NAME →

ISyE 3770 — Test 1b Solutions — Fall 2009

This test is 55 minutes long. You are allowed one cheat sheet.

Put your nice, simple answers here...

1. ______ 2. ______ 3. ______

4. ______ 5. ______ 6. ______

7. ______ 8. ______ 9. ______

10. ______ 11. ______ 12. ______

13. ______ 14. ______ 15. ______

16. ______ 17. ______ 18. ______

19. ______ 20. ______ 21. ______

22. ______ 23. ______ 24. ______

25. ______ 26. ______

1. What is any subset of an experiment's sample space called?

   Solution: Event. $\diamondsuit$

2. Suppose that $\mathsf{P}(\text{it rains today}) = 0.7$, $\mathsf{P}(\text{it rains tomorrow}) = 0.8$, and $\mathsf{P}(\text{it rains either day}) = 0.9$. What's the probability that it rains both days?

   Solution:

$$\begin{aligned}\mathsf{P}(\text{Today and Tomorrow}) &= \mathsf{P}(\text{Today}) + \mathsf{P}(\text{Tomorrow}) - \mathsf{P}(\text{Today or Tomorrow}) \\ &= 0.7 + 0.8 - 0.9 = 0.6. \quad \diamondsuit\end{aligned}$$

3. If $\mathsf{P}(A) = 0.3$, $\mathsf{P}(B) = 0.3$, and $\mathsf{P}(C) = 0.5$, and A, B, and C are independent, find the probability that *none* of A, B, and C occur.

   Solution:

$$\begin{aligned}\mathsf{P}(\bar{A} \cap \bar{B} \cap \bar{C}) &= \mathsf{P}(\bar{A})\mathsf{P}(\bar{B})\mathsf{P}(\bar{C}) \\ &= (0.7)(0.7)(0.5) \\ &= 0.245. \quad \diamondsuit\end{aligned}$$

4. TRUE or FALSE? $\mathsf{P}(A) = 1$ implies that A is the entire sample space.

   Solution: FALSE. For example, $S = [0,1]$, $A = [0,1] - \{0.5\}$. $\diamondsuit$

5. TRUE or FALSE? If (i) $\mathsf{P}(A \cap B) = \mathsf{P}(A)\mathsf{P}(B)$, (ii) $\mathsf{P}(A \cap C) = \mathsf{P}(A)\mathsf{P}(C)$, and (iii) $\mathsf{P}(B \cap C) = \mathsf{P}(B)\mathsf{P}(C)$, then A, B, and C are all independent of each other.

   Solution: FALSE. We also need $\mathsf{P}(A \cap B \cap C) = \mathsf{P}(A)\mathsf{P}(B)\mathsf{P}(C)$. $\diamondsuit$

6. Which is bigger, $P_{100,40}$ or $\binom{100}{60}$?

   Solution: $P_{100,40} = \frac{100!}{40!}$ and $\binom{100}{40} = \frac{100!}{40!60!}$. Thus, $P_{100,40}$ is bigger. $\diamondsuit$

7. Find $\sum_{i=0}^{19} \binom{20}{i}$.

   Solution: By a corollary of the Binomial Theorem, the sum is $2^{20} - 1$. $\diamondsuit$

8. If the conditional probability that the Braves will win tonight given that they won last night is 0.7, and if all of their games are independent, what is the probability that they'll win tonight's game?

   Solution: 0.7 since all of games are independent. $\diamondsuit$

9. Suppose that exactly half the people in a population are males. In addition, it is known that the probability that a male likes beer is 0.8, and the probability that a female likes beer is 0.6. Now, let's suppose that we observe a person enjoying beer. What is the probability that the person is a male?

   Solution: B : likes beer, M : male, F : female. By Bayes Theorem,

$$\begin{aligned} \mathsf{P}(M|B) &= \frac{\mathsf{P}(B|M)\mathsf{P}(M)}{\mathsf{P}(B|M)\mathsf{P}(M) + \mathsf{P}(B|F)\mathsf{P}(F)} \\ &= \frac{(0.8)(0.5)}{(0.8)(0.5) + (0.6)(0.5)} \\ &= \frac{4}{7}. \quad \diamondsuit \end{aligned}$$

10. I've written 5 letters with 5 accompanying envelopes, but sadly, all have been scrambled. What's the probability that there will be no correct matches?

    Solution: This is a straightforward application of the envelope problem.

$$1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!}\right) \quad = \quad 0.3667. \quad \diamondsuit$$

11. Suppose I toss 5 dice. What's the probability that I'll see exactly two pairs? (An example of such an outcome is 3,5,5,3,1.)

    Solution:
    The # ways to toss 5 dice is 6^5.

The # ways to pick the numbers for the two pairs is $\binom{6}{2}$.
The # ways to place the first pair is $\binom{5}{2}$.
The # ways to place the second pair is $\binom{3}{2}$.
The # ways to pick the last number is 4.
The # ways to place the last number is 1.
Thus, $\frac{15 \cdot 10 \cdot 3 \cdot 4 \cdot 1}{6^5} = 0.231 \quad \diamondsuit$

12. Pick 5 cards from a standard deck. What's the probability that I'll see a three-of-a-kind? (Example: $2\clubsuit, 2\diamondsuit, 10\heartsuit, 2\spadesuit, K\spadesuit$.)

    Solution:
    The # ways to choose the rank of the 3-of-a-kind is 13.
    The # ways to choose suits for the 3-of-a-kind is $\binom{4}{3}$.
    The # ways to choose 2 junk ranks is $\binom{12}{2}$.
    The # ways to choose suits from the 2 junk ranks is $\binom{4}{1}\binom{4}{1}$.
    Thus, $\frac{13\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = 0.0211 \quad \diamondsuit$

13. What does the "m" in "p.m.f." stand for?

    Solution: Mass $\diamondsuit$

14. Suppose that X is a continuous random variable with p.d.f. $f(x) = x^2$ for $c < x < 1$. Find c.

    Solution: $1 = \int_c^1 x^2 dx = \frac{1}{3} - \frac{c^3}{3}$, which implies $c = -2^{1/3} = -1.26 \quad \diamondsuit$

15. If X has p.d.f. $f(x) = (3/2)x^{1/2}$ for $0 < x < 1$, find $\mathsf{P}(-0.5 \le X \le 0.5)$.

    Solution: Since $0 < x < 1$, $\mathsf{P}(-0.5 \le X \le 0.5) = \mathsf{P}(0 \le X \le 0.5)$.
    We have $\mathsf{P}(0 \le X \le 0.5) = \int_0^{1/2}(3/2)x^{1/2}dx = x^{3/2}\big|_0^{1/2} = \frac{1}{8^{1/2}} \quad \diamondsuit$

16. If X has p.d.f. $f(x) = (3/2)x^{1/2}$ for $0 < x < 1$, find $\mathsf{E}[X]$.

Solution: $\mathsf{E}[X] = \int_0^1 (3/2)x^{3/2}dx = \frac{3}{5}$ $\diamondsuit$

17. If $\mathsf{E}[X] = 3$ and $\mathsf{E}[X^2] = 9$, find $\mathsf{Var}(X)$.

Solution: $9 - 3^2 = 0.$ $\diamondsuit$

18. If $X = 1$ with probability 0.3, $X = 2$ with probability 0.2, and X=3 with probability 0.5, find E[ℓn(X)].

Solution:

$$\begin{aligned} \mathsf{E}[\ell\text{n}(X)] &= \sum_x \ell\text{n}(x)\mathsf{P}(X = x) \\ &= (\ell\text{n}1)(0.3) + (\ell\text{n}2)(0.2) + (\ell\text{n}3)(0.5) \\ &= 0.688 \quad \diamondsuit \end{aligned}$$

19. Suppose X has p.d.f. $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for $-\infty < x < \infty$. Name the distribution of X.

Solution: Standard Normal $\diamondsuit$

20. Suppose that X has a Bernoulli distribution with parameter $p = 0.2$. Find $\mathsf{E}[-2X - 1]$.

Solution: $\mathsf{E}[-2X - 1] = -2\mathsf{E}[X] - 1 = -1.4$ $\diamondsuit$

21. Suppose the Braves play $n = 5$ games that we can regard as Bernoulli trials with success probability 0.7. Find the probability that they win exactly 4 out of these 5 games.

Solution: Here $n = 5$, $k = 4$ and We have

$$\begin{aligned} \mathsf{P}(X = 4) &= \binom{n}{k} p^k q^{n-k} \\ &= \binom{5}{4} (0.7)^4 (0.3)^1 = 0.36015 \quad \diamondsuit \end{aligned}$$

22. Suppose X has p.m.f. $f(x) = (1-p)^{x-1}p$, for $x = 1, 2, \ldots$. Which of the following choices is the correct expression for $\mathsf{E}[X]$?

    (a) p
    (b) $(1-p)/p$
    (c) $1/p$
    (d) p^2

    Solution: According to class notes, the answer is always $(c) = 1/p$. ♢

23. TRUE or FALSE? If X is discrete, then its p.m.f. is the derivative of its c.d.f.

    Solution: FALSE ♢

24. If $\mathsf{E}[X] = 3$ and $\mathsf{Var}(X) = 7$, find $\mathsf{Var}(-3X - 2)$.

    Solution: $\mathsf{Var}(-3X - 2) = 9 \cdot \mathsf{Var}(X) = 63$ ♢

25. TRUE or FALSE? $\mathsf{Var}(X^2) = \mathsf{E}[X^4] - [\mathsf{Var}(X) + (\mathsf{E}[X])^2]^2$.

    Solution: TRUE. We have $\mathsf{Var}(Y) = \mathsf{E}(Y^2) - (\mathsf{E}(Y))^2$. Let $Y = X^2$, so that

$$\begin{aligned} \mathsf{Var}(X^2) &= \mathsf{E}(X^4) - (\mathsf{E}(X^2))^2 \\ &= \mathsf{E}(X^4) - [\mathsf{Var}(X) + (\mathsf{E}(X))^2]^2 \quad \diamondsuit \end{aligned}$$

26. BONUS: In “I Ain’t Got You”, what kind of Maserati do the Yardbirds drive?

    Solution: GT ♢