NAME →

ISyE 3770 — Test 3 Solutions — Fall 2007

(revised 11/21/07)

This test is 80 minutes long. You are allowed one cheat sheet. Do not look at or start the test until you are told to do so. When we ask you to return the test, stop immediately, hand the test in, and do not utter a word to anyone. Do not show any work other than your answers on this sheet. Check your answers — we won't be giving any credit for any answers that you transfer incorrectly from your worksheets. Good luck!

Put your answers here...

1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________

7. ________ 8. ________ 9. ________

10. ________ 11. ________ 12. ________

13. ________ 14. ________ 15. ________

16. ________ 17. ________ 18. ________

19. ________ 20. ________

1. Suppose $X \sim \text{Bern}(0.6)$. Find the expected value of $Y = 4X^2 + 1$.

   Solution: Recall that $X \sim \text{Bern}(p)$ implies $\mathsf{E}[X^k] = p$ for any $k \geq 1$. Then $\mathsf{E}[Y] = 4\mathsf{E}[X^2] + 1 = 4p + 1 = 3.4$. ♢

2. Consider the joint probability mass function of (X, Y) for which $f(0,0) = 0.3$, $f(0,1) = 0.2$, $f(2,0) = 0.4$, and $f(2,1) = 0.1$. Find $\mathsf{P}(Y = 1)$.

   Solution: The joint p.m.f. $f(x, y)$ is

$Y \backslash X$	0	2	$f_Y(y)$
0	0.3	0.4	0.7
1	0.2	0.1	0.3
$f_X(x)$	0.5	0.5	1

   Thus, $\mathsf{P}(Y = 1) = f_Y(1) = 0.3$. ♢

3. Consider the joint p.m.f. from Question 2. Find $\mathsf{E}[Y]$.

   Solution: $\mathsf{E}[Y] = \sum_y y f_Y(y) = 0(0.7) + 1(0.3) = 0.3$. ♢

4. Consider the joint p.m.f. from Question 2. Find $\mathsf{Cov}(X, Y)$.

   Solution: First of all, $\mathsf{E}[Y] = 0.3$ from the solution to Problem 3. Similarly, $\mathsf{E}[X] = \sum_x x f_X(x) = 0(0.5) + 2(0.5) = 1$. Further,

$$\mathsf{E}[XY] = \sum_x \sum_y xy f(x, y) = 0(0)(0.3) + 0(1)(0.2) + 2(0)(0.4) + 2(1)(0.1) = 0.2.$$

   Thus, $\mathsf{Cov}(X, Y) = \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y] = 0.2 - 1(0.3) = -0.1$. ♢

5. True or False? If the joint p.m.f. of (X, Y) is given by $f(0,0) = 0.18$, $f(0,1) = 0.42$, $f(2,0) = 0.12$, and $f(2,1) = 0.28$, then X and Y are independent.

   Solution: The joint p.m.f. $f(x, y)$ is

$Y\backslash X$	0	2	$f_Y(y)$
0	0.18	0.12	0.3
1	0.42	0.28	0.7
$f_X(x)$	0.6	0.4	1

From this, we see that $f(x,y) = f_X(x)f_Y(y)$ for all four (x,y) choices. Thus, the answer is TRUE. $\diamondsuit$

6. True or False? If X and Y are independent, then the conditional p.d.f. $f(x|y) = f_Y(y)$.

   Solution: In fact, $f(x|y) = f_X(x)$, so the answer is FALSE. $\diamondsuit$

7. True or False? Suppose that X and Y have joint p.d.f. $f(x,y) = c/(x+y)^2$, $1 \le x \le 2$, $1 \le y \le 3$, for some constant c. Then X and Y are independent.

   Solution: Since you can't factor $f(x,y) = h(x)g(y)$ for some functions $h(x)$ and $g(y)$, the answer is FALSE. $\diamondsuit$

8. True or False? Suppose that X and Y have joint p.d.f. $f(x,y) = cxy$, $0 \le x \le y \le 1$, where c is the appropriate constant. Then X and Y are independent.

   Solution: Since you have funny limits, the answer is FALSE. $\diamondsuit$

9. Consider the bivariate p.d.f. $f(x,y) = 8xy$, $0 \le x \le y \le 1$. Find $\mathsf{E}[X^2Y^2]$.

   Solution: We have

$$
\begin{aligned}
\mathsf{E}[X^2Y^2] &= \int_0^1 \int_0^y x^2y^2 8xy \, dx \, dy \\
&= 8\int_0^1 y^3 \int_0^y x^3 \, dx \, dy \\
&= 2\int_0^1 y^7 \, dy \\
&= 1/4. \quad \diamondsuit
\end{aligned}
$$

10. Suppose $f(x,y) = 8xy$, $0 \le x \le y \le 1$. Find the conditional p.d.f. $f(y|x)$.

Solution: First of all,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\ &= \int_x^1 8xy\,dy \\ &= 4x(1-x^2),\ 0 \le x \le 1. \end{aligned}$$

Thus,

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2},\ 0 \le x \le y \le 1. \quad \diamondsuit$$

11. Suppose that $\mathsf{E}[X] = 3$, $\mathsf{E}[Y] = 4$, $\mathsf{Var}(X) = 5$, $\mathsf{Var}(Y) = 6$, and $\mathsf{Cov}(X,Y) = 4$. Find $\mathsf{Var}(X + 2Y)$.

Solution: $\mathsf{Var}(X+2Y) = \mathsf{Var}(X) + \mathsf{Var}(2Y) + 2\mathsf{Cov}(X,2Y) = \mathsf{Var}(X) + 4\mathsf{Var}(Y) + 4\mathsf{Cov}(X,Y) = 45.$ $\diamondsuit$

12. If $\mathsf{E}[X] = 3$, $\mathsf{E}[Y] = 4$, $\mathsf{Var}(X) = 5$, $\mathsf{Var}(Y) = 6$, and $\mathsf{Cov}(X,Y) = 4$, find $\mathsf{Corr}(X,Y)$.

Solution: $\mathsf{Corr}(X,Y) = \mathsf{Cov}(X,Y)/\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)} = 4/\sqrt{30} = 0.73.$ $\diamondsuit$

13. True or False? If $a > 0$ and $b > 0$ are constants, then $\mathsf{Corr}(X,Y) = \mathsf{Corr}(aX, bY)$.

Solution: We have

$$\begin{aligned} \mathsf{Corr}(aX,bY) &= \frac{\mathsf{Cov}(aX,bY)}{\sqrt{\mathsf{Var}(aX)\mathsf{Var}(bY)}} \\ &= \frac{ab\mathsf{Cov}(X,Y)}{\sqrt{a^2\mathsf{Var}(X)b^2\mathsf{Var}(Y)}} \\ &= \frac{\mathsf{Cov}(X,Y)}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} \\ &= \mathsf{Corr}(a,b), \end{aligned}$$

so the answer is TRUE. $\diamondsuit$

14. If X and Y are independent Exp(1) random variables, find $\mathsf{Var}(XY)$.

    Solution: We have

$$\begin{aligned}\mathsf{Var}(XY) &= \mathsf{E}[(XY)^2] - (\mathsf{E}[XY])^2 \\ &= \mathsf{E}[X^2]\mathsf{E}[Y^2] - (\mathsf{E}[X]\mathsf{E}[Y])^2 \quad \text{(you can factor since } X, Y \text{ are indep)} \\ &= 2(2) - (1 \cdot 1)^2 \quad \text{(by class notes and other sources)} \\ &= 3. \quad \diamondsuit\end{aligned}$$

15. True or False? If X and Y are independent, then $\mathsf{Corr}(X, Y) = 0$.

    Solution: TRUE. $\diamondsuit$

16. Toss a pair of dice 4 times. Let X denote the number of times we see a sum of 7. Find $\mathsf{Var}(X)$.

    Solution: $X \sim \text{Bin}(4, \frac{1}{6})$, which implies that $\mathsf{Var}(X) = npq = 4(\frac{1}{6})(\frac{5}{6}) = \frac{5}{9}$. $\diamondsuit$

17. What does "i.i.d." stand for?

    Solution: independent and identically distributed. $\diamondsuit$

18. If $\mathsf{P}(X = -1) = 0.6$ and $\mathsf{P}(X = 3) = 0.4$, find the m.g.f. of X.

    Solution: $M_X(t) = \mathsf{E}[e^{tX}] = \sum_x e^{tx} f(x) = e^{-t}(0.6) + e^{3t}(0.4)$. $\diamondsuit$

19. Suppose X has m.g.f. $3/(3 - 4t)$ for $t < 3/4$. Name (with any parameters) the distribution of X.

    Solution: We have $M_X(t) = \frac{3}{4}/(\frac{3}{4} - t)$, so that we immediately have $X \sim \text{Exp}(3/4)$. $\diamondsuit$

Note that $X \sim 4 \cdot \text{Exp}(3)$ would also have been an acceptable answer.

20. If X and Y are independent Bern(0.6) random variables, find the m.g.f. of $X + Y$.

    Solution: Since X and Y are i.i.d., we have $M_{X+Y}(t) = [M_X(t)]^2 = (pe^t + q)^2 = (0.6e^t + 0.4)^2$. $\diamondsuit$