

NAME →

ISyE 3770 — Test 1a Solutions — Fall 2009

This test is 55 minutes long. You are allowed one cheat sheet.

Put your nice, simple answers here...

1. _____ 2. _____ 3. _____

4. _____ 5. _____ 6. _____

7. _____ 8. _____ 9. _____

10. _____ 11. _____ 12. _____

13. _____ 14. _____ 15. _____

16. _____ 17. _____ 18. _____

19. _____ 20. _____ 21. _____

22. _____ 23. _____ 24. _____

25. _____ 26. _____

1. What is any subset of an experiment's sample space called?

Solution: Event. \diamond

2. Suppose that $P(\text{it rains today}) = 0.4$, $P(\text{it rains tomorrow}) = 0.8$, and $P(\text{it rains either day}) = 0.9$. What's the probability that it rains both days?

Solution:

$$\begin{aligned} P(\text{Today and Tomorrow}) &= P(\text{Today}) + P(\text{Tomorrow}) - P(\text{Today or Tomorrow}) \\ &= 0.4 + 0.8 - 0.9 = 0.3. \quad \diamond \end{aligned}$$

3. If $P(A) = 0.3$, $P(B) = 0.5$, and $P(C) = 0.5$, and A , B , and C are independent, find the probability that *none* of A , B , and C occur.

Solution:

$$\begin{aligned} P(\bar{A} \cap \bar{B} \cap \bar{C}) &= P(\bar{A})P(\bar{B})P(\bar{C}) \\ &= (0.7)(0.5)(0.5) \\ &= 0.175. \quad \diamond \end{aligned}$$

4. TRUE or FALSE? $P(A) = 1$ implies that A is the entire sample space.

Solution: FALSE. For example, $S = [0, 1]$, $A = [0, 1] - \{0.5\}$. \diamond

5. TRUE or FALSE? If (i) $P(A \cap B) = P(A)P(B)$, (ii) $P(A \cap C) = P(A)P(C)$, and (iii) $P(B \cap C) = P(B)P(C)$, then A , B , and C are all independent of each other.

Solution: FALSE. We also need $P(A \cap B \cap C) = P(A)P(B)P(C)$. \diamond

6. Which is bigger, $P_{100,50}$ or $\binom{100}{50}$?

Solution: $P_{100,50} = \frac{100!}{50!}$ and $\binom{100}{50} = \frac{100!}{50!50!}$. Thus $P_{100,50}$ is bigger. \diamond

7. Find $\sum_{i=0}^{18} \binom{18}{i}$.

Solution: By a corollary of the Binomial Theorem, the sum is $2^{18} = 262144$. \diamond

8. If the conditional probability that the Braves will win tonight given that they won last night is 0.6, and if all of their games are independent, what is the probability that they'll win tonight's game?

Solution: 0.6 since all of games are independent. \diamond

9. Suppose that exactly half the people in a population are males. In addition, it is known that the probability that a male likes beer is 0.8, and the probability that a female likes beer is 0.4. Now, let's suppose that we observe a person enjoying beer. What is the probability that the person is a male?

Solution: B : likes beer, M : male, F : female. By Bayes Theorem,

$$\begin{aligned} P(M|B) &= \frac{P(B|M)P(M)}{P(B|M)P(M) + P(B|F)P(F)} \\ &= \frac{(0.8)(0.5)}{(0.8)(0.5) + (0.4)(0.5)} \\ &= \frac{2}{3}. \quad \diamond \end{aligned}$$

10. I've written 6 letters with 6 accompanying envelopes, but sadly, all have been scrambled. What's the probability that there will be no correct matches?

Solution: This is a straightforward application of the envelope problem.

$$\begin{aligned} 1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!}\right) &= \frac{1}{720}(360 - 120 + 30 - 6 + 1) \\ &= \frac{265}{720} \\ &= 0.368. \quad \diamond \end{aligned}$$

11. Suppose I toss 5 dice. What's the probability that I'll see exactly one pair and one three-of-a-kind? (An example of such an outcome is 3,5,5,3,5.)

Solution:

The # ways to toss 5 dice is 6^5 .

The # ways to pick the number for the 3-of-a-kind is 6.

The # ways to pick the pair (after having chosen the 3-of-a-kind) is 5.

The # ways to place the 3-of-a-kind (or place the pair) is $\binom{5}{3}$.

Thus, $\frac{6 \cdot 5 \cdot \binom{5}{3}}{6^5} = \frac{300}{6^5} = 0.0386$ \diamond

12. Pick 5 cards from a standard deck. What's the probability that I'll see a three-of-a-kind? (Example: $2\clubsuit, 2\diamond, 10\heartsuit, 2\spadesuit, K\spadesuit$.)

Solution:

The # ways to choose the rank of the 3-of-a-kind is 13.

The # ways to choose suits for the 3-of-a-kind is $\binom{4}{3}$.

The # ways to choose 2 junk ranks is $\binom{12}{2}$.

The # ways to choose suits from the 2 junk ranks is $\binom{4}{1}\binom{4}{1}$.

Thus, $\frac{13\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = 0.0211$ \diamond

13. What does the “d” in “c.d.f.” stand for?

Solution: Distribution \diamond

14. Suppose that X is a continuous random variable with p.d.f. $f(x) = x^2$ for $0 < x < c$. Find c .

Solution: $1 = \int_0^c x^2 dx = \frac{c^3}{3}$, which implies $c = 3^{\frac{1}{3}} = 1.442$ \diamond

15. If X has p.d.f. $f(x) = 3x^2$ for $0 < x < 1$, find $P(-0.5 \leq X \leq 0.5)$.

Solution: Since $0 < x < 1$, $P(-0.5 \leq X \leq 0.5) = P(0 \leq X \leq 0.5)$.

We have $P(0 \leq X \leq 0.5) = \int_0^{0.5} 3x^2 dx = x^3 \Big|_0^{0.5} = \frac{1}{8}$ \diamond

16. If X has p.d.f. $f(x) = 3x^2$ for $0 < x < 1$, find $E[X]$.

Solution: $E[X] = \int_0^1 3x^3 dx = \frac{3}{4} \quad \diamond$

17. If $E[X] = 3$ and $E[X^2] = 10$, find $\text{Var}(X)$.

Solution: $10 - 3^2 = 1. \quad \diamond$

18. If $X = 5$ with probability 0.3 and $X = 10$ with probability 0.7, find $E[\ln(X)]$.

Solution:

$$\begin{aligned} E[\ln(X)] &= \sum_x \ln(x)P(X = x) \\ &= (\ln 5)(0.3) + (\ln 10)(0.7) \\ &= 0.483 + 1.612 \\ &= 2.095 \quad \diamond \end{aligned}$$

19. Suppose X has c.d.f. $F(x) = 1 - e^{-\lambda x}$ for $x > 0$. Name the distribution of X .

Solution: Exponential(λ) \diamond

20. Suppose that X has a Bernoulli distribution with parameter $p = 0.7$. Find $E[2X + 1]$.

Solution: $E[2X + 1] = 2E[X] + 1 = 2.4 \quad \diamond$

21. Suppose the Braves play $n = 6$ games that we can regard as Bernoulli trials with success probability 0.7. Find the probability that they win exactly 4 out of these 6 games.

Solution: Here $n = 6$, $k = 4$ and We have

$$\begin{aligned} P(X = 4) &= \binom{n}{k} p^k q^{n-k} \\ &= \binom{6}{4} (0.7)^4 (0.3)^2 = 0.324 \quad \diamond \end{aligned}$$

22. Suppose X has p.m.f. $f(x) = (1 - p)^{x-1}p$, for $x = 1, 2, \dots$. Which of the following choices is the correct expression for $E[X]$?

- (a) p
- (b) $(1 - p)/p$
- (c) $1/p$
- (d) p^2

Solution: According to class notes, the answer is always (c) = $1/p$. \diamond

23. TRUE or FALSE? If X is discrete, then its p.m.f. is the derivative of its c.d.f.

Solution: FALSE \diamond

24. If $E[X] = 3$ and $\text{Var}(X) = 7$, find $\text{Var}(-3X - 2)$.

Solution: $\text{Var}(-3X - 2) = 9 \cdot \text{Var}(X) = 63$ \diamond

25. TRUE or FALSE? $\text{Var}(X^2) = E[X^4] - [\text{Var}(X) + (E[X])^2]^2$.

Solution: TRUE. We have $\text{Var}(Y) = E(Y^2) - (E(Y))^2$. Let $Y = X^2$, so that

$$\begin{aligned}\text{Var}(X^2) &= E(X^4) - (E(X^2))^2 \\ &= E(X^4) - [\text{Var}(X) + (E(X))^2]^2 \quad \diamond\end{aligned}$$

26. BONUS: In “I Ain’t Got You”, what kind of Maserati do the Yardbirds drive?

Solution: GT \diamond