Real Options Analysis

1 Motivating Example

Example 1 A manufacturer is considering a new product line.

- Cost of the plant and equipment is estimated to be $500 million.
- The product line will be viable for 20 years.
- There are three possible, equally-likely market states:
  - *Excellent*. Expected after-tax cash flow given existing capacity will be $200 million/year.
  - *Average*. Expected after-tax cash flow will be $100 million/year.
  - *Failure*. Product line will generate no income.

Market outcome will be known at the end of the first year of operations.

- The company uses a 25% cost of capital for projects with this type of risk.

Should the company invest in the new product line?
Example 2 Upon further review, the CFO realizes it does have the following options:

- If the market outcome is excellent, an additional after-tax cash flow of $150 million/year can be generated if the facility is expanded. The cost of expansion will be $500 million.
- Company can liquidate its capital investment for 50% of its original value.

Additional information:

- The share price of a mutual fund that invests in similar companies is $50. The CFO’s assessment of how the fund’s share price will correlate with the product line outcome is provided in Table 1.

<table>
<thead>
<tr>
<th>Fund’s share price</th>
<th>Excellent (100% return)</th>
<th>Average (-20% return)</th>
<th>Failure (-40% return)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market outcome</td>
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<tr>
<td>Excellent</td>
<td>100</td>
<td>40</td>
<td>30</td>
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<tr>
<td>Average</td>
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<td>Failure</td>
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</table>

- The risk-free rate is 3%.

Should the company invest in the new product line with the embedded options?
2 The Twin Security

Recall that project value at a given state is the present value of expected future (after-tax) cash flow given all information known at that state. The discount rate is the appropriate risk-adjusted cost of capital.

Example 3 A company is deciding on whether to fund the following project.

- There are two possible market states: “good” or “bad.”
  - If the market state is good, the project value is estimated to be 1,500.
  - If the market state is bad, the project value is estimated to be 666.6.
- The objective probability of the good market state is 0.70.
- The project’s initial capital investment $I_0$ is 1,050.
- Risk-free rate is 5%.
- Market premium is 8%.
- There is a tradable non-dividend paying stock $S$ whose price process follows a binomial lattice with $U = 1.5$ and $D = U^{-1} = 0.6$. Current market price of $S$ is $S_0 = 100$.

Should the company fund this project?
3 The Marketed Asset Disclaimer (MAD)

Example 4 A company is deciding on whether to fund the following project.

- There are two possible market outcomes: “good” or “bad.”
  - If the market state is good, the project value is estimated to be 3,000.
  - If the market state is bad, the project value is estimated to be 500.

The market outcome will be known at the end of the first year.

- The objective probability of the good market state is 0.40.
- The project’s initial capital investment $I_0$ is 2,000.
- Risk-free rate is 2.5%.
- The appropriate risk-adjusted cost of capital is 25%.

*Should the company fund this project?*

**Analysis:**

1. **What is the project value (after investment)?**
   
   Project value is $\frac{0.4(3,000)+0.6(1,000)}{1.25} = 1,440$.

2. **What is the project’s NPV?**
   
   NPV is $1,440 - 2,000 = -560 < 0$. Do not invest.
**Example 5** Upon further review, the CFO realizes it does have the following options:

- If the market outcome is good, the company can expand operations with an additional capital expenditure of 1,000. Expansion will double the project’s value.
- If the market outcome is bad, the company can liquidate its capital investment for 60% of its original value.

*Should the company fund this “project with flexibility”?*

**Analysis:**

1. **What action should the company take given the market outcome?**
   - If the market outcome is good, the company could expand. If it did, the NPV is $2(3,000) - 1,000 = 5,000$.
     Since $5,000 > 3,000$, the value if no expansion takes place, the optimal action to take in this state is to expand.
   - If the market outcome is bad, liquidation generates a value of $(0.60)(2,000) = 1,200$.
     Since $1,200 > 1,000$, the value of continuing the project, the optimal action to take in this state is to liquidate.

2. **What are the values of the project with flexibility given the state-contingent optimal actions?**
   - 5,000 if the market outcome is good.
   - 1,200 if the market outcome is bad.

3. **What is the value of the project with flexibility (after investment)?**
   Project value is $\frac{0.4(5,000) + 0.6(1,200)}{1.25} = 2,176$.

4. **What is the project’s NPV?**
   The NPV is $2,176 - 2,000 = 176 > 0$.
   Yes, the company should fund the project with flexibility.
Refined Analysis:

1. **How does the riskiness of the values of the project with flexibility compare to the riskiness of the values of the project without flexibility?**
   
   (5,000, 1,200) vs. (3,000, 1,000). Riskier.

2. **Is the cost of capital that was used to value the project without flexibility appropriate to value the project with flexibility?**
   
   Certainly seems like it should be higher.

3. Assume the project without flexibility has a twin tradable security. This is called the **Marketed Asset Disclaimer** assumption. **How should this change the analysis?**
   
   - Project with the embedded options is a derivative security, $V$, whose underlying security is the project without the options, $P$.
     
     \[
     \begin{align*}
     P_0 &= 1,440 \quad \Rightarrow \quad P_1(g) &= 3,000 \\
     P_1(b) &= 1,000 \\
     V_1(g) &= 5,000 \\
     V_1(b) &= 1,200
     \end{align*}
     \]
     
     (1)
   
   - The payoffs of the project with the embedded options can be replicated by forming a portfolio of the project without flexibility, which we denote by $P$, and the risk-free asset.
   
   - The cost of the replicating portfolio less the initial capital investment is the **Real Options Analysis (ROA)** value of the project with flexibility (after investment).
   
   - The cost of replicating portfolio can be obtained via risk-neutral pricing or the state-price approach.

4. **State equations:**

   \[
   \begin{align*}
   3,000h + 1.025m &= 5,000 \\
   1,000h + 1.025m &= 1,200
   \end{align*}
   \]
   
   (2)

   (3)

   **Solution:**

   \[
   \begin{align*}
   h &= \frac{5,000 - 1,200}{3,000 - 1,000} = 1.9. \\
   m &= \frac{5,000 - 1.9(3,000)}{1.025} = -682.93.
   \end{align*}
   \]
   
   (4)

   (5)

5. **Cost of the replicating portfolio is**

   \[
   hP_0 + m = 1.9(1,440) + (-682.93)(1) = 2,053.07.
   \]

   **ROA value is** \(2,053.07 - 2,000 = 53.07\).
6. Risk-neutral pricing:

- The risk-neutral probability $q$ satisfies:
  \[
  1,440 = \frac{3,000q + 1,000(1-q)}{1.025} \implies q = 0.238.
  \] (7)

- Using the risk-neutral pricing, the cost of the replicating portfolio can be found via “discounted expectation” using the risk-neutral probability for the expectation and the risk-free rate for the discount rate:
  \[
  \text{Cost of the replicating portfolio} = E_q[V_1] \left( \frac{1 + r_f}{1 + r_f} \right)
  \]
  \[
  = \frac{5,000q + 1,200(1-q)}{1.025}
  \]
  \[
  = \frac{5,000(0.238) + 1,200(0.762)}{1.025}
  \]
  \[
  = 2,053.07 \text{ as before.}
  \] (10)

7. State-price approach:

- Let $P_{3000}$ denote a put option with strike price = 3,000 whose market price is $P_{3000}^0$. The payoff vector for $P_{3000}$ is $(0, 2,000)$.

- A portfolio of $(1P, 1P_{3000})$ will payoff 3,000 risk-free.
  Since $P_0 = 1,440$, it follows that $P_{3000}^0 = 2,926.83 - 1,440 = 1,486.83$.
  Since $1,486.83 = y(b) * 2,000$, it follows that $y(b) = 1,486.83/2,000 = 0.743415$.
  Since $y(g) + y(b) = 1/(1 + r_f) = 1/1.025$, it follows that $y(g) = 0.232195$.

- Let $C_{1000}$ denote a call option with strike price = 1,000 whose market price is $C_{1000}^0$. The payoff vector for $C_{1000}$ is $(2,000, 0)$.
  A portfolio of $(1P, -1C_{1000})$ will payoff 1,000 risk-free.
  Since $P_0 = 1,440$, it follows that $C_{1000}^0 = 1,440 - 975.61 = 464.39$.
  Since $464.39 = y(g) * 2,000$, it follows that $y(g) = 464.39/2,000 = 0.232195$.

- Cost of the project with the embedded options is
  \[
  V(g)y(g) + V(b)y(b) = (5,000)(0.232195) + (1,200)(0.743415)
  \]
  \[
  = 2,053.57 \text{ as before.}
  \] (12)

- It works because $q = y(g)(1 + r_f)$ and $1 - q = y(b) * (1 + r_f)$, and so
  \[
  V(g)y(g) + V(b)y(b) = \frac{1}{1 + r_f} [(1 + r_f)V(g)y(g) + (1 + r_f)V(b)y(b)]
  \]
  \[
  = \frac{1}{1 + r_f} E_q[V_1].
  \] (14)
8. What is the appropriate risk-adjusted cost of capital?

- Expected future payoffs of the project with flexibility is 
  \[ E_p[V_1] = 0.4(5,000) + 0.6(1,200) = 2,720. \]
- Value of the project with flexibility is \( V_0 = 2,053.07 \).
  Therefore, the appropriate cost of capital is 
  \[ \text{Risk-adjusted cost of capital} = \frac{E_p[V_1]}{V_0} = \frac{2,720}{2,053.07} - 1 = 32.5\%. \] \( \text{(16)} \)

9. What are the weights in the replicating portfolio?

- Replicating portfolio is \((1.9P, -682.93M)\).
- Its cost is, of course, \( 2,053.07 \).
- Cost of acquiring the \( P \)'s is \( 1.9(1,440) = 2,736 \).
- \( w_P = \frac{2,736}{2,053.07} = 1.3326 \) and \( w_M = -0.3326 \).

10. What is the expected return of the replicating portfolio using the objective probabilities?

- Expected return on \( P \) using the objective probabilities is 25%:
  \[ E[r_P] = \frac{E_p[P_1]}{P_0} = \frac{0.4(3,000) + 0.6(1,000)}{1,440} - 1 = 25\%. \] \( \text{(17)} \)
  This is how we found the value 1,440.
- Expected return on the risk-free asset is 2.5%.
- Expected return of the replicating portfolio \( RP \) is
  \[ E[r_{RP}] = (1.3326)[25\%] + (-0.3326)[2.5\%] = 32.5\%. \] \( \text{(18)} \)

11. How does the expected return on the replicating portfolio using the objective probabilities compare to the appropriate risk-adjusted cost of capital for the project with flexibility?

The replicating portfolio IS equivalent to the project with flexibility, and so its expected return calculated as in (18) will always equal the appropriate risk-adjusted cost of capital for the project with flexibility computed in (16).
4 Another Single-Period Example

Example 6 A company is deciding on whether to fund the following project.

- There are two possible market outcomes: “good” or “bad.”
  - If the market state is good, the project value is estimated to be 8,000.
  - If the market state is bad, the project value is estimated to be 3,000.

The market outcome will be known at the end of the first year.

- The objective probability of the good market state is 0.30.
- The project’s initial capital investment $I_0$ is 4,000.
- Risk-free rate is 5%.
- The appropriate risk-adjusted cost of capital is 25%.

Should the company fund this project?

Analysis:

1. What is the project value (after investment)?
2. What is the project’s NPV?
Example 7 Upon further review, the CFO realizes it does have the following options:

- If the market outcome is good, the company can expand operations with an additional capital expenditure of 1,000. Expansion will increase the project’s value by 50%.
- If the market outcome is bad, the company can liquidate its capital investment for 50% of its original value.

Should the company fund this project with the embedded options?

Analysis:

1. What are the values of the project with flexibility given the state-contingent optimal action?
2. What are the values of the project with flexibility (after investment) and the project’s NPV using traditional analysis?
3. Is the original cost of capital appropriate given the riskiness of the values for the project with flexibility?
4. How does adopting the Marketed Asset Disclaimer assumption change the analysis?
5. What are the state equations and what is the replicating portfolio?
6. What is the cost of the replicating portfolio?
7. What is the ROA value of the project with flexibility?
8. What is the value (after investment) of the project with flexibility using risk-neutral pricing?
9. What are the state prices (using the Law of One Price approach)?
10. What is the value (after investment) of the project with flexibility using the state-price approach?
11. What is the appropriate risk-adjusted cost of capital?
12. What are the weights in the replicating portfolio?
13. What is the expected return of the replicating portfolio using the objective probabilities?
14. How does the expected return of the replicating portfolio using the objective probabilities compare to the appropriate risk-adjusted cost of capital for the project with the embedded options?
5 A Delay Option (Value of Learning)

Example 8 A real estate developer owns a parcel of undeveloped land. Here is the relevant information:

- Price today for a one-unit condominium $P_0 = $100 thousand.
- Price next year for a one-unit condominium:
  - $150 thousand, if the market moves favorably, or
  - $90 thousand, if the market moves unfavorably.
- Construction cost today and next year:
  - $80 thousand per unit for a 6-unit building, or
  - $90 thousand per unit for a 12-unit building.
- Construction is instantaneous.
- Risk-free rate is 5%.

What is the value of this real estate development project?
6  Multi-Year Delay Options

Example 9  A real estate developer has the opportunity to develop a parcel of land over the next three years. Here is the relevant information:

- Project today is valued at $216 million.
- Development cost today is $200 million. These costs increase 5% per year.
- It costs $10 million, payable at the beginning of each year, to maintain the land if not developed. This cost is not incurred if the land is developed.
- At any time the land can be sold for $20 million.
- Project’s value follows a binomial lattice with \( U = 1.5 \) and \( D = 2/3 \).
- Objective probability of the market going up is 0.70.
- Risk-free rate is 2.5%.

What is the value of this real estate development project? When should it be developed?
Analysis:

At each project state there are three possible actions: (1) develop now – assumes that development has not yet occurred; (2) do not develop now, i.e., wait, or (3) abandon. The optimal action is the one that generates the largest value.

Table 2: Project without flexibility value event tree.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</thead>
<tbody>
<tr>
<td>Project values</td>
<td>216</td>
<td>324</td>
<td>486</td>
<td>729</td>
</tr>
<tr>
<td></td>
<td>144</td>
<td>216</td>
<td>324</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>96</td>
<td>144</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Investment</td>
<td>200</td>
<td>210</td>
<td>221</td>
<td>232</td>
</tr>
<tr>
<td>Maintenance</td>
<td>10</td>
<td>10</td>
<td>10</td>
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Table 3: Project with flexibility value event tree.

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<th>3</th>
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</thead>
<tbody>
<tr>
<td>Project values</td>
<td>53</td>
<td>123</td>
<td>265</td>
<td>497</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>40</td>
<td>92</td>
<td></td>
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<tr>
<td></td>
<td>20</td>
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<tr>
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<td></td>
<td></td>
<td>20</td>
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</tbody>
</table>

Calculations:

1. \(1 = \frac{1.5q + 0.5(1-q)}{1.025} \implies q = 0.43\).
2. \(V_2(\text{uu}) = \max\{486 - 221 = 265, \frac{0.43(497) + 0.57(92)}{1.025} - 10 = 249.66, 20\} = 265\).
3. \(V_2(\text{ud}) = \max\{216 - 221 = -5, \frac{0.43(92) + 0.57(20)}{1.025} - 10 = 39.72, 20\} = 39.72\).
4. \(V_2(\text{dd}) = \max\{96 - 221 = -125, \frac{0.43(20) + 0.57(20)}{1.025} - 10 = 9.51, 20\} = 20\).
5. \(V_1(\text{u}) = \max\{324 - 210 = 114, \frac{0.43(265) + 0.57(39.72)}{1.025} - 10 = 123.26, 20\} = 123.26\).
6. \(V_1(\text{d}) = \max\{144 - 210 = -66, \frac{0.43(39.72) + 0.57(20)}{1.025} - 10 = 17.78, 20\} = 20\).
7. \(V_0 = \max\{216 - 200 = 16, \frac{0.43(123.26) + 0.57(20)}{1.025} - 10 = 52.83, 20\} = 52.83\).
Example 10 Consider this investment in a tree farm. Here is the relevant information:

- Initial value of the trees, if harvested immediately, is $10 million.
- Price of lumber follows a binomial lattice with $U = 1.20$ and $D = 0.90$.
- Costs $4 million each year to maintain the forest land.
  Maintenance costs are payable at the beginning of the year.
- If not harvested, the number of trees will grow by 50% the first year, 30% the second year, and 15% in the third year.
- The trees may be harvested at the beginning of each year over the next three years.
  If harvested, there is no maintenance cost.
- Risk-free rate is 5%.

What is the value of this investment, and what is the optimal policy?

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<tbody>
<tr>
<td>$\max{15.739 - 4, 16} = 11.739$</td>
<td>$\max{23.501 - 4, 18.8} = 19.501$</td>
<td>$\max{32.292 - 4, 28.080} = 28.292$</td>
<td>$\max{38.750, 38.750} = 38.750$</td>
</tr>
</tbody>
</table>

Table 4: Project with flexibility value event tree.

Verification:

\[
\text{Full Delay Value} = \frac{E^Q[P_T]}{(1 + r_f)^T} - \text{PV of maintenance cost cash flows} \quad (19)
\]

\[
= \frac{E^Q[S_T G_T]}{(1 + r_f)^T} - \left(4 + \frac{4}{1.05} + \frac{4}{1.05^2}\right) \quad (20)
\]

\[
= G_T \frac{E^Q[S_T]}{(1 + r_f)^T} - 11.4376 \quad (21)
\]

\[
= G_T P_0 - 11.4376 = 10.988 \quad (22)
\]

\[
\text{Expected ROA Savings} = \frac{0.5(21.06 - 20.219) + 0.25(15.795 - 14.164)}{1.05^2} = 0.751 \quad (23)
\]

\[
\text{ROA Value} = \text{Fully Delay Value} + \text{Expected Savings} = 11.739. \quad (24)
\]
7 Growth Options

Example 11 A company is considering replacing each of its existing 6 machines with newer machines embedding the latest technology. A company analyst was tasked with making a recommendation. She prepared this data (in units of 000’s).

- Cost of the new machine is 1000.
- New machine has an economic life of 5 years.
- Incremental cash flows for replacing one old machine with one newer machine include the benefits for better productivity, lower maintenance costs, lower energy costs, in addition to the tax shield on incremental depreciation, the sale of the old machine and salvage value of the new machine at the end of its 5-year horizon. These cash flows are not known with certainty. The analyst has arrived at these expected values:

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<th>4</th>
<th>5</th>
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</thead>
<tbody>
<tr>
<td>Expected cash flow</td>
<td>-1000</td>
<td>220</td>
<td>300</td>
<td>400</td>
<td>200</td>
<td>150</td>
</tr>
</tbody>
</table>

- Estimated value of risk-adjusted cost of capital for this project is 12%.
- The new machines will not be replaced during their useful life.

Questions:

- Should the company replace the machines?
- Is there an embedded option here? Assumptions?

\[ P_0 = 932.52, \ r_f = 6\%, \ \sigma = 0.40 \implies C(T, K) = C(1, 100) = 143.98. \]
A project with an embedded growth option involves a pioneer venture stage to first prove the new technology in order to establish its viability for future commercial development of spin-off products. The pioneer venture involves high initial costs and insufficient projected cash inflows. If the technology proves successful, subsequent product commercialization can be many times the size of the pioneer venture.

**Example 12** Consider this data for a particular biotech project.

- **Pioneer venture.**
  - Requires an initial investment outlay of \( I_0 = \$100 \text{ million} \).
  - Expected cash flows are \( C_1 = \$54 \text{ million} \) and \( C_2 = \$36 \text{ million} \).

- **Follow-on commercialization stage.**
  - Will become available at the end of the pioneer stage, i.e., year 2.
  - Requires an initial investment outlay of \( I_2 = \$1,000 \text{ million} \).
  - Expected cash flows are \( C_3 = \$540 \text{ million} \) and \( C_4 = \$360 \text{ million} \).

- Cost of capital for projects with this type of risk is 20%.

- Risk-free rate is 2.5%.

*Should the company invest in this project?*
8 An Installment Option (Staged Investment)

Example 13 A company has an opportunity to invest in this project.

- It will take two years of capital investment before the project realizes any cash flow.
- The present value of the expected cash flow after year two at the appropriate risk-adjusted cost of capital is $100 million. This is the project’s current value.
- Capital investment schedule is *staged* over time, as follows:
  - $60 million at time 0,
  - $15 million at time 1,
  - $30 million at time 2.

Capital investments must be made to continue the project for the upcoming year.

- Company has the option to “default” on its scheduled investment at which point the project is terminated.
- Project value (twin security) has an annual volatility of 80%.
- The risk-free rate is 5%.

**Analysis – Full Commitment:**

Suppose the company must commit to the investment schedule should it undertake the project. In this case, the investment schedule cash flows should be discounted at 5%, which yields a present value of $101.5 million. The NPV is -$1.5 million, and so this project should not be undertaken.
Analysis – Option to Default:

Calculations for Tables 5 and 6:

1. \( U = e^{0.80} = 2.2255 \) and \( D = 1/U = 0.4493 \).

2. \( q = \frac{1.05 - 0.4493}{2.2255 - 0.4493} = 0.3382 \).

3. \( V_2(\text{uu}) = \max\{495.30 - 30, 0\} = 465.3 \)

4. \( V_2(\text{ud}) = \max\{100.00 - 30, 0\} = 70.00 \)

5. \( V_2(\text{dd}) = \max\{20.19 - 30, 0\} = 0.00 \)

6. \( V_1(\text{u}) = \max\left\{ \frac{0.3382(465.3) + 0.6618(70)}{1.05} - 15, 0 \right\} = 178.99 \)

7. \( V_1(\text{d}) = \max\left\{ \frac{0.3382(70) + 0.6618(0)}{1.05} - 15, 0 \right\} = 7.55 \)

8. \( V_0 = \max\left\{ \frac{0.3382(178.99) + 0.6618(7.55)}{1.05} - 60, 0 \right\} = 2.41 \)

9. Option value = 2.41 - (-1.5) = 3.91 = \( \frac{(0.6618)^2(30 - 20.19)}{1.05^2} \).

Table 5: Project without flexibility value event tree.

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<tbody>
<tr>
<td></td>
<td>100</td>
<td>222.55</td>
<td>495.30</td>
</tr>
<tr>
<td></td>
<td>44.93</td>
<td>100.00</td>
<td>20.19</td>
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Table 6: Project with flexibility value event tree.

<table>
<thead>
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<th>0</th>
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<tbody>
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<td></td>
<td>2.41</td>
<td>178.99</td>
<td>465.30</td>
</tr>
<tr>
<td></td>
<td>7.55</td>
<td>70.00</td>
<td>*0.00</td>
</tr>
</tbody>
</table>
Example 14 Suppose the project investment schedule of the previous example could be changed to:

- $20 million at time 0,
- $30 million at time 1,
- $70 million at time 2.

The present value of this investment schedule cash flow @ 5% is $112.06 million. “Back-loading” the investment schedule increases the cost (if fully implemented).

How does it affect the project value?

Analysis – Option to Default:

Calculations for Table 7:

1. \( V_2(uu) = \max\{495.30 - 70, 0\} = 425.3 \)
2. \( V_2(ud) = \max\{100.00 - 70, 0\} = 70.0 \)
3. \( V_2(dd) = \max\{20.19 - 70, 0\} = 0.0 \)
4. \( V_1(u) = \max\{\frac{0.3382(425.30) + 0.6618(30)}{1.05} - 30, 0\} = 125.9. \)
5. \( V_1(d) = \max\{\frac{0.3382(30) + 0.6618(0)}{1.05} - 30, 0\} = 0.0 \)
6. \( V_0 = \max\{\frac{0.3382(125.90) + 0.6618(0)}{1.05} - 20, 0\} = 20.55. \)
7. Option value = 20.55 - (-12.06) = 32.61 = \frac{0.6618(30+70/(1.05))-44.93)}{1.05}.

Table 7: Project with flexibility value event tree.

<table>
<thead>
<tr>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
<th>( t = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.55</td>
<td>125.90</td>
<td>425.30</td>
</tr>
<tr>
<td>0</td>
<td>30</td>
<td>0</td>
</tr>
</tbody>
</table>

19
9 A Case Study

Detailed project cash flow projections are shown in Table 8. Company uses a 13% cost of capital for the first six years, 12% after year 6. Terminal value assumes a 4% growth in free cash flow after year 6. All rates are continuously compounded.

Table 8: Expected cash flows for project without flexibility.

<table>
<thead>
<tr>
<th>Assumptions:</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>30</td>
<td>27.67</td>
<td>25.51</td>
<td>23.53</td>
<td>21.7</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quantity</td>
<td>200</td>
<td>230</td>
<td>264</td>
<td>303</td>
<td>349</td>
<td>400</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unit Cost</td>
<td>9</td>
<td>8.6</td>
<td>8.1</td>
<td>7.7</td>
<td>7.4</td>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue</td>
<td>6000</td>
<td>6364</td>
<td>6735</td>
<td>7130</td>
<td>7573</td>
<td>8024</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cost</td>
<td>1800</td>
<td>1978</td>
<td>2138</td>
<td>2333</td>
<td>2583</td>
<td>2807</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gross Income</td>
<td>4200</td>
<td>4386</td>
<td>4597</td>
<td>4797</td>
<td>4990</td>
<td>5217</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rent</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SGA</td>
<td>600</td>
<td>637</td>
<td>676</td>
<td>718</td>
<td>763</td>
<td>810</td>
<td></td>
<td></td>
</tr>
<tr>
<td>EBITDA</td>
<td>3400</td>
<td>3549</td>
<td>3721</td>
<td>3879</td>
<td>4027</td>
<td>4207</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Depreciation</td>
<td>3500</td>
<td>3500</td>
<td>3500</td>
<td>3500</td>
<td>3500</td>
<td>3500</td>
<td></td>
<td></td>
</tr>
<tr>
<td>EBIT</td>
<td>-100</td>
<td>49</td>
<td>221</td>
<td>379</td>
<td>527</td>
<td>707</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Taxes</td>
<td>0</td>
<td>20</td>
<td>88</td>
<td>152</td>
<td>211</td>
<td>283</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Net Income</td>
<td>(100)</td>
<td>29</td>
<td>133</td>
<td>227</td>
<td>316</td>
<td>424</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adjustments:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Add depreciation</td>
<td>3500</td>
<td>3500</td>
<td>3500</td>
<td>3500</td>
<td>3500</td>
<td>3500</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Less investment</td>
<td>(35000)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Free Cash Flow</td>
<td>(35000)</td>
<td>3400</td>
<td>3529</td>
<td>3633</td>
<td>3727</td>
<td>3816</td>
<td>3924</td>
<td>51012</td>
</tr>
<tr>
<td>Present Value</td>
<td>34707</td>
<td>36125</td>
<td>37611</td>
<td>39199</td>
<td>40914</td>
<td>42778</td>
<td>44793</td>
<td></td>
</tr>
<tr>
<td>Net Present Value</td>
<td>(293)</td>
<td>39525</td>
<td>41140</td>
<td>42832</td>
<td>44641</td>
<td>46594</td>
<td>48717</td>
<td></td>
</tr>
<tr>
<td>Free Cash Flow %</td>
<td>0.086</td>
<td>0.0858</td>
<td>0.0848</td>
<td>0.0835</td>
<td>0.0819</td>
<td>0.0805</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Options:

- **Expansion.** Increases future cash flow by 30% at a fixed cost of 10,500.
- **Abandonment.** Liquidate initial capital investment for 15,000.

---

1Adapted from Chapter 11 in *Real Options: A Practitioner’s Guide* by Copeland and Antikarov, 2001.
Analysis:

- Simulated volatility of project without flexibility is 36%.

- $U = e^{0.36} = 1.43$. $D = e^{-0.36} = 0.6977$. $q = \frac{e^{0.025} e^{-0.36}}{e^{0.36} e^{-0.36}} = 0.4454$.

- Expansion value = $[1.3(V_t - 0.085V_t) + 0.085V_t] - 10,500 = 1.2745V_t - 10,500$.

Table 9: **ROA value of project with flexibility.**

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>34,707</td>
<td>49,747</td>
<td>65,243</td>
<td>85,566</td>
<td>112,220</td>
<td>147,176</td>
<td>193,021</td>
</tr>
<tr>
<td>0</td>
<td>4.228</td>
<td>5.547</td>
<td>7.273</td>
<td>9.539</td>
<td>12,510</td>
<td>16,407</td>
</tr>
<tr>
<td>39,308</td>
<td>54,750</td>
<td>72,652</td>
<td>98,544</td>
<td>132,524</td>
<td>177,076</td>
<td>235,505</td>
</tr>
<tr>
<td>24,214</td>
<td>31,757</td>
<td>41,649</td>
<td>54,623</td>
<td>71,638</td>
<td>93,953</td>
<td></td>
</tr>
<tr>
<td>2,058</td>
<td>2,699</td>
<td>3,540</td>
<td>4,643</td>
<td>6,089</td>
<td>7,986</td>
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</tr>
<tr>
<td>28,703</td>
<td>35,057</td>
<td>41,616</td>
<td>59,117</td>
<td>80,803</td>
<td>109,243</td>
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</tr>
<tr>
<td>15,458</td>
<td>20,273</td>
<td>26,588</td>
<td>34,870</td>
<td>45,732</td>
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<tr>
<td>1,314</td>
<td>1,723</td>
<td>2,260</td>
<td>2,964</td>
<td>3,887</td>
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<td>21,106</td>
<td>23,991</td>
<td>28,464</td>
<td>35,762</td>
<td>47,785</td>
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<tr>
<td>9,868</td>
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<td></td>
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<tr>
<td>839</td>
<td>1,100</td>
<td>1,443</td>
<td>1,892</td>
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</tr>
<tr>
<td>17,324</td>
<td>18,309</td>
<td>19,724</td>
<td>22,260</td>
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<tr>
<td>6,299</td>
<td>8,261</td>
<td>10,836</td>
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<tr>
<td>535</td>
<td>702</td>
<td>921</td>
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<td></td>
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<td>15,773</td>
<td>15,974</td>
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<td>4,021</td>
<td>5,274</td>
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<td>342</td>
<td>448</td>
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<tr>
<td>15,342</td>
<td>15,448</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>2,567</td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>218</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15,218</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
10 Estimating Project Volatility

**Background:** For a stochastic process \( \{Y_\tau, \tau \geq 0\} \) that follows Geometric Brownian Motion:

\[
Y_t = Y_0 e^{X_t}, \text{ where } X_t \sim N(\nu t, \sigma^2 t).
\]  
\[
\tilde{Y}_t := E[Y_t] = Y_0 e^{\mu t}, \text{ where } \mu = \nu + 0.5\sigma^2.
\]

\[
P\{Y_t \leq y_t\} = P\{Y_0 e^{X_t} \leq y_t\}
\]
\[
= P\{X_t \leq \ln(y_t/Y_0)\}
\]
\[
= \Phi\left(\frac{\ln(y_t/Y_0) - \nu t}{\sigma \sqrt{t}}\right).
\]

**Confidence intervals:** For each \( \alpha \in (0, 1) \), let \( Y_t^{L_\alpha} \) denote the lower limit of a 100\( \alpha \)% confidence interval so that

\[
P\{Y_t \leq Y_t^{L_\alpha}\} = (1 - \alpha)/2,
\]
and define \( k_\alpha := \Phi^{-1}((1 - \alpha)/2) \) so that

\[
k_\alpha := \frac{\ln(Y_t^{\alpha}/Y_0) - \nu t}{\sigma \sqrt{t}}.
\]

For example, if \( \alpha = 0.95 \), then \( k_\alpha = 1.96 \).

**Deriving \( \sigma \) from \( \tilde{Y}_t \) and \( Y_t^{L_\alpha} \):**

\[
k_\alpha = \frac{\ln(Y_t^{\alpha}/Y_0) - \nu t}{\sigma \sqrt{t}}
\]
\[
= \frac{\ln(Y_t^{L_\alpha}/Y_0) - (\mu - 0.5\sigma^2)t}{\sigma \sqrt{t}}
\]
\[
= \frac{(\ln(Y_t^{L_\alpha}/Y_0) - \mu t) + 0.5\sigma^2 t}{\sigma \sqrt{t}}
\]
\[
= \frac{(\ln(Y_t^{L_\alpha}/Y_0) - \ln(\tilde{Y}_t/Y_0)) + 0.5\sigma^2 t}{\sigma \sqrt{t}}
\]
\[
= \frac{\ln(Y_t^{L_\alpha}/\tilde{Y}_t)}{\sigma \sqrt{t}} + \sigma \sqrt{t}
\]
\[
\implies \sigma \sqrt{t} = \sqrt{k_\alpha^2 + 2\ln(\tilde{Y}_t/Y_t^{L_\alpha})} - k_\alpha.
\]
**Example 15** For the case study assume that:

- The price process follows Geometric Brownian Motion that begins at time \( t = 1 \).
- A 95% lower confidence interval value for the price at time 6 is 15.

*What are the values for the parameters \( \mu_P, \sigma_P \) and \( \nu_P \)?*

**Analysis:**

- From the case study cash flow estimates, \( P_1 = 30 \) and \( \bar{P}_6 = 20 \).
  Since \( \bar{P}_6 = P_1 e^{5\mu} \), it follows that \( \mu_P = \frac{\ln(20/30)}{5} = -0.081 \).
- Applying (38),
  \[
  \sigma_P = \frac{\sqrt{(1.96)^2 + 2\ln(20/15)} - 1.96}{\sqrt{5}} = 0.0634 \text{ or } 6.34\%.
  \]
  (39)
- \( \nu_P = \mu_P - 0.5\sigma_P^2 = -0.081 - 0.5(0.0634)^2 = -0.083 \).
- The price increment satisfies \( \ln P_{t+1} - \ln P_t \sim N(-0.083, (0.0634)^2) \).
  So, to generate a sample path of prices, start with \( P_1 = 30 \), simulate independent standard variates \( z_t \), then calculate next year’s price \( P_{t+1} = P_t e^{-0.083 + 0.0634z_t} \).
Example 16 For the case study assume that:

- The quantity process follows Geometric Brownian Motion that begins at time $t = 1$, and this process is independent of the price process.
- A 95% lower confidence interval value for the price at time 6 is 190.

What are the values for the parameters $\mu_Q$, $\sigma_Q$ and $\nu_Q$?

How does one simulate a quantity path?

Answers:

- $\mu_Q = 0.1386$.
- $\sigma_Q = 0.156$.
- $\nu_Q = 0.1264$.
- $\ln Q_{t+1} - \ln Q_t \sim N(0.1264, (0.156)^2)$. 
11 Advanced Topic

Example 17 A company has the option to acquire a ten-year lease to extract gold from a gold mine. The lease should be acquired if the cost is less than the value of the lease.

- Mine capacity is $I_0$ ounces.
- When the begin-of-year mine inventory is $I$, then the cost to mine $z$ ounces for the upcoming year is $Cz^2/I$.
- Production occurs instantaneously. Profits accrue at the beginning of each year.
- Price of gold follows a binomial lattice.
- Risk-free rate is $r_f$.

What is the value of the lease?
What is the optimal production policy?