NAME →

ISyE 3770 — Test 3 Solutions — Fall 2008

This test is 90 minutes long. You are allowed three cheat sheets. Do not look at or start the test until you are told to do so. When we ask you to return the test, stop immediately, hand the test in, and do not utter a word to anyone. Do not communicate via mind power with anyone. Do not show any work other than your answers on this sheet. Good luck and have a holly jolly test!

1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________

7. ________ 8. ________ 9. ________

10. ________ 11. ________ 12. ________

13. ________ 14. ________ 15. ________

16. ________ 17. ________ 18. ________

19. ________ 20. ________ 21. ________

22. ________ 23. ________ 24. ________

25. ________ 26. ________

1. Suppose U_1, U_2 are i.i.d. U(0,1). Find $\mathsf{E}[\frac{U_1}{U_1+U_2}]$.

   Solution: $\frac{1}{2}$ (by symmetry) $\square$

2. Let X be the outcome of a die toss. Find $\mathsf{E}[\ell\text{n}(X)]$ to 3 significant digits.

   Solution: $\sum_{i=1}^{6} \frac{1}{6}\ell\text{n}(i) = 1.096$ $\square$

3. Suppose that the lifetime of a light bulb is exponential with a mean of 10000 hours. Further suppose that the bulb has already survived 20000 hours. Find the expected number of hours that it will survive beyond the 20000 it has already survived.

   Solution: 10000 (Memoryless property) $\square$

4. Suppose that X and Y are both Exp(1) random variables with $\mathsf{Cov}(X, Y) = 1/2$. Find $\mathsf{Var}(3X + Y)$.

   Solution: $9\mathsf{Var}(X) + \mathsf{Var}(Y) + 6\mathsf{Cov}(X, Y) = 9 + 1 + 3 = 13$ $\square$

5. Suppose that the number of typos in a book is Poisson with rate 1/page. What's the probability that there will be exactly 4 typos on the next two pages?

   Solution: $X \sim \text{Poisson}(2/2\,\text{pages})$, so that $P(X = 4) = \frac{e^{-2}2^4}{4!} = 0.09224$ $\square$

6. Suppose X has m.g.f. of $(0.3e^t + 0.7)^4$. Name the distribution of X (with parameters).

   Solution: Bin(4, 0.3) $\square$

7. Suppose $X_1, X_2, \ldots, X_{300}$ are i.i.d. Pois(3). Further, define the sample mean $\bar{X} \equiv \sum_{i=1}^{300} X_i/300$. Find an approximate expression for $\mathsf{P}(2.9 \le \bar{X} \le 3.1)$.

   Solution: By the CLT, $\bar{X} \approx \text{Nor}(3, \frac{3}{300}) = \text{Nor}(3,\ 0.01)$, so that

   $$\mathsf{P}(2.9 \leqq \bar{X} \leqq 3.1) \approx \mathsf{P}(-1 \leqq Z \leqq 1) = 2\Phi(1) - 1 = 2(0.8413) - 1 = 0.6826 \quad \square$$

8. If $\alpha = 0.025$, find the normal quantile value z_α.

   Solution: 1.96 $\square$

9. TRUE or FALSE? If $\Phi(x)$ is the standard normal c.d.f. and Z is a standard normal random variable, then $\mathsf{P}(a \le Z \le b) = \Phi(b) - \Phi(a)$, as long as $a \le b$.

Solution: True $\square$

10. Find $\chi^2_{0.05,4}$.

    Solution: 9.49 $\square$

11. If T has the Student's t distribution with 8 degrees of freedom. Find x such that $\mathsf{P}(T < x) = 0.05$.

    Solution: -1.86 $\square$

12. TRUE or FALSE? A T random variable with 200 degrees of freedom is approximately normal.

    Solution: True $\square$

13. Find the sample standard deviation of -1, 1, 0, 0, -1, and 1.

    Solution: $S^2 = \frac{\sum X_i^2 - n\bar{X}^2}{n-1} = \frac{4}{5}$, so that $S = \sqrt{\frac{4}{5}} = 0.8944$ $\square$

14. If $X_1, \ldots, X_n$ are i.i.d. Pois(10), what is the expected value of the sample variance S^2?

    Solution: $\mathsf{E}(S^2) = \mathsf{Var}(X_i) = 10$ $\square$

15. TRUE or FALSE? The mean squared error of an estimator is its $(\text{variance} + \text{bias})^2$.

    Solution: False. It is $(\text{Variance} + \text{bias}^2)$ $\square$

16. TRUE or FALSE? Suppose $X_1, \ldots, X_n$ are i.i.d. Exp(λ). Then $1/\bar{X}$ is unbiased for λ.

    Solution: False $\square$

17. If $X_1, \ldots, X_5$ are i.i.d. Nor$(-4, 10)$, what is the expected value of the maximum likelihood estimator for the variance σ^2?

    Solution: $\mathsf{E}[\widehat{\sigma}^2] = \frac{n-1}{n}\mathsf{E}[S^2] = \frac{n-1}{n}\mathsf{Var}(X_i) = 10 \times \frac{4}{5} = 8$ $\square$

18. Consider the observations 18, 17, 83, 34. Suppose these observations are from a U$(0, \theta)$ distribution. Find the maximum likelihood estimate for θ.

Solution: $\max X_i = 83$

19. Suppose X_1, X_2, X_3 are i.i.d. Exp(λ), and that we observe the realizations $X_1 = 1.0$, $X_2 = 2.0$, and $X_3 = 3.0$. What is the maximum likelihood estimate of $\mathsf{P}(X_1 > 2)$?

   Solution: $\hat{\lambda} = 1/\bar{X} = 1/2$, so that $\widehat{\mathsf{P}}(X_1 > 2) = e^{-2\hat{\lambda}} = e^{-1} = 0.368$ $\square$

20. Suppose $X_1, \ldots, X_n$ are i.i.d. with p.d.f.

$$f(x) \; = \; \frac{1}{x\tau\sqrt{2\pi}} \exp\left\{ -\frac{[\ell\text{n}(x)]^2}{2\tau^2} \right\}, \quad x \ge 0.$$

   Find the MLE for τ^2.

   Solution: We have

$$L(\tau) \; = \; \frac{1}{\left(\prod_{i=1}^n x_i\right)(\tau\sqrt{2\pi})^n} \times \exp\left\{ \frac{-1}{2\tau^2} \sum_{i=1}^n [\ell\text{n}(x_i)]^2 \right\}$$

   so that

$$\ell\text{n}\, L \; = \; -\ell\text{n}\left(\prod_{i=1}^n x_i \right) - n\ell\text{n}(\tau\sqrt{2\pi}) - \frac{1}{2\tau^2} \sum_{i=1}^n [\ell\text{n}(x_i)]^2.$$

   Taking the derivative, we obtain

$$\frac{\partial}{\partial\tau} \ell\text{n}\, L \; = \; -\frac{n}{\tau} + \frac{2}{2\tau^3} \sum_{i=1}^n [\ell\text{n}(x_i)]^2 = 0.$$

   Solving this equation (and using the invariance property of MLE's, we finally get

$$\widehat{\tau^2} \; = \; \frac{1}{n} \sum_{i=1}^n [\ell\text{n}(x_i)]^2. \quad \square$$

21. FILL IN THE BLANK. "MOM" stands for "Method of ____________________".

   Solution: Moments $\square$

22. Suppose that $X_1 = 2$, $X_2 = 0$, and $X_3 = -2$ are i.i.d. normal observations with *known* variance $\sigma^2 = 4$. Find a 95% two-sided confidence interval for the mean μ.

Solution: $\mu \in \bar{X} \pm z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}} = 0 \pm 1.96\sqrt{\frac{4}{3}} = 0 \pm 2.2632 = [-2.26,\ 2.26]$ $\square$

23. Suppose that $X_1, \ldots, X_n$ are i.i.d. normal with unknown mean μ, but *known* variance of 9. How big would n have to be in order for a two-sided 95% confidence interval to have a half-length of 1? (Give the smallest such number.)

    Solution: The form of the confidence interval is $\mu \in \bar{X} \pm z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}$. Then the half-length requirement is $z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}} \le 1 \;\Leftrightarrow\; 1.96\sqrt{\frac{9}{n}} \le 1$. A little algebra eventually yields $n \geqq 35$. $\square$

24. Suppose that $X_1 = 2$, $X_2 = 0$, and $X_3 = -2$ are i.i.d. normal observations with *unknown* variance σ^2. Find a 95% two-sided confidence interval for the mean μ.

    Solution: $S^2 = \frac{1}{n-1}(\sum X_i^2 - n\overline{X}^2) = \frac{1}{2}(8.0) = 4$. This implies that the desired confidence interval is

    $$\mu \;\in\; \bar{X} \pm t_{\alpha/2,n-1}\sqrt{\frac{S^2}{n}} \;=\; 0 \pm t_{0.025,2}\sqrt{\frac{4}{3}} \;=\; \pm 4.303\sqrt{\frac{4}{3}} \;=\; [-4.97,\ 4.97]. \quad \square$$

25. Suppose $[0, 2]$ is a 95% confidence interval for the mean μ based on 5 i.i.d. normal observations with *unknown* σ^2. Now the boss has decided that she wants a 90% CI based on those same 5 observations. What is it?

    Solution: $\mu \in \bar{X} \pm t_{0.05/2,\,n-1}\sqrt{\frac{S^2}{n}} = 1 \pm 1$ implies

    $$1 \;=\; t_{0.025,\,n-1}\sqrt{\frac{S^2}{n}} \;=\; t_{0.025,\,4}\sqrt{\frac{S^2}{5}} \;=\; 2.776\sqrt{\frac{S^2}{5}},$$

    so that $S^2 = 0.6488$. Now we can get the 90% confidence interval:

    $$\mu \in \bar{X} \pm t_{0.10/2,n-1}\sqrt{\frac{S^2}{n}} \;=\; 1 \pm t_{0.05,4}\sqrt{\frac{S^2}{5}} \;=\; 1 \pm 2.132\sqrt{\frac{0.649}{5}} \;=\; 1 \pm 0.768,$$

    so that $\mu \in [0.232,\ 1.768]$. $\square$

26. TRUE or FALSE? When forming a confidence interval for the difference in the means from two highly correlated normal distributions with unknown variances, it's a great idea to use a *paired* variance estimator.

    Solution: True $\square$

Table 1: Standard normal values

z	$\mathsf{P}(Z \leq z)$
1	0.8413
1.28	0.9000
1.5	0.9332
1.645	0.9500
1.96	0.9750
2	0.9773

Table 2: $\chi^2_{\alpha,\nu}$ values

$\nu \setminus \alpha$	0.50	0.10	0.05	0.025
4	3.36	7.78	9.49	11.14
5	4.35	9.24	11.07	12.83
6	5.35	10.65	12.59	14.45

Table 3: $t_{\alpha,\nu}$ values

$\nu \setminus \alpha$	0.10	0.05	0.025
1	3.078	6.314	12.706
2	1.886	2.920	4.303
3	1.638	2.353	3.182
4	1.533	2.132	2.776
5	1.476	2.015	2.571
6	1.440	1.943	2.447
7	1.415	1.895	2.365
8	1.397	1.860	2.306
9	1.383	1.833	2.262
10	1.372	1.812	2.228