NAME →

ISyE 3770 — Test 4 Solutions — Fall 2007

This test is 90 minutes long. You are allowed one cheat sheet. Do not look at or start the test until you are told to do so. When we ask you to return the test, stop immediately, hand the test in, and do not utter a word to anyone. Do not show any work other than your answers on this sheet. Check your answers — we won't be giving any credit for any answers that you transfer incorrectly from your worksheets. Good luck!

1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________

7. ________ 8. ________ 9. ________

10. ________ 11. ________ 12. ________

13. ________ 14. ________ 15. ________

16. ________ 17. ________ 18. ________

19. ________ 20. ________ 21. ________

22. ________ 23. ________ 24. ________

25. ________ 26. ________ 27. ________

28. ________ 29. ________ 30. ________

1. Suppose U_1, U_2, U_3 are i.i.d. U(0,1). Find $\mathsf{P}(U_1 < U_2 < U_3)$.

   Solution: There are six possible random orderings of the U_i's. Therefore, the probability of this particular ordering is 1/6. $\diamondsuit$

2. Let X be the outcome of a die toss. Find $\mathsf{E}[1/X]$ to 3 significant digits.

   Solution: $\mathsf{E}[1/X] = \sum_{i=1}^{6}(1/i)\mathsf{P}(X = i) = \frac{1}{6}\sum_{i=1}^{6}\frac{1}{i} = 0.408$ $\diamondsuit$

3. Suppose that the lifetime of a light bulb is exponential with a mean of 10000 hours. Further suppose that the bulb has already survived 20000 hours. Find the probability that it survives at least another 10000 hours.

   Solution: By the memoryless property,

   $$\mathsf{P}(X > 30000|X > 20000) = \mathsf{P}(X > 10000) = e^{-\lambda t} = e^{-1} = 0.368. \quad \diamondsuit$$

4. Suppose that X and Y are both Nor(1,4) random variables with $\mathsf{Cov}(X, Y) = 1$. Find $\mathsf{E}[XY]$.

   Solution: $\mathsf{E}[XY] = \mathsf{Cov}(X, Y) + \mathsf{E}[X]\mathsf{E}[Y] = 1 + 1 = 2.$ $\diamondsuit$

5. If X and Y are both $\text{Nor}(0, 1)$ with $\mathsf{Cov}(X, Y) = 0.5$. What's the distribution of $X + Y + 1$?

   Solution: $\mathsf{E}[X + Y + 1] = \mathsf{E}[X] + \mathsf{E}[Y] + 1 = 1$ and $\mathsf{Var}(X + Y + 1) = \mathsf{Var}(X) + \mathsf{Var}(Y) + 2\mathsf{Cov}(X, Y) = 3$. Thus, $X + Y + 1 \sim \text{Nor}(1, 3)$. $\diamondsuit$

6. Suppose that the number of accidents at a factory is Poisson with rate 2/month. What's the probability that there will be exactly 4 accidents during the next two-month period?

   Solution: The number of accidents in a two-month period is $X \sim \text{Pois}(4)$. Thus,

   $$\mathsf{P}(X = 4) = \frac{e^{-\lambda}\lambda^4}{4!} = \frac{e^{-4}4^4}{4!} = 0.195. \quad \diamondsuit$$

7. If $X \sim \text{Bern}(0.6)$, what's the m.g.f. of $2X - 1$?

   Solution: The m.g.f. of X is $M_X(t) = (0.6e^t + 0.4)$, so the m.g.f. of $2X - 1$ is

   $$e^{bt}M_X(at) = e^{-t}M_X(2t) = e^{-t}(0.6e^{2t} + 0.4) = 0.6e^t + 0.4e^{-t}. \quad \diamondsuit$$

8. TRUE or FALSE? If $X_1, X_2, \ldots, X_{100}$ are i.i.d. Poisson(3), then the sample mean $\bar{X}$ is approximately normal.

   Solution: TRUE (by the CLT). $\diamondsuit$

9. Suppose that the high temperature on February 1 is $\text{Nor}(20, 50)$ and the high temperature on March 1 is $\text{Nor}(30, 50)$. Assuming the two days are independent of each other, find the probability that the high temperature on February 1 is lower than the high on March 1.

   Solution:

$$\begin{aligned}\mathsf{P}(F < M) &= \mathsf{P}(F - M < 0)\\ &= \mathsf{P}(\text{Nor}(20-30, 50+50) < 0)\\ &= \mathsf{P}(\text{Nor}(-10, 100) < 0)\\ &= \mathsf{P}(\text{Nor}(0,1) < 10/\sqrt{100})\\ &= \mathsf{P}(Z < 1) = 0.8413. \quad \diamondsuit\end{aligned}$$

10. If $\alpha = 0.05$, find the normal quantile value $z_{\alpha/2}$.

    Solution: $z_{0.025} = 1.96$. $\diamondsuit$

11. TRUE or FALSE? If $\Phi(x)$ is the standard normal c.d.f. and Z is a standard normal random variable, then $\mathsf{P}(-x \le Z \le x) = 2\Phi(x) - 1$ for any x.

    Solution: TRUE. $\diamondsuit$

12. If $X \sim \chi^2(5)$, find $\mathsf{P}(X > 9.24)$.

    Solution: From table, 0.10. $\diamondsuit$

13. If T has the Student's t distribution with 8 degrees of freedom, find the probability that it is less than 2.306.

    Solution: From table, 0.975. $\diamondsuit$

14. If $X \sim t(3000)$, find $\mathsf{P}(X > 1.96)$.

    Solution: By CLT, $t(3000) \approx \text{Nor}(0,1)$. Thus, $\mathsf{P}(X > 1.96) \approx \mathsf{P}(Z > 1.96) = 0.025$. $\diamondsuit$

15. Find the sample variance of 0, -1, 1, and 0.

    Solution:

$$S^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right] = \frac{1}{3}(2-0) = 2/3. \quad \diamondsuit$$

16. Suppose $X_1, \ldots, X_6$ are i.i.d. $\text{Bern}(p)$, and we observe $X_1 = 0$, $X_2 = 0$, $X_3 = 1$, $X_4 = 1$, $X_5 = 0$, and $X_6 = 0$. What is the maximum likelihood estimate of p?

    Solution: $\hat{p} = \bar{X} = 1/3$. $\diamondsuit$

17. Suppose $X_1, \ldots, X_6$ are i.i.d. Bern(p), and we observe $X_1 = 0$, $X_2 = 0$, $X_3 = 1$, $X_4 = 1$, $X_5 = 0$, and $X_6 = 0$. What is the maximum likelihood estimate of $\sqrt{p}$?

    Solution: By the invariance property, the MLE of $\sqrt{p}$ is $\sqrt{\hat{p}} = \sqrt{1/3} = 0.577$. ♢

18. TRUE or FALSE? The mean squared error of an estimator is simply its bias plus its variance.

    Solution: FALSE. It's the bias^2 plus the variance. ♢

19. Suppose that X_1, X_2, X_3 are i.i.d. Nor(μ, σ^2), and that the sample mean and sample variance are $\bar{X} = 25$ and $S^2 = 30$, respectively. What is the MLE of σ^2?

    Solution: $\widehat{\sigma^2} = \frac{n-1}{n} S^2 = 20$. ♢

20. TRUE or FALSE? If an estimator T is unbiased for a parameter θ, then $1/T$ is unbiased for $1/\theta$.

    Solution: FALSE. ♢

21. TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. $U(0, \theta)$, then the MLE for θ is unbiased.

    Solution: FALSE. By class notes, the MLE is $\hat{\theta} = \max_i X_i$, which is slightly biased. (See below.) ♢

22. If $X_1, \ldots, X_n$ are i.i.d. Exp(3), find $\mathsf{E}[S^2]$.

    Solution: Since S^2 is unbiased for the variance, we have $\mathsf{E}[S^2] = \mathsf{Var}(X_i) = 1/9$. ♢

23. TRUE or FALSE? If $X_1, \ldots, X_n$ are i.i.d. $U(0, \theta)$, then *both* $2\bar{X}$ and $\frac{n+1}{n} \max_{1 \le i \le n}\{X_i\}$ are unbiased for θ.

    Solution: TRUE. ♢

24. Suppose we have 2 estimators, T_1 and T_2, for some parameter θ. Further suppose that $\mathsf{Bias}(T_1) = 15$, $\mathsf{Var}(T_1) = 40$, $\mathsf{Bias}(T_2) = 5$, and $\mathsf{Var}(T_2) = 220$. Which estimator has the larger mean squared error?

    Solution: $\mathsf{MSE}(T_1) = 15^2 + 40 = 265$ and $\mathsf{MSE}(T_2) = 5^2 + 220 = 245$; so the answer is T_1. ♢

25. Which denotes another type of estimator — MOM or DAD?

    Solution: MOM (method of moments). ♢

26. Suppose that $X_1, \ldots, X_9$ are i.i.d. normal with unknown mean and variance. Further suppose that $\bar{X} = 100$ and $S^2 = 300$. Find a 95% two-sided confidence interval for μ.

Solution:

$$\begin{aligned} \mu &\in \bar{X} \pm t_{\alpha/2,n-1}\sqrt{S^2/n} \\ &= 100 \pm t_{0.025,8}\sqrt{300/9} \\ &= 100 \pm 2.306\sqrt{300/9} \\ &= 100 \pm 13.31 \\ &= [86.69, 113.31]. \quad \diamondsuit \end{aligned}$$

27. Suppose that $X_1, \ldots, X_n$ are i.i.d. normal with unknown mean μ, but *known* variance of 100. How big would n have to be in order for a two-sided 90% confidence interval to have a half-length of 1? (Give the smallest such number.)

Solution: The half-length is

$$H = 1 = z_{\alpha/2}\sqrt{\sigma^2/n} = z_{0.05}\sqrt{100/n} = (1.645)(10)/\sqrt{n}.$$

Solving for n, we have $n = 270.6$, so we take $n = 271$. $\diamondsuit$

28. Suppose $[0, 5]$ is a 90% t-distribution confidence interval for the mean μ based on 10 i.i.d. normal observations. Now the boss has decided that she wants a 95% confidence interval based on those same 10 observations. What is it?

Solution: The confidence interval is of the form $\mu \in 2.5 \pm 2.5$. In particular,

$$\begin{aligned} \mu &\in \bar{X} \pm t_{\alpha/2,n-1}\sqrt{S^2/n} \\ &= 2.5 \pm t_{0.05,9}\sqrt{S^2/10} \\ &= 2.5 \pm 1.833\sqrt{S^2/10}. \end{aligned}$$

Thus, if we set the half-length $1.833\sqrt{S^2/10} = 2.5$, we find that $S^2 = 18.60$. So finally, the desired confidence interval is

$$\begin{aligned} \mu &\in \bar{X} \pm t_{\alpha/2,n-1}\sqrt{S^2/n} \\ &= 2.5 \pm t_{0.025,9}\sqrt{18.60/10} \\ &= 2.5 \pm 2.262\sqrt{18.60/10} \\ &= 2.5 \pm 3.08. \end{aligned}$$

So the desired 95% CI is $\mu = [-0.58, 5.58]$. $\diamondsuit$

29. TRUE or FALSE? When forming a confidence interval for the difference in the means from two normal distributions with unknown variances, it's OK to use a *pooled* variance estimator if we think the two variances are approximately equal.

    Solution: TRUE. $\diamondsuit$

30. Flip a coin (quietly). If it comes up heads, answer only Question A; if it comes up tails, answer only Question B.

    - Question A: Flip the coin again. Did it come up heads this time?
    - Question B: Have you ever cheated on a test at Georgia Tech?

    Your answer to this problem will be simply one word: either "yes" or "no". Don't tell me what happened with your coin flip(s) — I don't want to know which question you're answering so that I have no way of knowing whether you have cheated or not. There is no trick involved. Please do not re-toss the coin(s) if you didn't get the outcome that you desired. Simply flip the coin(s), and answer the appropriate question honestly.

    Solution: Either YES or NO is acceptable. For the Fall 2077 final, we received 150 NO answers and 64 YES answers. $\diamondsuit$

Table 1: Standard normal values

z	$\mathsf{P}(Z \leq z)$
1	0.8413
1.28	0.9000
1.5	0.9332
1.645	0.9500
1.96	0.9750
2	0.9773

Table 2: $\chi^2_{\alpha,\nu}$ values

$\nu \setminus \alpha$	0.50	0.10	0.05	0.025
4	3.36	7.78	9.49	11.14
5	4.35	9.24	11.07	12.83
6	5.35	10.65	12.59	14.45

Table 3: $t_{\alpha,\nu}$ values

$\nu \setminus \alpha$	0.10	0.05	0.025
8	1.397	1.860	2.306
9	1.383	1.833	2.262
10	1.372	1.812	2.228