first stage of the procedure. Each system was run for $n_0 = 20$ macroreplications with each macroreplication consisting of the averages of $m = 20$ simulations of the system.

Rinott table with $k = 4$ and $P^* = 0.90$ gives $g = 2.720$.

The total sample sizes $N_i$ are computed for each system and are displayed in the summary table.
### Find the Normal Distribution with the Largest Mean

#### Two-Stage Procedure

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**Summary of Airline Rez Example**
E.g., System 2 requires an additional $N_2 - 20 = 465$ macroreplications in the second stage (each macroreplication again being the average of $m = 20$ system simulations).

In all, a total of about 40,000 simulations of the four systems were required to implement procedure $N_R$. The combined sample means for each system are listed in row 4 of the summary table.

Clearly establish System 1 as having the largest $E[TTF]$.  □
Multi-Stage Procedure $\mathcal{N}_{KN}$ (Kim & Nelson 2001)

Very efficient procedure. Takes observations from each population one-at-a-time, and eliminates populations that appear to be noncompetitive along the way.

Assumes popns have unknown (unequal) variances.

For the given $k$, specify $(\delta^*, P^*)$, and a common initial sample size from each scenario $n_0 \geq 2$.

To begin with, calculate the constant

$$\eta \equiv \frac{1}{2} \left[ \left( \frac{2(1 - P^*)}{k - 1} \right)^{-2/(n_0 - 1)} - 1 \right].$$

Initialize $I = \{1, 2, \ldots, k\}$ and let $h^2 \equiv 2\eta(n_0 - 1)$. 

Take an initial random sample of \( n_0 \geq 2 \) observations \( Y_{ij} \) \((1 \leq j \leq n_0)\) from population \( i \) \((1 \leq i \leq k)\).

For population \( i \), compute the sample mean based on the \( n_0 \) observations, \( \bar{Y}_i(n_0) = \frac{\sum_{j=1}^{n_0} Y_{ij}}{n_0} \) \((1 \leq i \leq k)\).

For all \( i \neq \ell \), compute the sample variance of the difference between treatments \( i \) and \( \ell \),

\[
S_{i\ell}^2 = \frac{1}{n_0 - 1} \sum_{j=1}^{n_0} (Y_{ij} - Y_{\ell j} - [\bar{Y}_i(n_0) - \bar{Y}_\ell(n_0)])^2.
\]

For all \( i \neq \ell \), set \( N_{i\ell} = \left\lfloor h^2 S_{i\ell}^2 / \delta^*^2 \right\rfloor \) and then \( N_i = \max_{\ell \neq i} N_{i\ell} \).
If $n_0 > \max_i N_i$, stop and select the population with the largest sample mean $\bar{Y}_i(n_0)$ as one having the largest mean. Otherwise, set the sequential counter $r = n_0$ and go to the Screening phase of the procedure.

**Screening:** Set $I^{\text{old}} = I$ and re-set

$$I = \{ i : i \in I^{\text{old}} \text{ and } \bar{Y}_i(r) \geq \bar{Y}_\ell(r) - W_{i\ell}(r), \text{ for all } \ell \in I^{\text{old}}, \ell \neq i \},$$

where

$$W_{i\ell}(r) = \max \left\{ 0, \frac{\delta^*}{2r} \left( \frac{h^2 S_{i\ell}^2}{(\delta^*)^2} - r \right) \right\}.$$  

Keep those surviving populations that aren’t “too far” from the current leader.
**Stopping Rule:** If $|I| = 1$, then stop and select the treatment with index in $I$ as having the largest mean.

If $|I| > 1$, take one additional observation $Y_{i,r+1}$ from each treatment $i \in I$.

Increment $r = r + 1$ and go to the screening stage if $r < \max_i N_i + 1$.

If $r = \max_i N_i + 1$, then stop and select the treatment associated with the largest $\bar{Y}_i(r)$ having index $i \in I$. 

Normal Extensions

Correlation *between* populations.

Better fully sequential procedures.

Better elimination of popns that aren’t competitive.

Different variance estimators.
Outline

1. Introduction
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   - Sequential Procedure
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4. Find the Most Probable Multinomial Cell
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   - Curtained Procedure
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Examples:

- Which anti-cancer drug is most effective?
- Which simulated system is most likely to meet design specs?

There are 100’s of such procedures. Highlights:

- Single-Stage Procedure (Sobel and Huyett 1957)
- Sequential Procedure (Bechhofer, Kiefer, Sobel 1968)
- “Optimal” Procedures (Bechhofer, et al., 1980’s)

Again use the *indifference-zone* approach.
A Single-Stage Procedure (Sobel and Huyett 1957)

We have \(k\) competing Bern populations with success parameters \(p_1, p_2, \ldots, p_k\). Denote the ordered \(p\)'s by \(p[1] \leq p[2] \leq \cdots \leq p[k]\).

Probability Requirement: For specified constants \((P^*, \Delta^*)\) with \(1/k < P^* < 1\) and \(0 < \Delta^* < 1\), we require

\[
P\{CS\} \geq P^* \quad \text{whenever} \quad p[k] - p[k-1] \geq \Delta^*.
\]

The \(P\{CS\}\) depends on the entire vector \(p\) and on the number \(n\) of observations taken from each of the \(k\) treatments.

Note that the probability requirement is defined in terms of the difference \(p[t] - p[t-1]\), and we interpret \(\Delta^*\) as the “smallest difference worth detecting.”
**Procedure** $B_{SH}$

For the specified $(\Delta^*, P^*)$, find $n$ from a table.

Take a sample of $n$ observations $X_{ij} \ (1 \leq j \leq n)$ in a single stage from each population (1 $\leq i \leq k$).

Calculate the $k$ sample sums $Y_{in} = \sum_{j=1}^{n} X_{ij}$.

Select the treatment that yielded the largest $Y_{in}$ as the one associated with $p[k]$; in the case of ties, randomize.
Find the Bernoulli Distrn with the Largest Success Prob

Single-Stage Procedure

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\Delta^*$</th>
<th>$P^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.60 0.75 0.80 0.85 0.90 0.95 0.99</td>
</tr>
<tr>
<td>3</td>
<td>0.10</td>
<td>20     52     69     91    125   184   327</td>
</tr>
<tr>
<td></td>
<td>0.20</td>
<td>5      13     17     23    31    46     81</td>
</tr>
<tr>
<td></td>
<td>0.30</td>
<td>3      6      8      10    14    20     35</td>
</tr>
<tr>
<td></td>
<td>0.40</td>
<td>2      4      5      6     8     11     20</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>2      3      3      4     5     7      12</td>
</tr>
<tr>
<td>4</td>
<td>0.10</td>
<td>34     71     90     114   150   212   360</td>
</tr>
<tr>
<td></td>
<td>0.20</td>
<td>9      18     23     29    38    53     89</td>
</tr>
<tr>
<td></td>
<td>0.30</td>
<td>4      8      10     13    17    23     39</td>
</tr>
<tr>
<td></td>
<td>0.40</td>
<td>3      5      6      7     9     13     21</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>2      3      4      5     6     8      13</td>
</tr>
</tbody>
</table>

Smallest $n$ for $B_{SH}$ to Guarantee Prob Reqt
Example: Suppose we want to select the best of $k = 4$ treatments with probability at least $P^* = 0.95$ whenever $p[4] - p[3] \geq 0.10$.

The table shows that we need $n = 212$ observations.

Suppose that, at the end of sampling, we have $Y_{1,212} = 70$, $Y_{2,212} = 145$, $Y_{3,212} = 95$ and $Y_{4,212} = 102$.

Then we select population 2 as the best. □
A Sequential Procedure (BKS 1968)

New Probability Requirement: For specified constants \((P^*, \theta^*)\) with \(1/k < P^* < 1\) and \(\theta^* > 1\), we require \(P\{CS\} \geq P^*\) whenever the \textit{odds ratio}

\[
\frac{p_{[k]}/(1 - p_{[k]})}{p_{[k-1]}/(1 - p_{[k-1]})} \geq \theta^*.
\]

Procedure \(B_{BKS}\)

For the given \(k\), specify \((P^*, \theta^*)\).

At the \(m\)th stage of experimentation \((m \geq 1)\), observe the random Bernoulli vector \((X_{1m}, \ldots, X_{km})\).

Let \(Y_{im} = \sum_{j=1}^{m} X_{ij}\) \((1 \leq i \leq k)\) and denote the ordered \(Y_{im}\)-values by \(Y_{[1]m} \leq \cdots \leq Y_{[k]m}\).
After the $m$th stage of experimentation, compute

\[ Z_m = \sum_{i=1}^{k-1} \left( \frac{1}{\theta^*} \right)^{Y_{[k]}_m - Y_{[i]}_m}. \]

Stop at the first value of $m$ (call it $N$) for which $Z_m \leq (1 - P^*)/P^*$. Note that $N$ is a random variable.

Select the treatment that yielded $Y_{[k]}_N$ as the one associated with $p_{[k]}$. 
Example: For $k = 3$ and $(P^*, \theta^*) = (0.75, 2)$, suppose the following sequence of vector-observations is obtained using $\mathcal{B}_{BKS}$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$x_{1m}$</th>
<th>$x_{2m}$</th>
<th>$x_{3m}$</th>
<th>$y_{1m}$</th>
<th>$y_{2m}$</th>
<th>$y_{3m}$</th>
<th>$z_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0.75</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>0.375</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>0.375</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>6</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Since $z_6 \leq (1 - P^*)/P^* = 1/3$, sampling stops at stage $N = 6$ and treatment 3 is selected as best. $\square$
Bernoulli Extensions

Correlation *between* populations.

More-efficient sequential procedures.

Elimination of populations that aren’t competitive.
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Examples:

- Who is the most popular political candidate?
- Which television show is most watched during a particular time slot?
- Which simulated warehouse configuration is most likely to maximize throughput?

Yet again, use the indifference-zone approach.
 Experimental Set-Up:

• $k$ possible outcomes (categories).

• $p_i$ is the probability of the $i$th category.

• $n$ independent replications of the experiment.

• $Y_i$ is the number of outcomes falling in category $i$ after the $n$ observations have been taken.
Definition: If the $k$-variate discrete vector random variable $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_k)$ has the probability mass function

$$P\{Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k\} = \frac{n!}{\prod_{i=1}^k y_i!} \prod_{i=1}^k p_i^{y_i},$$

then $\mathbf{Y}$ has a \textit{multinomial} distribution with parameters $n$ and $\mathbf{p} = (p_1, \ldots, p_k)$, where $\sum_{i=1}^k p_i = 1$ and $p_i > 0$ for all $i$. 
Example: Suppose three of the faces of a fair die are red, two are blue, and one is green, i.e., \( p = (3/6, 2/6, 1/6) \).

Toss it \( n = 5 \) times. Then the probability of observing exactly three reds, no blues and two greens is

\[
P\{Y = (3, 0, 2)\} = \frac{5!}{3!0!2!} \left(\frac{3}{6}\right)^3 \left(\frac{2}{6}\right)^0 \left(\frac{1}{6}\right)^2 = 0.03472. \]

Example (continued): Suppose we did not know the probabilities for red, blue, and green in the previous example and that we want to select the most probable color.

The selection rule is to choose the color that occurs the most frequently during the five trials, using randomization to break ties.
Let $\mathbf{Y} = (Y_r, Y_b, Y_g)$ denote the number of occurrences of (red, blue, green) in five trials. The probability that we correctly select red is given by

$$P\{\text{red wins in 5 trials}\} = P\{Y_r > Y_b \text{ and } Y_g\} + 0.5P\{Y_r = Y_b, Y_r > Y_g\}$$
$$+ 0.5P\{Y_r > Y_b, Y_r = Y_g\}$$

$$= P\{\mathbf{Y} = (5, 0, 0), (4, 1, 0), (4, 0, 1), (3, 2, 0), (3, 1, 1), (3, 0, 2)\}$$
$$+ 0.5P\{\mathbf{Y} = (2, 2, 1)\} + 0.5P\{\mathbf{Y} = (2, 1, 2)\}.$$

We can list the outcomes favorable to a correct selection (CS) of red, along with the associated probabilities of these outcomes, randomizing for ties.
Find the Most Probable Multinomial Cell

Motivation

<table>
<thead>
<tr>
<th>Outcome (red, blue, green)</th>
<th>Contribution to $P{\text{red wins in 5 trials}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,0,0)</td>
<td>0.03125</td>
</tr>
<tr>
<td>(4,1,0)</td>
<td>0.10417</td>
</tr>
<tr>
<td>(4,0,1)</td>
<td>0.05208</td>
</tr>
<tr>
<td>(3,2,0)</td>
<td>0.13889</td>
</tr>
<tr>
<td>(3,1,1)</td>
<td>0.13889</td>
</tr>
<tr>
<td>(3,0,2)</td>
<td>0.03472</td>
</tr>
<tr>
<td>(2,2,1)</td>
<td>(0.5)(0.13889)</td>
</tr>
<tr>
<td>(2,1,2)</td>
<td>(0.5)(0.06944)</td>
</tr>
</tbody>
</table>

The probability of correctly selecting red as the most probable color based on $n = 5$ trials is 0.6042. This $P\{\text{CS}\}$ can be increased by increasing the sample size $n$. □
Example: The most probable alternative might be preferable to that having the largest expected value.

Consider two inventory policies, $A$ and $B$, where

$$\text{Profit from } A = \$5 \text{ with probability 1}$$

and

$$\text{Profit from } B = \begin{cases} 
0 & \text{with probability 0.99} \\
1000 & \text{with probability 0.01}
\end{cases}.$$ 

Then

$$E[\text{Profit from } A] = \$5 < E[\text{Profit from } B] = \$10$$

but

$$P\{\text{Profit from } A > \text{Profit from } B\} = 0.99.$$ 

So $A$ has a lower expected value than $B$, but $A$ will win almost all of the time. □
Assumptions and Notation

- $X_j = (X_{1j}, \ldots, X_{kj})$ ($j \geq 1$) are independent observations taken from a multinomial distribution having $k \geq 2$ categories with associated unknown probabilities $p = (p_1, \ldots, p_k)$.

- $X_{ij} = 1$ [0] if category $i$ does [does not] occur on the $j$th observation.

- The (unknown) ordered $p_i$’s are $p[1] \leq \cdots \leq p[k]$.

- The category with $p[k]$ is the most probable or best.

- The cumulative sum for category $i$ after $m$ multinomial observations have been taken is $y_{im} = \sum_{j=1}^{m} x_{ij}$.

- The ordered $y_{im}$’s are $y[1]m \leq \cdots \leq y[k]m$. 
Indifference-Zone Procedures

Goal: Select the category associated with $p[k]$.

A *correct selection* (CS) is made if the Goal is achieved.

Probability Requirement: For specified constants $(P^*, \theta^*)$ with $1/k < P^* < 1$ and $\theta^* > 1$, we require

$$P\{\text{CS}\} \geq P^* \text{ whenever } p[k]/p[k-1] \geq \theta^*.$$  (3)

The probability in (3) depends on the entire vector $p$ and on the number $n$ of independent multinomial observations to be taken.

$\theta^*$ is the “smallest $p[k]/p[k-1]$ ratio worth detecting.”

Now we will consider a number of procedures to guarantee probability requirement (3).
Single-Stage Procedure $\mathcal{M}_{BEM}$ (Bechhofer, Elmaghraby, and Morse 1959):

For the given $k$, $P^\ast$ and $\theta^\ast$, find $n$ from the table.

Take $n$ multinomial observations $X_j = (X_{1j}, \ldots, X_{kj})$ ($1 \leq j \leq n$) in a single stage.

Calculate the ordered sample sums $y_{[1]}n \leq \cdots \leq y_{[k]}n$. Select the category with the largest sample sum, $y_{[k]}n$, as the one associated with $p_{[k]}$, randomizing to break ties.

Remark: The $n$-values are computed so that $\mathcal{M}_{BEM}$ achieves $P\{CS\} \geq P^\ast$ when the cell probabilities $\mathbf{p}$ are in the least-favorable (LF) configuration (Kesten and Morse 1959),

$$p_{[1]} = p_{[k-1]} = 1/(\theta^\ast + k - 1) \quad \text{and} \quad p_{[k]} = \theta^\ast/(\theta^\ast + k - 1). \quad (4)$$
Example: A soft drink producer wants to find the most popular of $k = 3$ proposed cola formulations.

The company will give a taste test to $n$ people.

The sample size $n$ is to be chosen so that $P\{CS\} \geq 0.95$ whenever the ratio of the largest to second largest true (but unknown) proportions is at least 1.4.

Entering the table with $k = 3$, $P^* = 0.95$, and $\theta^* = 1.4$, we find that $n = 186$ individuals must be interviewed.  \qed
Find the Most Probable Multinomial Cell

Single-Stage Procedure

<table>
<thead>
<tr>
<th>$P^*$</th>
<th>$\theta^*$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
<th>$k = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$n$</td>
<td>$n_0$</td>
<td>$n$</td>
<td>$n_0$</td>
</tr>
<tr>
<td>2.0</td>
<td>5.0</td>
<td>5</td>
<td>5</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>1.8</td>
<td>5.0</td>
<td>5</td>
<td>7</td>
<td>17</td>
<td>18</td>
</tr>
<tr>
<td>0.75</td>
<td>9.0</td>
<td>9</td>
<td>9</td>
<td>26</td>
<td>32</td>
</tr>
<tr>
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<td>17</td>
<td>19</td>
<td>52</td>
<td>71</td>
</tr>
<tr>
<td>1.4</td>
<td>55.0</td>
<td>55</td>
<td>67</td>
<td>181</td>
<td>285</td>
</tr>
<tr>
<td>1.2</td>
<td>15.0</td>
<td>15</td>
<td>15</td>
<td>29</td>
<td>34</td>
</tr>
<tr>
<td>0.90</td>
<td>0.0</td>
<td>19</td>
<td>27</td>
<td>40</td>
<td>50</td>
</tr>
<tr>
<td>1.8</td>
<td>4.0</td>
<td>31</td>
<td>41</td>
<td>64</td>
<td>83</td>
</tr>
<tr>
<td>0.90</td>
<td>1.0</td>
<td>59</td>
<td>79</td>
<td>126</td>
<td>170</td>
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<td>267</td>
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<td>670</td>
</tr>
<tr>
<td>1.2</td>
<td>2.0</td>
<td>23</td>
<td>27</td>
<td>42</td>
<td>52</td>
</tr>
<tr>
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<td>0.0</td>
<td>33</td>
<td>35</td>
<td>59</td>
<td>71</td>
</tr>
<tr>
<td>1.8</td>
<td>4.0</td>
<td>49</td>
<td>59</td>
<td>94</td>
<td>125</td>
</tr>
<tr>
<td>0.95</td>
<td>1.0</td>
<td>97</td>
<td>151</td>
<td>186</td>
<td>266</td>
</tr>
<tr>
<td>1.6</td>
<td>327.0</td>
<td>327</td>
<td>455</td>
<td>645</td>
<td>960</td>
</tr>
</tbody>
</table>

Sample Size $n$ for $\mathcal{M}_{BEM}$ and Truncation Numbers $n_0$ for $\mathcal{M}_{BG}$ to Guarantee (3)
A Curtailed Procedure $\mathcal{M}_{BK}$ (Bechhofer and Kulkarni 1984)

For the given $k$, specify $n$ prior to the start of sampling.

At the $m$th stage of experimentation ($m \geq 1$), take the random observation $X_m = (X_{1m}, \ldots, X_{km})$.

Calculate the sample sums $y_{im}$ through stage $m$ ($1 \leq i \leq k$). Stop sampling at the first stage $m$ for which there exists a category satisfying

$$y_{im} \geq y_{jm} + n - m \quad \text{for all } j \neq i \ (1 \leq i, j \leq k). \quad (5)$$

Let $N$ (a random variable) denote the value of $m$ at the termination of sampling. Select the category having the largest sum as the one associated with $p[k]$, randomizing to break ties.
Remark: The LHS of (5) is the current total number of occurrences of category \(i\); the RHS is the current total of category \(j\) plus the additional number of potential occurrences of \(j\) if all of the \((n - m)\) remaining outcomes after stage \(m\) were also to be associated with \(j\).

Thus, curtailment takes place when one of the categories has sufficiently more successes than all of the other categories, i.e., sampling stops when the leader can do no worse than tie.

Procedure \(M_{BK}\) saves observations and achieves the same \(P\{CS\}\) as does \(M_{BEM}\) with the same \(n\). In fact,…

\[
P\{\text{CS using } M_{BK} \mid p\} = P\{\text{CS using } M_{BEM} \mid p\}
\]

and

\[
E\{N \text{ using } M_{BK} \mid p\} \leq n \text{ using } M_{BEM}.
\]
Example: For $k = 3$ and $n = 2$, stop sampling if

<table>
<thead>
<tr>
<th>$m$</th>
<th>$x_1m$</th>
<th>$x_2m$</th>
<th>$x_3m$</th>
<th>$y_1m$</th>
<th>$y_2m$</th>
<th>$y_3m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

and select category 1 because $y_{1m} = 1 \geq y_{jm} + n - m = 0 + 2 - 1 = 1$ for $j = 2$ and 3. \(\square\)

Example: For $k = 3$ and $n = 3$ or 4, stop sampling if

<table>
<thead>
<tr>
<th>$m$</th>
<th>$x_1m$</th>
<th>$x_2m$</th>
<th>$x_3m$</th>
<th>$y_1m$</th>
<th>$y_2m$</th>
<th>$y_3m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

and select category 2 because $y_{2m} = 2 \geq y_{jm} + n - m = 0 + n - 2$ for $n = 3$ or $n = 4$ and both $j = 1$ and 3. \(\square\)
Example: For $k = 3$ and $n = 3$ suppose that

<table>
<thead>
<tr>
<th>$m$</th>
<th>$x_{1m}$</th>
<th>$x_{2m}$</th>
<th>$x_{3m}$</th>
<th>$y_{1m}$</th>
<th>$y_{2m}$</th>
<th>$y_{3m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Because $y_{13} = y_{23} = y_{33} = 1$, we stop sampling and randomize among the three categories. $\Box$
Sequential Procedure with Curtailment $\mathcal{M}_{BG}$ (Bechhofer and Goldsman 1986)

For the given $k$ and specified $(P^*, \theta^*)$, find the truncation number $n_0$ from the table.

At the $m$th stage of experimentation ($m \geq 1$), take the random observation $X_m = (X_{1m}, \ldots, X_{km})$.

Calculate the ordered category totals $y_{[1]}m \leq \cdots \leq y_{[k]}m$ and

$$z_m = \sum_{i=1}^{k-1} \left( \frac{1}{\theta^*} \right) (y_{[k]}m - y_{[i]}m).$$

Stop sampling at the first stage when either

$$z_m \leq \frac{(1 - P^*)}{P^*} \quad \text{or} \quad y_{[k]}m - y_{[k-1]}m \geq n_0 - m. \quad (6)$$
Let $N$ denote the value of $m$ at the termination of sampling. Select the category that yielded $y[k]N$ as the one associated with $p[k]$; randomize in the case of ties.

Remark: The truncation numbers $n_0$ given in the previous table are calculated assuming that Procedure $\mathcal{M}_{BG}$ has the same LF-configuration (3) as does $\mathcal{M}_{BEM}$. (This hasn’t been proven yet.)

Example: Suppose $k = 3$, $P^* = 0.75$, and $\theta^* = 3.0$. The table tells us to truncate sampling at $n_0 = 5$ observations. For the data

<table>
<thead>
<tr>
<th>$m$</th>
<th>$x_1m$</th>
<th>$x_2m$</th>
<th>$x_3m$</th>
<th>$y_1m$</th>
<th>$y_2m$</th>
<th>$y_3m$</th>
</tr>
</thead>
<tbody>
<tr>
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</tbody>
</table>

we stop sampling by the first criterion in (6) because

$z_2 = (1/3)^2 + (1/3)^2 = 2/9 \leq (1 - P^*)/P^* = 1/3$, and we select category 2. □
Example: Again suppose $k = 3$, $P^* = 0.75$, and $\theta^* = 3.0$ (so that $n_0 = 5$). For the data

<table>
<thead>
<tr>
<th>$m$</th>
<th>$x_{1m}$</th>
<th>$x_{2m}$</th>
<th>$x_{3m}$</th>
<th>$y_{1m}$</th>
<th>$y_{2m}$</th>
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</tbody>
</table>

we stop sampling by the second criterion in (6) because $m = n_0 = 5$ observations, and we select category 1. □
Example: Yet again suppose $k = 3$, $P^* = 0.75$, and $\theta^* = 3.0$ (so that $n_0 = 5$). For the data

<table>
<thead>
<tr>
<th>$m$</th>
<th>$x_{1m}$</th>
<th>$x_{2m}$</th>
<th>$x_{3m}$</th>
<th>$y_{1m}$</th>
<th>$y_{2m}$</th>
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<td>1</td>
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</tbody>
</table>

we stop according to the second criterion in (6) because $m = n_0 = 5$. However, we now have a tie between $y_{1,5}$ and $y_{2,5}$ and thus randomly select between categories 1 and 2.  □
Example: Still yet again suppose \( k = 3, P^* = 0.75, \) and \( \theta^* = 3.0 \) (so that \( n_0 = 5 \)). Suppose we observe

\[
\begin{array}{c|ccc|ccc}
  m & x_{1m} & x_{2m} & x_{3m} & y_{1m} & y_{2m} & y_{3m} \\
  1 & 0 & 1 & 0 & 0 & 1 & 0 \\
  2 & 1 & 0 & 0 & 1 & 1 & 0 \\
  3 & 0 & 1 & 0 & 1 & 2 & 0 \\
  4 & 0 & 0 & 1 & 1 & 2 & 1 \\
\end{array}
\]

Because categories 1 and 3 can do no better than tie category 2 (if we were to take the potential remaining \( n_0 - m = 5 - 4 = 1 \) observation), the second criterion in (6) tells us to stop; we select category 2. \( \square \)
Remark: Procedure $\mathcal{M}_{BG}$ usually requires fewer observations than $\mathcal{M}_{BEM}$.

Example: Suppose $k = 4$, $\theta^* = 1.6$, $P^* = 0.75$.

The single-stage procedure $\mathcal{M}_{BEM}$ requires 46 observations to guarantee (3).

Procedure $\mathcal{M}_{BG}$ (with a truncation number of $n_0 = 57$) has $E\{N|\text{LF}\} = 31.1$ and $E\{N|\text{EP}\} = 37.7$ for $p$ in the LF-configuration (4) and equal-probability (EP) configuration, $p[1] = p[t]$, respectively.
Suppose we take i.i.d. vector-observations $W_j = (W_{1j}, \ldots, W_{kj})$ ($j \geq 1$), where the $W_{ij}$ can be either discrete or continuous random variables.

For a particular vector-observation $W_j$, suppose the experimenter can determine which of the $k$ observations $W_{ij}$ ($1 \leq i \leq k$) is the “most desirable.” The term “most desirable” is based on some criterion of goodness designated by the experimenter, and it can be quite general, e.g., . . .

- The largest crop yield based on a vector-observation of $k$ agricultural plots using competing fertilizers.
- The smallest sample average customer waiting time based on a simulation run of each of $k$ competing queueing strategies.
- The smallest estimated variance of customer waiting times (from the above simulations).
For a particular vector-observation $W_j$, suppose $X_{ij} = 1$ or 0 according as $W_{ij} (1 \leq i \leq k)$ is the “most desirable” of the components of $W_j$ or not. Then $(X_{1j}, \ldots, X_{kj}) (j \geq 1)$ has a multinomial distribution with probability vector $p$, where

$$p_i = P\{W_{i1} \text{ is the “most desirable” component of } W_1\}.$$

Selection of the category corresponding to the largest $p_i$ can be thought of as that of finding the component having the highest probability of yielding the “most desirable” observation of those from a particular vector-observation. This problem can be approached using the multinomial selection methods described in this module.
Example: Suppose we want to find which of \( k = 3 \) job shop configurations is most likely to give the shortest expected times-in-system for a certain manufactured product. Because of the complicated configurations of the candidate job shops, it is necessary to simulate the three competitors. Suppose the \( j \)th simulation run of configuration \( i \) yields \( W_{ij} (1 \leq i \leq 3, j \geq 1) \), the proportion of 1000 times-in-system greater than 20 minutes.

Management has decided that the “most desirable” component of \( W_j = (W_{1j}, W_{2j}, W_{3j}) \) will be that component corresponding to \( \min_{1 \leq i \leq 3} W_{ij} \).
If $p_i$ denotes the probability that configuration $i$ yields the smallest component of $W_j$, then we seek to select the configuration corresponding to $p[3]$. Specify $P^* = 0.75$ and $\theta^* = 3.0$. The truncation number from the table for $\mathcal{M}_{BG}$ is $n_0 = 5$. We apply the procedure to the data

<table>
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<tr>
<th>$m$</th>
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<th>$w_{3m}$</th>
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<th>$x_{2m}$</th>
<th>$x_{3m}$</th>
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...and select shop configuration 2. □