Benchmarking Warehousing and Distribution
Operations Example Problem Solution
ISyE 6202—Fall 2003

Data has been collected on the following warehouses:

<table>
<thead>
<tr>
<th>Warehouse</th>
<th>Capital</th>
<th>Labor</th>
<th>Lines Shipped</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>20.00</td>
<td>2.0</td>
<td>2.00</td>
</tr>
<tr>
<td>B</td>
<td>3.00</td>
<td>1.0</td>
<td>0.50</td>
</tr>
<tr>
<td>C</td>
<td>30.00</td>
<td>2.5</td>
<td>7.50</td>
</tr>
<tr>
<td>D</td>
<td>0.75</td>
<td>1.0</td>
<td>0.25</td>
</tr>
<tr>
<td>E</td>
<td>20.00</td>
<td>4.0</td>
<td>2.50</td>
</tr>
</tbody>
</table>

Question 1. Determine the input efficiency of warehouse E. (Assume the technology exhibits constant returns-to-scale, free disposability of input and convexity, as discussed in class.)

Solution. The transformed data will be:

<table>
<thead>
<tr>
<th>Warehouse</th>
<th>Capital</th>
<th>Labor</th>
<th>Lines Shipped</th>
<th>Capital-Labor Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>25.0</td>
<td>2.5</td>
<td>2.5</td>
<td>10.0</td>
</tr>
<tr>
<td>B</td>
<td>15.0</td>
<td>5.0</td>
<td>2.5</td>
<td>3.0</td>
</tr>
<tr>
<td>C</td>
<td>10.0</td>
<td>7.5</td>
<td>2.5</td>
<td>1.33</td>
</tr>
<tr>
<td>D</td>
<td>7.5</td>
<td>10.0</td>
<td>2.5</td>
<td>0.75</td>
</tr>
<tr>
<td>E</td>
<td>20.0</td>
<td>4.0</td>
<td>2.5</td>
<td>5.00</td>
</tr>
</tbody>
</table>

The ray emanating from the origin that passes through the point $x_E = (4, 20)$ will have a slope of 5.00, and therefore will intersect the line segment joining warehouses A and B. Since the equation of the line joining warehouses A and B is $y = -4x + 35$, the intersection of it and the line $y = 5x$ occurs at the point $(35/9, 175/9)$. Since the ratio of $35/9$ to $4$ (or the ratio of $175/9$ to $20$) is $35/36$, the value of $\theta^*$ is thus $35/36 = 0.9722$.

The linear programming perspective. The corresponding linear program, as outlined in the paper, would be as follows:

$$\min \theta \tag{1}$$
subject to:

$$20\lambda_1 + 3\lambda_2 + 30\lambda_3 + 0.75\lambda_4 \leq 20\theta \tag{2}$$
$$2\lambda_1 + 1\lambda_2 + 22.5\lambda_3 + 1\lambda_4 \leq 4\theta \tag{3}$$
\begin{align*}
2\lambda_2 + 0.5\lambda_2 + 7.5\lambda_3 + 0.25\lambda_4 &\geq 2.5 \quad (4) \\
\lambda_i &\geq 0, \ i = 1, 2, 3, 4 \quad (5)
\end{align*}

Now we can transform this linear program by defining

\[
\mu_i := \left(\frac{y_E}{y_i}\right)\lambda_i
\]

for each \(i\). In the \(\mu\) variables, the linear program becomes:

\[
\min \theta 
\]

subject to:

\[
\begin{align*}
2[1.25\mu_1] + 3[5\mu_2] + 30[1/3\mu_3] + 0.75[10\mu_4] &\leq 20\theta \quad (7) \\
2[1.25\mu_1] + 1[5\mu_2] + 22.5[1/3\mu_3] + 1[10\mu_4] &\leq 4\theta \quad (8) \\
2[1.25\mu_1] + 0.5[5\mu_2] + 7.5[1/3\mu_3] + 0.25[10\mu_4] &\geq 2.5 \quad (9)
\end{align*}
\]

\[
\mu_i \geq 0, \ i = 1, 2, 3, 4 \quad (10)
\]

or, in reduced form,

\[
\min \theta 
\]

subject to:

\[
\begin{align*}
25\mu_1 + 15\mu_2 + 10\mu_3 + 7.5\mu_4 &\leq 20\theta \quad (12) \\
2.5\mu_1 + 5\mu_2 + 7.5\mu_3 + 10\mu_4 &\leq 4\theta \quad (13) \\
2.5\mu_1 + 2.5\mu_2 + 2.5\mu_3 + 2.5\mu_4 &\geq 2.5 \quad (14)
\end{align*}
\]

\[
\mu_i \geq 0, \ i = 1, 2, 3, 4 \quad (15)
\]

Note that the last equation (14) is obviously equivalent to

\[
\sum_i \mu_i \geq 1;
\]

however, since all of entries in the tableau are non-negative, it would never pay to have slack in this equation, and so the last equation may be equivalently represented as

\[
\sum_i \mu_i = 1. \quad (16)
\]

Thus, the linear program is looking for a convex combination of the transformed input vectors that will lie on the ray emanating from the origin that passes through the point \((4, 20)\) and which is closest to the origin. We
know an extreme point solution will have the property that at most 2 of the
\( \mu_i \)'s can enter the basis. (The variable \( \theta \) will be positive.) Moreover, the
\( \mu_i \)'s must be also be non-negative. Thus, it boils down to examining all line
segments joining a pair of (transformed) input vectors that intersect the ray,
and picking the one whose intersection is closest to the origin. Since we seek
the closest point to the origin, and since the slopes of the line segments join-
ing warehouses A to B, B to C and C to D are negative and increasing—the
Efficient Frontier, viewed as a function, is convex—it follows then from the
labor-capital ratios that only \( \mu_1 \) and \( \mu_2 \) will be positive, i.e., only the line
segment joining warehouse A to B needs to be examined here.

**Question 2.** Suppose the dataset is enlarged to include warehouse F, which
uses 6 units of labor and 27 units of capital to produce 4 units of lines
shipped.

a. What is the input efficiency of warehouse F?

**Solution.** The transformed input vector corresponding to warehouse
F is \((3.75, 16.875)\). (All input vectors to follow are assumed trans-
formed.) The slope connecting this point to warehouse A is -6.75,
which is steeper than the slope of the line segment joining warehouse
A to warehouse B. Thus, warehouse F lies below the line joining A
to B. Now, the equation of the line joining B to C is \( y = -2x + 25 \);
plugging in the value of \( x = 3.75 \), we get a \( y \) value of 17.5, which is
above 16.875. Thus, warehouse F lies below the line joining B to C.
The equation of the line joining C to D is \( y = -x + 17.5 \); plugging
in the value of \( x = 3.75 \), we get a \( y \) value of 13.75, which is below
16.875. Thus, warehouse F lies above the line joining C to D. From
this information, we may deduce that the boundary has been changed,
and is given symbolically by \( A \rightarrow F \rightarrow C \rightarrow D \). As a consequence, we
see that F lies on the boundary and so is input efficient, and we also
see that B is now inefficient.

b. Does the addition of warehouse F change the assessment of input effi-
ciency of warehouse E? If so, by how much?

**Solution.** Since the Efficient Frontier has been “lowered” so that the
line segment joining A to B lies in the interior, the input efficiency for
E has been affected, too. The capital-labor ratio of F is 4.5, and so
we need only examine the line segment joining A to F to assess the
efficiency of E. The equation of the line joining A to F is \( y = -1.5 +
22.5 \), which intersects the line \( y = 5x \) at the point \((3.4615, 17.3077)\).
This implies a radial input efficiency of \( 3.4615/4 = 17.3077/20 =
0.8654 \), which is indeed lower than its value 0.9722 before.