You have 80 minutes. You are allowed 2 cheat sheets. Good luck!

1. TRUE or FALSE? An Arena resource can belong to multiple resource sets.

   Solution: TRUE.

2. TRUE or FALSE? In Arena, it is possible to have multiple arrivals occur at the same time.

   Solution: TRUE.

3. In Arena, what does an ALTER block alter?

   Solution: The number of servers on duty.

4. YES or NO? Consider the Arena call center example that we worked on in class. Were there ever any customers left stranded in the system at the end of the day?

   Solution: YES.

5. Which template do you need to access the Expression “spreadsheet” in an Arena program?
   (a) Basic Process.
   (b) Advanced Process.
   (c) Basic Transfer.
   (d) Advanced Transfer.
6. Consider the Arena expression \texttt{DISC(0.2,0.5,0.7,0.8,1.0,2)}. What is the probability that this expression will generate a value of 0.8?

**Solution:** 0.5

7. During office hours, students arrive to ask questions according to a Poisson process with rate \( \lambda \), and the professor answers any question in an exponential time with mean \( 1/\mu \). If the students balk when the professor is busy, what is the probability for an arriving student to find the professor available? (Assume steady state.)

**Solution:** This is an \( M/M/1/1 \) system. \[ P_0 = \frac{\mu}{\lambda + \mu}. \]

8. George runs his own one-man ice-cream shop. During this week he has noticed that people arrive according to a Poisson process at rate 6 per hour. Each customer needs to choose his/her ice-cream and pay for it. Choosing takes 3 minutes and paying takes 5 minutes (both tasks require George). Which standard queueing system best describes the ice-cream shop?

**Solution:** \( M/D/1 \).

9. Under the same set up as in Question 8, what is the expected number of people in the ice-cream shop (in line + in service)?

**Solution:** \[ L = \rho + \frac{\rho^2}{2(1 - \rho)} \] and \( \rho = \frac{\lambda}{\mu} = \frac{0.1}{0.125} = 0.8 \). So \( L = 2.4 \).

10. Under the same set up as in Question 8, what is the expected time in system (in line + in service)?

**Solution:** \[ w = \frac{L}{\lambda} = 24 \text{ minutes}. \]

11. Consider again the set up as in Question 8, but now the choice and service times are i.i.d. exponential with average 3 minutes. Which system best describes the
ice-cream shop?

**Solution:** $M/E_2/1$ □

12. **TRUE** or **FALSE**? The mid-square method provides independent PRNs.

**Solution:** FALSE. □

13. **TRUE** or **FALSE**? The LCG $X_i = (4X_{i−1} + 2) \mod 8$ is full-period.

**Solution:** FALSE. (There are never any odd numbers.) □

14. What is $(1 \text{ XOR } 1) \text{ XOR } 1$?

**Solution:** $(1 \text{ XOR } 1) \text{ XOR } 1 = 0 \text{ XOR } 1 = 1$. □

15. Suppose $r = 3$ and $q = 5$ for the Tausworthe generator, and $B_1 = 1$, $B_2 = 0$, $B_3 = 0$, $B_4 = 0$, $B_5 = 1$. Find $B_7$.

**Solution:** $B_7 = (B_{7−3} + B_{7−5}) \mod 2 = B_4 \text{ XOR } B_2 = 0 \text{ XOR } 0 = 0$. □

16. If the Tausworthe generator gives a 5-bit “word” 10011, what is the value of the resulting $U_i$?

**Solution:** $\frac{1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0}{2^5} = \frac{19}{32}$ □

17. Consider the PRN generator $X_i = 16807X_{i−1}\text{mod}(2^{31} − 1)$, with $X_0 = 456789$. Obtain $X_1$.

**Solution:** Using the usual algorithm, we find that

$$X_1 = (16807 \cdot 456789)\text{mod}(2^{31} − 1) = 1234801782$. □
Moreover, note that the resulting PRN is then
\[ U_1 = \frac{1234801782}{2^{31} - 1} = 0.5750. \]

18. Use your answer from Question 17 to generate a random variate whose distribution is exponential with rate 1.

**Solution:** Use the Inverse Transform Theorem: 
\[ -\ell n(1 - U_1) = -\ell n(1 - 0.5750) = 0.8557 \]
\[ \text{or } -\ell n(U_1) = -\ell n(0.5750) = 0.5534. \] ✔

19. Which error arises when you Reject \( H_0 \) even though it’s true?

**Solution:** Type I error. ✔

20. Suppose we sample 160 PRN’s and we wish to conduct a \( \chi^2 \) goodness-of-fit test at level \( \alpha = 0.10 \) of the hypothesis that the numbers are \( \text{Unif}(0,1) \). Using the following table, find the value of the g-o-f statistic, \( \chi^2_0 \).

<table>
<thead>
<tr>
<th>Interval</th>
<th>Observed frequency (( O_i ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, 0.2))</td>
<td>23</td>
</tr>
<tr>
<td>([0.2, 0.4))</td>
<td>31</td>
</tr>
<tr>
<td>([0.4, 0.6))</td>
<td>25</td>
</tr>
<tr>
<td>([0.6, 0.8))</td>
<td>37</td>
</tr>
<tr>
<td>([0.8, 1])</td>
<td>44</td>
</tr>
</tbody>
</table>

**Solution:** We have \( k = 5 \) equiprobable intervals, so that \( E_i = \frac{n}{k} = \frac{160}{5} = 32 \). Then
\[ \chi^2_0 = \sum_{i=1}^{5} \frac{(O_i - E_i)^2}{E_i} = 9.375. \] ✔

21. Referring to Question 20, what is the \( \chi^2 \) quantile you should use?

**Solution:** \( \chi^2_{n,k-1} = \chi^2_{0.1,4} = 7.78 \). ✔
22. Referring to Question 20, do you accept or reject $H_0$?

**Solution:** $\chi_0^2 = 9.375 > \chi_{\alpha,k-1}^2 = 7.78$ so we reject uniformity. □

23. We want to make a test of independence on the following numbers:

<table>
<thead>
<tr>
<th>0.55</th>
<th>0.46</th>
<th>0.31</th>
<th>0.41</th>
<th>0.51</th>
<th>0.76</th>
<th>0.42</th>
<th>0.62</th>
<th>0.73</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.81</td>
<td>0.25</td>
<td>0.58</td>
<td>0.05</td>
<td>0.86</td>
<td>0.36</td>
<td>0.22</td>
<td>0.07</td>
<td>0.03</td>
</tr>
</tbody>
</table>

How many runs “Up and Down” are there?

**Solution:** Let’s write + for “Up” and - for “Down”:

```
−− + − − + + − + + − + − − − − −
```

There are 11 runs. □

24. Referring to Question 23, obtain the test statistic $Z_0$ for a runs test “Up and Down”.

**Solution:**

\[
Z_0 = \frac{A - E[A]}{\sqrt{\text{Var}(A)}}
\]

where $A = 11$, $E[A] = \frac{2 \cdot 20 - 1}{3} = 13$ and

\[
\sqrt{\text{Var}(A)} = \sqrt{\frac{16 \cdot 20 - 29}{90}} = \sqrt{3.23} = 1.80.
\]

Then, $Z_0 = \frac{11 - 13}{1.80} = -1.112$. □

25. Referring to Question 24 and using $\alpha = 0.05$, do you accept or reject $H_0$ for the runs test “Up and Down”?

**Solution:** $z_{\alpha/2} = 1.96$. Then, $|Z_0| = 1.12 < z_{\alpha/2} = 1.96$ so we accept $H_0$. □

26. If $X \sim \text{Exp}(\lambda)$, find the distribution of the nasty function $1 - e^{-\lambda X}$.

**Solution:** Note that the c.d.f. of $X$ is $F(x) = 1 - e^{-\lambda x}$. Thus, the “nasty” function is actually $F(X)$, which is $U(0,1)$. □
27. Suppose $Z_1$ and $Z_2$ are i.i.d. $\text{Nor}(0,1)$ with c.d.f. $\Phi(z)$. What is the distribution of $\Phi(Z_1) + \Phi(Z_2)$?

**Solution:** $U(0,1) + U(0,1) \sim \text{TRIA}(0,1,2)$. □

28. Use the inverse transform method with $U = 0.975$ to generate a realization from a $\text{Nor}(-1,4)$ random variate.

**Solution:** $\Phi^{-1}(0.975) = 1.96$ gives a random variate from a standard normal. A variate from a $\text{Nor}(-1,4)$ can be obtained by the transformation

$$-1 + 2Z = -1 + 2\Phi^{-1}(0.975) = -1 + 2(1.96) = 2.92.$$ □

29. Suppose $U_1$ and $U_2$ are i.i.d. $\text{Unif}(0,1)$ with $U_1 = 0.1$ and $U_2 = 0.8$. Use Box–Muller to generate a single $\text{Nor}(-1,4)$ random variate.

**Solution:** $Z = \sqrt{-2\ln(U_1) \cos(2\pi U_2)} = 2.146(0.309) = 0.663$. Again we transform $Z$ to get $-1 + 2(0.663) = 0.326$ as a realization of $\text{Nor}(-1,4)$. (We also accepted $-5.082$, $-0.215$, and $0.081$ since you could use sin or a different order of the uniforms for your calculations.) □

30. Suppose $U_1, U_2, U_3$ are i.i.d. $\text{Unif}(0,1)$. What is the distribution of

$$X = -3 \ln(U_1^2(1 - U_2)^2U_3^2)?$$

**Solution:** $X = -3 \ln(U_1^2(1 - U_2)^2U_3^2) \sim -6 \ln(U_1) - 6 \ln(U_2) - 6 \ln(U_3)$, a sample of the sum of three exponentials with rate $1/6$. This distribution is $\text{Erlang}_{k=3}(1/6)$. □

31. Suppose $X$ and $Y$ are i.i.d. $\text{Exp}(\lambda)$ random variables. Show how to use inverse transform to generate a realization of $\max(X,Y)$ using a single $\text{Unif}(0,1)$ PRN.

**Solution:** To find the c.d.f. of the max, notice that

$$\Pr(\max(X,Y) < m) = \Pr(X < m, Y < m) = \Pr(X < m)\Pr(Y < m),$$
which follows because $X$ and $Y$ are independent. Then $\Pr(\max(X,Y) < m) = (1 - e^{-\lambda m})^2$. Using inverse transform, we set $U = (1 - e^{-\lambda X})^2$; so after a little algebra, we find that $X = -\frac{1}{\lambda} \ln(1 - \sqrt{U})$. □

32. Suppose $X$ has p.d.f.

$$f(x) = \begin{cases} \frac{e^x}{(e^2 - 1)} & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$ 

Use the inverse transform method with $U = 0.75$ to generate a realization of $X$.

**Solution:** The c.d.f. is

$$F(x) = \int_{-\infty}^{x} f(t) \, dt = \frac{1}{e^2 - 1} \int_{0}^{x} e^t \, dt = \frac{e^x - 1}{e^2 - 1}, \quad \text{for } x \in [0, 2].$$

Then, by inverse transform, $U = \frac{e^x - 1}{e^2 - 1}$ implies $X = \ell n(1 + (e^2 - 1)U) = 1.756$. □

33. Suppose $U \sim U(0, 1)$. Name the distribution of $\cot(2\pi U)$.

**Solution:** Note that

$$\cot(2\pi U) = \frac{\sqrt{-2\ell n(U_1)} \cos(2\pi U_2)}{\sqrt{-2\ell n(U_1)} \sin(2\pi U_2)} = \frac{\text{Nor}(0,1)}{\text{Nor}(0,1)}.$$ 

We know this to be the Cauchy distribution. □

34. Suppose we sample $n = 30$ PRN’s and obtain a sample mean of $\bar{X} = 0.53$. Use this result and our desert-island central limit theorem method to generate a single Nor(0,1) random variate.

**Solution:**

$$\frac{\sum_{i=1}^{30} U_i - n/2}{\sqrt{n/12}} = \frac{30(0.53) - 15}{\sqrt{30/12}} = 0.569.$$ □
35. Consider the i.i.d. Unif(0,1) numbers $U_1 = 0.94, U_2 = 0.82, U_3 = 0.08, U_4 = 0.23,$ and $U_5 = 0.01$. Use the acceptance-rejection method to generate a Poisson(3.8) random variable. (You may not need all of the uniforms to do this.)

**Solution:** Stop when the product of uniforms first falls below $e^{-3.8} = 0.0234$. $U_1U_2U_3 = 0.0617$ and $U_1U_2U_3U_4 = 0.0142$, so $N = 3$. □

36. (BONUS) Who did you recently vote for to be inducted into the Rock and Roll Hall of Fame?

**Solution:** The Zombies. □