NAME →

ISyE 3770 — Test 2a

Spring 2007

This test is 85 minutes long. You are allowed two cheat sheets. Whenever possible, please give answers in simplified form.

Put your answers here...

1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________

7. ________ 8. ________ 9. ________

10. ________ 11. ________ 12. ________

13. ________ 14. ________ 15. ________

16. ________ 17. ________ 18. ________

19. ________ 20. ________ 21. ________

1. If X is a random variable with c.d.f. $F(x) = 1 - e^{-3x}$, $x \geq 0$, find $\mathsf{Var}(X)$.
   Solution: $\frac{1}{9}$
   $$X \sim Exp(3) \Rightarrow \mathsf{Var}(X) = \frac{1}{9}$$
   □

2. What does the "m" in "m.g.f." stand for?
   Solution: moment (moment generating function)
   □

3. If X is a random variable with m.g.f. $0.4e^t + 0.6$, find $\mathsf{Var}(X)$.
   Solution: 0.24
   $$\text{If } M_x(t) = pe^t + q \Rightarrow X \sim Bern(p) \Rightarrow \mathsf{Var}(X) = pq$$
   $$\text{Thus, } X \sim Bern(0.4) \text{ in this example } \Rightarrow \mathsf{Var}(X) = 0.24$$
   □

4. True or False? If X is a discrete random variable with m.g.f. $M_X(t)$, then $\mathsf{E}[X] = \frac{d}{dt}M_X(t)$.
   Solution: False
   $$\mathsf{E}(X) = \frac{\mathrm{d}}{\mathrm{dt}}M_x(t)|_{t=0}$$
   □

5. If X and Y are i.i.d. with m.g.f. $(1-t)^{-1}$, what's the m.g.f. of $X + Y$?
   Solution: $(1-t)^{-2}$
   $$M_{X+Y}(t) = M_X(t)M_Y(t) = (1-t)^{-1}(1-t)^{-1} = (1-t)^{-2}$$
   □

6. True or False? If X and Y are independent, then $\mathsf{E}[X^2|Y = y] = \mathsf{E}[X^2]$.
   Solution: Ture
   □

7. True or False? Suppose that X and Y have joint p.d.f. $f(x, y) = cx^2\ell\text{n}(y)$, $1 \leq x \leq 2$, $1 \leq y \leq 3$, where c is the appropriate constant. Then X and Y are independent.
   Solution: True
   $$f(x, y) = cx^2 \ln y, \quad 1 \leq x \leq 2, \ 1 \leq y \leq 3,$$
   since I can factor $f(x, y) \Rightarrow X, Y$ are independent.
   □

8. True or False? Suppose that X and Y have joint p.d.f. $f(x,y) = cx^2\ell\text{n}(y)$, $1 \le x \le y \le 2$, where c is the appropriate constant. Then X and Y are independent.
Solution: False

$$1 \le x \le y \le 2, \text{ Funny limits } \Rightarrow \text{ not independent}$$

□

9. Consider the bivariate p.d.f. $f(x,y) = 10x^2y$, $0 \le y \le x \le 1$. Find $\mathsf{E}[XY]$.
Solution: $\frac{10}{21}$

$$\begin{aligned} \mathsf{E}(XY) &= \iint_{\mathbb{R}^2} xyf(x,y)\,dxdy \\ &= \int_0^1 \int_0^x xy10x^2y\,dydx = \frac{10}{21} \end{aligned} \tag{1}$$

□

10. Consider the bivariate p.d.f. $f(x,y) = 10x^2y$, $0 \le y \le x \le 1$. Find the conditional p.d.f. $f(x|y)$.
Solution: $\frac{3x^2}{1-y^3}$, $0 \le y \le x \le 1$

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \int_y^1 10\,x^2y\,dx = \frac{10}{3}y(1-y^3), \;\; 0 < y < 1$$

$$\Rightarrow f_{X|y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{10x^2y}{\frac{10}{3}y(1-y^3)} = \frac{3x^2}{1-y^3}, \;\; 0 \le y \le x \le 1$$

□

11. If $\mathsf{E}[X] = 3$, $\mathsf{E}[Y] = 4$, $\mathsf{Var}(X) = 5$, $\mathsf{Var}(Y) = 6$, and $\mathsf{Cov}(X,Y) = 4$, find $\mathsf{E}[\mathsf{E}(X|Y)]$.
Solution: 3

$$\mathsf{E}(\mathsf{E}(X|Y)) = \mathsf{E}(X) = 3$$

□

12. True or False? $\mathsf{E}[\mathsf{E}(X^2|Y)] = \mathsf{E}[X^2]$ whether or not X and Y are independent.
Solution: True
□

13. If X and Y are i.i.d. Bern(p), find $\mathsf{E}[X^2Y^2]$.
Solution: p^2

$$\mathsf{E}(X^2Y^2) = \mathsf{E}(X^2)\mathsf{E}(Y^2) = (\mathsf{E}(X^2))^2 = p^2$$

□

14. True or False? If $\mathsf{Cov}(X, Y) = 0$, then X and Y are independent.
Solution: False (see example in class notes)
□

15. Suppose X and Y are discrete random variables having joint p.m.f. $f(0, 1) = 0.3$, $f(0, 2) = 0.4$, $f(1, 1) = 0.1$, and $f(1, 2) = 0.2$. Find $\mathsf{E}[X]$.
Solution: 0.3

$y \backslash x$	0	1	$f_Y(y)$
0	0.3	0.1	0.4
1	0.4	0.2	0.6
$f_X(x)$	0.7	0.3	1

$$\mathsf{E}(X) = \sum_x x f_X(x) = 0(0.7) + 1(0.3) = 0.3$$
$$\mathsf{E}(Y) = \sum_y y f_Y(y) = 1(0.4) + 2(0.6) = 1.6$$

□

16. Suppose X and Y are discrete random variables having joint p.m.f. $f(0, 1) = 0.3$, $f(0, 2) = 0.4$, $f(1, 1) = 0.1$, and $f(1, 2) = 0.2$. Find $\mathsf{Cov}(X, Y)$.
Solution: 0.02

$$\mathsf{E}(XY) = \sum_x \sum_y xy\, f(x, y) = 0.5$$
$$\mathsf{Cov}(X, Y) = \mathsf{E}(XY) - \mathsf{E}(X)\mathsf{E}(Y) = 0.5 - 0.3(1.6) = 0.02$$

□

17. If $\mathsf{E}[X] = \mathsf{E}[Y] = 3$, $\mathsf{Var}(X) = \mathsf{Var}(Y) = 5$, and $\mathsf{Var}(X + Y) = 8$, find $\mathsf{Cov}(X, Y)$.
Solution: -1

$$\mathsf{Var}(X + Y) = \mathsf{Var}(X) + \mathsf{Var}(Y) + 2\mathsf{Cov}(X, Y)$$
$$8 = 5 + 5 + 2\mathsf{Cov}(X, Y) \Rightarrow \mathsf{Cov}(X, Y) = -1$$

□

18. True or False? $\mathsf{Cov}(X,Y) = \mathsf{Corr}(X,Y)\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}$.
    Solution: True
    □

19. If $X_1, X_2, \ldots, X_{100}$ are i.i.d. with mean 5 and variance 100, find the variance of the sample mean, $\mathsf{Var}(\bar{X})$.
    Solution: 1
    $$\mathsf{Var}(\bar{X}) = \frac{\mathsf{Var}(X_i)}{n} = \frac{100}{100} = 1$$
    □

20. Suppose the probability that a random person likes vanilla ice cream is 0.6. Let X denote the number out of 4 random people who like vanilla. Find $\mathsf{Var}(X)$.
    Solution: 0.96
    $$X \sim Bin(4, 0.6) \Rightarrow \mathsf{Var}(X) = npq = 4(0.6)(0.4)$$
    □

21. BONUS: Complete the following lyric from the Zombies' "Time of the Season": "What's your name? ____________________" [Hint: This is a famous phrase that is still in use today. Amazingly, it refers to the song "Summertime", by George Gershwin.]
    Solution: Who's Your Daddy?
    □