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Some Properties of Convex Hulls of Integer Points Contained in General Convex Sets

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Abstract In this paper, we study properties of general closed convex sets that determine the closedness and polyhedrality of the convex hull of integer points contained in it. We first present necessary and sufficient conditions for the convex hull of integer points contained in a general convex set to be closed. This leads to useful results for special classes of convex sets such as pointed cones, strictly convex sets, and sets containing integer points in their interior. We then present a sufficient condition for the convex hull of integer points in general convex sets to be a polyhedron. This result generalizes the well-known result due to Meyer [8]. Under a simple technical assumption, we show that these sufficient conditions are also necessary for the convex hull of integer points contained in general convex sets to be a polyhedron.

Keywords Convex Integer Program · Convex Hull · Polyhedron

1 Introduction

An important goal in the study of mathematical programming is to analyze properties of the convex hull of feasible solutions. The Fundamental Theorem of Integer Programming (see Section 2.5 in [2]), due to Meyer [8], states

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that the convex hull of feasible points in a mixed integer linear set defined by rational data is a polyhedron. The proof of this result relies on (i) the Minkowski-Weyl Representation Theorem for polyhedra and (ii) the fact that the recession cone is a rational polyhedral cone and thus generated by a finite number of integer vectors. In the world of mixed integer linear programming (MILP) problems, these sufficient conditions for polyhedrality of the convex hull of feasible solutions are reasonable since we expect most instances to be described using rational data.

A convex integer program is an optimization problem where the feasible region is of the form $K \cap \mathbb{Z}^n$ where $K \subseteq \mathbb{R}^n$ is a closed convex set and $\mathbb{Z}^n$ denotes the set of n-dimensional integral vectors. Let $\text{conv}(K \cap \mathbb{Z}^n)$ represent the convex hull of $K \cap \mathbb{Z}^n$. In this setting we do not have Minkowski-Weyl Representation Theorem for K or nice properties of recession cone of K. Therefore a natural question is to generalize Meyer's Theorem, in order to understand properties of the set K that lead to $\text{conv}(K \cap \mathbb{Z}^n)$ being a polyhedron. Note that [4] presents condition about the set $K \cap \mathbb{Z}^n$ (and more generally any subset of $\mathbb{Z}^n$) such that elements of $K \cap \mathbb{Z}^n$ have a finite integral generating set. In contrast, here we are interested in properties of the set K that allow us to deduce that $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron.

Observe that if $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron, then $\text{conv}(K \cap \mathbb{Z}^n)$ is a closed set. To the best of our knowledge, even the basic question of conditions that lead to $\text{conv}(K \cap \mathbb{Z}^n)$ being closed is not well-understood. (See [9] for some sufficient conditions for closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ when K is a polyhedron that is not necessarily described by rational data). We therefore divide this paper into two parts: (a) conditions for closedness and (b) conditions for polyhedrality. All the main results of these two parts are collected in Section 2.

In the first part of this paper (Section 3), we present necessary and sufficient conditions for $\text{conv}(K \cap \mathbb{Z}^n)$ to be closed when K contains no lines (Theorem 1). This characterization also leads to useful results for special classes of convex sets such as sets containing integer points in their interior (Theorem 2), strictly convex sets (Theorem 3), and pointed cones (Theorem 4). The necessary and sufficient conditions we present in Theorem 2 generalize the result presented in [9]. The case where K contains lines is then dealt separately (Theorem 5).

In the second part of this paper (Section 4), we present sufficient conditions for the convex hull of integer points contained in general convex sets to be polyhedra (Theorem 6). These sufficient conditions generalize the result presented in [8]. For a general convex set K containing at least one integer point in its interior, we show that these sufficient conditions are also necessary for $\text{conv}(K \cap \mathbb{Z}^n)$ to be a polyhedron (Theorem 6).

We conclude with some remarks in Section 5.

2 Main results.

2.1 Notation

We denote the standard scalar product on $\mathbb{R}^n$ as $\langle \cdot, \cdot \rangle$ and the norm corresponding to this scalar product as $\| \cdot \|$. For $u \in \mathbb{R}^n$ and $\epsilon > 0$, we use the notation $B(u, \epsilon)$ to denote the set $\{x \in \mathbb{R}^n \,|\, \|x - u\| \leq \epsilon\}$. Let $K \subseteq \mathbb{R}^n$. In this paper $\overline{K}$ represents the closure of K, $\text{int}(K)$ represents the interior of K, $\text{bd}(K)$ denotes the boundary of K, $\text{rel.int}(K)$ denotes the relative interior of K, $\dim(K)$ represents the dimension of K, $\text{rec.cone}(K)$ represents the recession cone of K, $\text{lin.space}(K)$ represents the lineality space of K, $\text{conv}(K)$ denotes the convex hull of K, $\overline{\text{conv}}(K)$ denotes the closure of $\text{conv}(K)$ and $\text{aff}(K)$ represents the affine hull of K. Note that we use the definition of recession cone for general convex sets given in Section 8 of [10]: $\text{rec.cone}(K) = \{d \in \mathbb{R}^n \,|\, x + \lambda d \in K \ \forall x \in K, \forall \lambda \geq 0\}$.

2.2 Results on closedness of $\text{conv}(K \cap \mathbb{Z}^n)$

Definition 1 **($u(K)$)** Given a convex set $K \subseteq \mathbb{R}^n$ and $u \in K \cap \mathbb{Z}^n$, we define $u(K) = \{d \in \mathbb{R}^n \,|\, u + \lambda d \in \text{conv}(K \cap \mathbb{Z}^n) \ \forall \lambda \geq 0\}$.

The main result is the following characterization of closedness of $\text{conv}(K \cap \mathbb{Z}^n)$.

Theorem 1 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line. Then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$.*

Furthermore, we present some refinements and consequences of this result when the closed convex set K contains an integer point in its interior (Theorem 2), K is a strictly closed convex set (Theorem 3) and K is a pointed closed cone (Theorem 4).

Theorem 2 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line and containing an integer point in its interior. Then the following are equivalent.*

1. $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed.*
2. $u(K) = \text{rec.cone}(K) \ \forall u \in K \cap \mathbb{Z}^n$.
3. *The following property holds for every proper exposed face F of K: If $F \cap \mathbb{Z}^n \neq \emptyset$, then for all $u \in F \cap \mathbb{Z}^n$ and for all $r \in \text{rec.cone}(F)$, $\{u + \lambda r \,|\, \lambda \geq 0\} \subseteq \text{conv}(F \cap \mathbb{Z}^n)$.*

Theorem 3 *If $K \subseteq \mathbb{R}^n$ is a full-dimensional closed strictly convex set, then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed.*

Theorem 4 *Let K be a full-dimensional pointed closed convex cone in $\mathbb{R}^n$. Then $\overline{\text{conv}}(K \cap \mathbb{Z}^n) = K$. In particular, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if every extreme ray of K is rational scalable (i.e. it can be scaled to be an integral vector).*

Finally, we present an extension of Theorem 1 for the case of the closed convex set K contains lines. Given $L \subseteq \mathbb{R}^n$ a linear subspace, we denote by $L^\perp$ the linear subspace orthogonal to L and we denote by $P_{L^\perp}$ the projection onto the set $L^\perp$. L is said to be a rational linear subspace if there exists a basis of L formed by rational vectors.

Theorem 5 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set such that the lineality space $L = \text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$ is not trivial. Then, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if the following two conditions hold:*

1. *$\text{conv}(K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n))$ is closed.*
2. *L is a rational linear subspace.*

2.3 Results on polyhedrality of $\text{conv}(K \cap \mathbb{Z}^n)$

Definition 2 (Thin Convex set) Let $K \subseteq \mathbb{R}^n$ be a closed convex set. We say K is *thin* if the following holds for all $c \in \mathbb{R}^n$: $\min\{\langle c, x\rangle \,|\, x \in K\} = -\infty$ if and only if there exist $d \in \text{rec.cone}(K)$ such that $\langle d, c\rangle < 0$.

The main result is a sufficient and necessary condition for $\text{conv}(K \cap \mathbb{Z}^n)$ to be a polyhedron.

Theorem 6 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. If K is thin and recession cone of K is a rational polyhedral cone, then $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron. Moreover, if $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ and $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron, then K is thin and $\text{rec.cone}(K)$ is a rational polyhedral cone.*

3 Closedness of $\text{conv}(K \cap \mathbb{Z}^n)$

Before presenting the results of this section, we develop some intuition by examining some examples.

Example 1 If K is a bounded convex set, then $\text{conv}(K \cap \mathbb{Z}^n)$ is a polytope. Therefore properties of the recession cone play an important role in determining the closedness of $\text{conv}(K \cap \mathbb{Z}^n)$. Intuitively, it appears that irrational extreme recession directions of K may cause $\text{conv}(K \cap \mathbb{Z}^n)$ to be not closed. However this is not entirely true as illustrated in the next few examples.

1. First consider the set $K^1 = \{x \in \mathbb{R}^2 \,|\, x_2 - \sqrt{2}x_1 \leq 0, x_2 \geq 0\}$. It is easily verified that in this case $\text{conv}(K^1 \cap \mathbb{Z}^2)$ is not closed (see Theorem 2, Section 2; also see Figure 1). In particular, the half-line $\{x \in \mathbb{R}^2 \,|\, x_2 - \sqrt{2}x_1 = 0, x_2 > 0\}$ is contained in $\overline{\text{conv}}(K \cap \mathbb{Z}^2)$ but not in $\text{conv}(K \cap \mathbb{Z}^2)$. In this case it is clear that the irrational data describing the polyhedron causes $\text{conv}(K^1 \cap \mathbb{Z}^2)$ to be not closed.

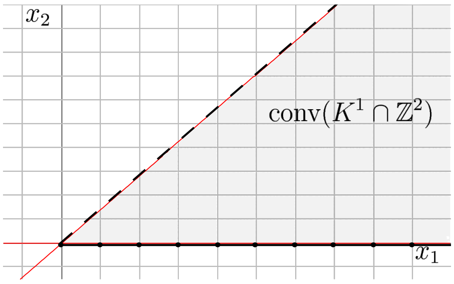

Fig. 1 K^1 and $\mathrm{conv}(K^1 \cap \mathbb{Z}^2)$.

2. Now consider the set $K^2 = \{x \in \mathbb{R}^2 \,|\, x_2 - \sqrt{2}x_1 \leq 0, x_2 \geq 0, x_1 \geq 1\}$. Notice that the recession cone of K^1 and K^2 are the same. In fact $(K^1 \cap \mathbb{Z}^2) = (K^2 \cap \mathbb{Z}^2) \cup \{(0,0)\}$. However, we can verify (see Theorem 2, Section 2; also see Figure 2) that $\mathrm{conv}(K^2 \cap \mathbb{Z}^2)$ is closed.

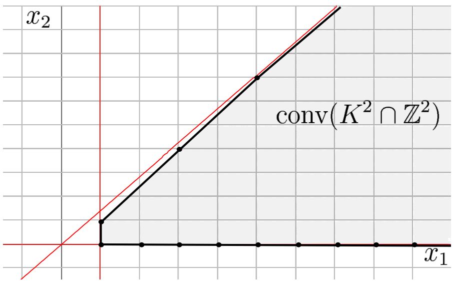

Fig. 2 K^2 and $\mathrm{conv}(K^2 \cap \mathbb{Z}^2)$.

We next illustrate a similar observation (i.e. recession cone of K has irrational extreme ray, but $\mathrm{conv}(K \cap \mathbb{Z}^n)$ is closed) using non-polyhedral sets.

3. Let $K^3 = \{x \in \mathbb{R}^2 \,|\, x_2 \geq x_1^2\}$. The recession cone of K^3 is $\{x \in \mathbb{R}^2 \,|\, x_1 = 0, x_2 \geq 0\}$. It can be shown that $\mathrm{conv}(K^3 \cap \mathbb{Z}^2)$ is closed (see Theorem 3, Section 2; also see Figure 3).

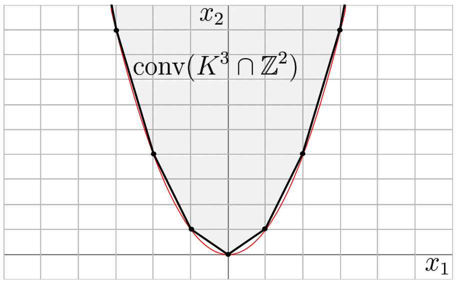

Fig. 3 K^3 and $\text{conv}(K^3 \cap \mathbb{Z}^2)$.

4. Now consider the set where we rotate the parabola K^3 such that the new recession cone is $\{x \in \mathbb{R}^2 \,|\, \sqrt{2}x_1 = x_2, x_2 \geq 0\}$, i.e., consider the set $K^4 = \{x \in \mathbb{R}^2 \,|\, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} x \in K^3\}$. In this case, even though the recession cone is a non-rational polyhedral set, it can be verified that $\text{conv}(K^4 \cap \mathbb{Z}^2)$ is closed (see Theorem 3, Section 2).

Observe that all the sets discussed above have polyhedral recession cones. However, sets whose recession cone are non-polyhedral can also lead to $\text{conv}(K \cap \mathbb{Z}^n)$ being closed.

5. Consider the set $K^5 = \{(0,0,1)\} \cup \{(0,1,1)\} \cup \left\{(\frac{1}{n}, \frac{1}{n^2}, 1) \,|\, n \in \mathbb{Z}, n \geq 1\right\}$. Then K^5 is closed, since it contains all its limit points. Therefore K^5 is a compact set and thus $\text{conv}(K^5)$ is compact (Theorem 17.2 [10]). Therefore, $K^6 = \text{conv}\left(\{\sum_{u \in K^5} \lambda_u u \,|\, \lambda_u \geq 0 \;\forall u \in K^5\}\right)$ is a closed convex cone. Finally, it can be verified that $\text{conv}(K^6 \cap \mathbb{Z}^3) = K^6$ is closed (see Theorem 4, Section 2).

3.1 Necessary and sufficient conditions for closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ for sets with no lines

In this section we will prove the following result (for definition of $u(K)$ see Definition 1, Section 2).

Theorem 1 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line. Then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$.*

Note here that when $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$, Theorem 1 implies that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed and therefore we obtain $u(K) = \text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n))$ is closed for every $u \in K \cap \mathbb{Z}^n$.

It is not difficult to verify that,

$$\text{conv}(K \cap \mathbb{Z}^n) = \text{conv}\left(\bigcup_{u \in K \cap \mathbb{Z}^n} (u + u(K))\right). \tag{1}$$

Hence Theorem 1 states that if the recession cone of each $u + u(K)$ is identical, then the convex hull of their union is closed. Therefore Theorem 1 is very similar in flavor to the following result.

Lemma 1 (Corollary 9.8.1 in [10]) *If K_1, ..., K_m are non-empty closed convex sets in $\mathbb{R}^n$ all having the same recession cone C, then* $\mathrm{conv}(K_1 \cup \ldots \cup K_m)$ *is closed and has C as its recession cone.*

Note however that Lemma 1 is not directly useful in verifying the 'sufficient part' of Theorem 1 since the number of integer points in a general convex set in not necessarily finite and thus the union in the right-hand-side of equation (1) is possibly over a countably infinite number of sets. Lemma 1 does not extend to infinite unions, in fact it does not hold even if the individual sets are polyhedra with the same recession cone. (Consider for example $\mathrm{conv}(\bigcup_{i\in\mathbb{Z}, i\geq 1} K_i)$ where $K_i = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 = \frac{1}{i}, x_2 \geq 0\}$.) However, we note here that the proof of Theorem 1 presented here will eventually use Lemma 1 is some cases, by suitably converting the set $\mathrm{conv}\left(\bigcup_{u\in K\cap\mathbb{Z}^n}(u + u(K))\right)$ to the convex hull of the union of a finite number of appropriate sets.

We begin by presenting some results that are required for the proof of Theorem 1.

Lemma 2 (Corollary 8.3.1 in [10]) *Let $K \subseteq \mathbb{R}^n$ be a convex set. Then*

$$\mathrm{rec.cone}(\mathrm{rel.int}(K)) = \mathrm{rec.cone}(\overline{K}) \supseteq \mathrm{rec.cone}(K).$$

The following crucial result is a direct consequence of Theorem 3.5 in [6].

Lemma 3 ([6]) *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed set. Then every extreme point of $\overline{\mathrm{conv}}(K)$ belongs to K.*

Lemma 4 *Let U be a $n \times n$ unimodular matrix and let $K \subseteq \mathbb{R}^n$ be a closed convex set. Then* $\mathrm{conv}(K \cap \mathbb{Z}^n)$ *is closed if and only if* $\mathrm{conv}((UK) \cap \mathbb{Z}^n)$ *is closed.*

Theorem 7 (Theorem 18.5 [10]) *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line. Let S be the set of extreme points of K and let D be the set of extreme rays of* $\mathrm{rec.cone}(K)$*. Then* $K = \mathrm{conv}(S) + \mathrm{cone}(D)$*. (Where* $\mathrm{cone}(D) = \{\sum_{i=1}^{N} \mu_i d_i \,|\, N \in \mathbb{Z}_+,\ d_i \in D,\ \mu_i \geq 0,\ \forall\ i = 1, \ldots, N\}$*.)*

A convex set $K \subseteq \mathbb{R}^n$ is called *lattice-free*, if $\mathrm{int}(K) \cap \mathbb{Z}^n = \emptyset$. A lattice-free convex set $K \subseteq \mathbb{R}^n$ is called *maximal lattice-free convex set* if there does not exist a lattice-free convex set $K' \subseteq \mathbb{R}^n$ satisfying $K' \supsetneq K$.

We note here that every lattice-free convex set is contained in a maximal lattice-free convex set. The following characterization of maximal lattice-free convex set is from [7]. See also [1] for a related result.

Theorem 8 ([1], [7]) *A full-dimensional lattice-free convex set $Q \subseteq \mathbb{R}^n$ is a maximal lattice-free convex set if and only if Q is a polyhedron of the form $Q = P + L$, where P is a polytope and L is a rational linear subspace and every facet of Q contains a point of $\mathbb{Z}^n$ in its relative interior.*

We now present the proof of the main result of this section.

Proof of Theorem 1 If $K \cap \mathbb{Z}^n = \emptyset$, then the result is trivial. Therefore, we will assume that $K \cap \mathbb{Z}^n \neq \emptyset$.

Let us prove "$\Rightarrow$". If $\text{conv}(K \cap \mathbb{Z}^n)$ is closed, then $\forall\ u \in K \cap \mathbb{Z}^n$, $u(K) = \text{rec.cone}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$. Thus $u(K)$ is identical for all $u \in K \cap \mathbb{Z}^n$.

Let us prove "$\Leftarrow$". Observe that for all $u \in K \cap \mathbb{Z}^n$ we have that

$$\text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) \subseteq u(K) \subseteq \text{rec.cone}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)). \tag{2}$$

The first inclusion follows directly by definition of $\text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n))$ and $u(K)$. The second inclusion is due to the fact that for a closed convex set, its recession cone gives the recession directions for every point in the set.

Assume now that $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$. We first claim that $u(K) = \text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n))\ \forall\ u \in K \cap \mathbb{Z}^n$. Let $r \in u(K)$ and $x \in \text{conv}(K \cap \mathbb{Z}^n)$. We can write $x = \sum_{i=1}^N \alpha_i z_i$, where $z_i \in K \cap \mathbb{Z}^n$, $\alpha_i \geq 0$ for all $i = 1, \dots, N$ and $\sum_{i=1}^N \alpha_i = 1$. Since $r \in z_i(K),\ \forall\ i = 1, \dots, N$, we have $z_i + \lambda r \in \text{conv}(K \cap \mathbb{Z}^n)$ for all $\lambda \geq 0$. Since $x + \lambda r = \sum_{i=1}^N \alpha_i(z_i + \lambda r)$, we obtain that $x + \lambda r \in \text{conv}(K \cap \mathbb{Z}^n)$ for all $\lambda \geq 0$. Thus, $u(K) \subseteq \text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n))$ and by (2) we obtain that

$$u(K) = \text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) \quad \forall u \in K \cap \mathbb{Z}^n. \tag{3}$$

We will now show that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. There are two cases:

- **Case 1:** $\text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n \neq \emptyset$. We will verify that $\text{conv}(K \cap \mathbb{Z}^n) \supseteq \overline{\text{conv}}(K \cap \mathbb{Z}^n)$. Let $u \in \text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n$. Since $u \in \text{rel.int}(\text{conv}\,(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n$ we obtain that if $r \in \text{rec.cone}(\text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n))$, then $u + \lambda r \in \text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)) \subseteq \text{conv}(K \cap \mathbb{Z}^n)$ for all $\lambda \geq 0$. Therefore, $\text{rec.cone}(\text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)) \subseteq u(K)$. Since $\text{rec.cone}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) = \text{rec.cone}(\text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n))$ (by Lemma 2), by using (2) we conclude that $u(K) = \text{rec.cone}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$. Therefore, by using (3) we obtain that $\text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec.cone}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$. Observer that, by Lemma 3, the extreme points of $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ belong to $K \cap \mathbb{Z}^n$. Since $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \subseteq K$, it does not contain any lines. Thus, by Theorem 7, $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ is given by the convex hull of its extreme points plus its recession cone. Since the extreme points of $\overline{\text{conv}}(K \cap \mathbb{Z}^n)$ belongs to $\text{conv}(K \cap \mathbb{Z}^n)$ and $\text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec.cone}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$, we obtain that $\text{conv}(K \cap \mathbb{Z}^n) \supseteq \overline{\text{conv}}(K \cap \mathbb{Z}^n)$. Therefore, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed.
- **Case 2:** $\text{rel.int}(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n = \emptyset$. We will use induction on the dimension of $\text{conv}(K \cap \mathbb{Z}^n)$. The base case, $\dim(\text{conv}(K \cap \mathbb{Z}^n)) = 1$ is straightforward to verify.
  Suppose now the property is true for every closed convex set K' such that $\dim(\text{conv}(K' \cap \mathbb{Z}^n)) < \dim(\text{conv}(K \cap \mathbb{Z}^n))$ and $\text{rel.int}(\text{conv}(K' \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n = \emptyset$.
  First for convenience, notice that we may assume $K = K \cap \text{aff}(K \cap \mathbb{Z}^n)$. Therefore $\dim(K \cap \mathbb{Z}^n) = \dim(K)$. Let $z \in K \cap \mathbb{Z}^n$. We now translate K as

$K-\{z\}$ and note that it is sufficient to show that $\text{conv}(K\cap\mathbb{Z}^n)$ is closed for this new set K. Observe that $\text{aff}(K\cap\mathbb{Z}^n)$ is a rational linear subspace, since it is generated by integer vectors. Now by selecting a suitable unimodular matrix (see [11]) and by the application of Lemma 4, we may assume that $\text{aff}(K\cap\mathbb{Z}^n)$ is of the form $\{x\,|\,x_i=0\ \forall i=k+1,...,n\}$. Finally, we can project out the last $n-k$ components (every point in K has zero in these components) and note that it is sufficient to show that $\text{conv}(K\cap\mathbb{Z}^k)$ is closed for this new set $K\subseteq\mathbb{R}^k$. In particular, without loss of generality, we may assume that $\text{conv}(K\cap\mathbb{Z}^n)$ is full-dimensional.
Note now that $\text{conv}(K\cap\mathbb{Z}^n)$ is lattice-free, and therefore there exists a full-dimensional maximal lattice-free polyhedron $Q\subseteq\mathbb{R}^n$ such that $\text{conv}(K\cap\mathbb{Z}^n)\subseteq Q$ and $Q=P+L$, where P is a polytope and L is a rational linear subspace.
Let $F_i,\ i=1,\dots,N$ be the facets of Q such that $K\cap F_i\cap\mathbb{Z}^n\neq\emptyset$. We will verify that

$$\text{conv}(K\cap\mathbb{Z}^n)\cap F_i=\text{conv}(K\cap F_i\cap\mathbb{Z}^n). \tag{4}$$

Since $\text{conv}(K\cap\mathbb{Z}^n)\cap F_i$ is a convex set and contains $K\cap F_i\cap\mathbb{Z}^n$ we have $\text{conv}(K\cap\mathbb{Z}^n)\cap F_i\supseteq\text{conv}(K\cap F_i\cap\mathbb{Z}^n)$. On the other hand, let $x\in\text{conv}(K\cap\mathbb{Z}^n)\cap F_i$. Therefore $x=\sum_{j=1}^M\alpha_j z_j$, where $z_j\in K\cap\mathbb{Z}^n$, $\alpha_j\geq 0$ for all $j=1,\dots,M$, and $\sum_{j=1}^M\alpha_j=1$. Since $K\cap\mathbb{Z}^n\subseteq Q$ and $x\in F_i$, we must have $z_j\in F_i,\ \forall,\ j=1,\dots,M$, so $x\in\text{conv}(K\cap F_i\cap\mathbb{Z}^n)$.
Next, for all $i=1,\dots,N$, we verify that

$$u(K\cap F_i)=u(K)\cap L\qquad\forall\ u\in K\cap F_i\cap\mathbb{Z}^n. \tag{5}$$

Let $r\in u(K\cap F_i)$. Then, by definition we have that $u+\lambda r\in\text{conv}(K\cap F_i\cap\mathbb{Z}^n)\ \forall\ \lambda\geq 0$. By (4), this is equivalent to $u+\lambda r\in\text{conv}(K\cap\mathbb{Z}^n)\cap F_i$ $\forall\ \lambda\geq 0$. This is also equivalent to $u+\lambda r\in\text{conv}(K\cap\mathbb{Z}^n)\ \forall\ \lambda\geq 0$ and $u+\lambda r\in F_i\ \forall\ \lambda\geq 0$. Thus equivalently we obtain that $r\in u(K)$ and $r\in\text{rec.cone}(F_i)$. By the form of Q (see Theorem 8) we have, $\forall\ i=1,\dots,N$, that $\text{rec.cone}(F_i)=\text{rec.cone}(Q)=L$. We obtain $r\in u(K)\cap L$. Therefore, we conclude $u(K\cap F_i)=u(K)\cap L$.
Since $u(K)$ is identical for all $u\in K\cap\mathbb{Z}^n$, (5) implies that $u(K\cap F_i)$ is identical for every $u\in K\cap F_i\cap\mathbb{Z}^n$ and $\forall\ i=1,\dots,N$. Moreover, since $\text{conv}(K\cap F_i\cap\mathbb{Z}^n)\subseteq F_i$, we obtain $\dim(\text{conv}(K\cap F_i\cap\mathbb{Z}^n))<\dim(\text{conv}(K\cap\mathbb{Z}^n))$. So we can use either case 1 or the induction hypothesis to conclude that $\text{conv}(K\cap F_i\cap\mathbb{Z}^n)$ is a closed set.
We now have that $u(K\cap F_i)=\text{rec.cone}(\text{conv}(K\cap F_i\cap\mathbb{Z}^n))=u(K)\cap L$ for all $i=1,\dots,N$. So the recession cone of $\text{conv}(K\cap F_i\cap\mathbb{Z}^n)$ is the same for all $i=1,\dots,N$. Observe that,

$$\text{conv}(K\cap\mathbb{Z}^n)=\text{conv}\left[\bigcup_{i=1}^N\text{conv}(K\cap F_i\cap\mathbb{Z}^n)\right].$$

Since the convex hull of a finite union of closed convex sets with the same recession cone is closed (Lemma 1), we conclude that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. □

We note here that the condition that K contains no line in the statement of Theorem 1 is not artificial. This is illustrated in the next Example.

Example 2 Consider the set $K^7 = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, 0.5 \leq x_2 - \sqrt{2}x_1 \leq 0.7\}$ which contains a line and let $u \in K^7 \cap \mathbb{Z}^2$. Since $\text{bd}(K^7) \cap \mathbb{Z}^2 = \emptyset$, we have that $u \in \text{int}(K^7)$. Since $u \in \text{int}(K^7)$, it can be verified that $u(K^7) = \text{rec.cone}(K^7)$ (see Lemma 8 in Section 3.2). Thus $u(K^7)$ is identical for all $u \in K^7 \cap \mathbb{Z}^2$. However, $\text{conv}(K^7 \cap \mathbb{Z}^2)$ is not closed, since $\text{conv}(K^7 \cap \mathbb{Z}^2) = \text{int}(K^7)$. To see this, first observe that the lines defining the boundary of K^7 do not contain any integer point, so $\text{conv}(K^7 \cap \mathbb{Z}^2) \subseteq \text{int}(K^7)$. The other inclusion is a consequence of the Dirichlet Diophantine Approximation Theorem.

3.2 Closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ where $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$

In this section, we simplify the conditions of Theorem 1 for the case where $\text{int}(\text{conv}(K \cap \mathbb{Z}^n)) \cap \mathbb{Z}^n \neq \emptyset$. We will assume that K is full-dimensional throughout this section. In particular if K is not full-dimensional, then by application of Lemma 4 as in the proof of Theorem 1, we can modify K and subsequently apply projection to achieve full-dimensionality of K and $K \cap \mathbb{Z}^n$.

In this section, we prove the following result.

Theorem 2 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line and containing an integer point in its interior. Then the following are equivalent.*

1. $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed.*
2. $u(K) = \text{rec.cone}(K)$ $\forall u \in K \cap \mathbb{Z}^n$.
3. *The following property holds for every proper exposed face F of K: If $F \cap \mathbb{Z}^n \neq \emptyset$, then for all $u \in F \cap \mathbb{Z}^n$ and for all $r \in \text{rec.cone}(F)$, $\{u + \lambda r \,|\, \lambda \geq 0\} \subseteq \text{conv}(F \cap \mathbb{Z}^n)$.*

Theorem 2 converts the question of verification of closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ to the verification of a somewhat simpler property of the faces of the set K. To see a simple application of Theorem 2 consider the cases (1.) and (2.) presented in Example 1. Note that both K^1 and K^2 contain integer points in their interior and $\text{conv}(K \cap \mathbb{Z}^n)$ is full-dimensional. In (1.), the facet $F := \{x \in \mathbb{R}^2 \,|\, x_2 = \sqrt{2}x_1, x_2 \geq 0\}$ contains only the point $(0, 0)$ and thus does not satisfy the property presented in Theorem 2. Hence we deduce that $\text{conv}(K^1 \cap \mathbb{Z}^n)$ is not closed. On the other hand, since the facet $\{x \in \mathbb{R}^2 \,|\, x_2 = \sqrt{2}x_1, x_2 \geq 0, x_1 \geq 1\}$ contains no integer point and all other faces of K^2 also satisfy the property presented in Theorem 2, we can deduce that $\text{conv}(K^2 \cap \mathbb{Z}^n)$ is closed.

We note that Theorem 2 generalizes sufficient conditions for closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ presented in [9]. [9] shows that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if K is a polyhedron that contains no lines, $\text{rec.cone}(K)$ is full-dimensional and every face of K satisfies the conditions described in the statement of Theorem 2. We note here that Theorem 2 is not true if the condition $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ is removed. The set K^{10} in Example 4 (Section 4) illustrates this.

Before we present the proof of Theorem 2, we first present a sequence of preliminary lemmas.

The following lemma is a direct consequence of the Dirichlet Diophantine Approximation Theorem (see Theorem 6.1 in [11]) and was proven in this form in [1].

Lemma 5 ([1]) *If $x \in \mathbb{Z}^n$ and $r \in \mathbb{R}^n$, then for all $\epsilon > 0$ and $\gamma \geq 0$, there exists a point of $\mathbb{Z}^n$ at a distance less than ϵ from the half line $\{x + \lambda r \mid \lambda \geq \gamma\}$.*

Lemma 6 *Let $V \subseteq \mathbb{R}^n$ be a linear subspace of $\mathbb{R}^n$ of dimension m. Let $\{a_1, \ldots, a_m\}$ be a basis of V. Then there exists $\delta > 0$ such that if $b_i \in B(-a_i, \delta) \cap V$, for $i = 1, \ldots, m$, then we have $0 \in \text{conv}(\{b_1, \ldots, b_m, a_1, \ldots, a_m\})$.*

Proof Without loss of generality, we may assume $V = \mathbb{R}^n$ and that $\{a_1, \ldots, a_n\}$ is the canonical basis of $\mathbb{R}^n$. Let $\delta > 0$, such that $\delta < \frac{1}{n}$. For $i = 1, \ldots, n$, consider $b_i \in B(-a_i, \delta)$, that is, there exist $v_i \in \mathbb{R}^n$ with $\|v_i\| \leq 1$ such that $b_i = -a_i + \delta v_i$.

Let $\mu \in \mathbb{R}^n$ be the vector with all components equal to 1 and let $\lambda = \mu - \delta \sum_{i=1}^n v_i$. Since $\delta < \frac{1}{n}$ and $\|v_i\| \leq 1$, we have that $\lambda \geq 0$. We observe that

$$0 = \lambda - \mu + \delta \sum_{i=1}^n v_i = \sum_{i=1}^n \lambda_i a_i + \sum_{i=1}^n \mu_i b_i.$$

Thus, we conclude $0 \in \text{conv}(\{b_1, \ldots, b_n, a_1, \ldots, a_n\})$. □

We will call a vector $r \in \mathbb{R}^n$ *rational scalable* if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\lambda r \in \mathbb{Z}^n$.

Lemma 7 *Let $r \in \mathbb{R}^n$ be a vector that is not rational scalable and $\gamma > 0$. Let $\mathcal{P}$ be the projection of the set $\{x \in \mathbb{Z}^n \mid \langle x, r \rangle \geq \gamma\}$ on the subspace $r^\perp := \{x \in \mathbb{R}^n \mid \langle x, r \rangle = 0\}$. Then for all $\epsilon > 0$, $0 \in \text{conv}(B(0, \epsilon) \cap \mathcal{P})$.*

Proof Let V be the linear subspace of $r^\perp$ generated by $\text{int}(B(0, \epsilon)) \cap \mathcal{P}$ and let $m = \dim(V)$. Notice that since r is not rational scalable, by Lemma 5 we have that $m \geq 1$. Let $\{a_1, \ldots, a_m\} \subseteq \text{int}(B(0, \epsilon)) \cap \mathcal{P}$ be a basis of V. For $i = 1, \ldots, m$, let $x_i \in \mathbb{Z}^n$ such that a_i is the projection on $r^\perp$ of x_i. Let $\delta > 0$ be such that if $b_i \in B(-a_i, \delta) \cap V$, for $i = 1, \ldots, m$, then we have $0 \in \text{conv}(\{b_1, \ldots, b_m, a_1, \ldots, a_m\})$ (Lemma 6).

Since $\{-a_1, \ldots, -a_m\} \subseteq \text{int}(B(0, \epsilon))$ there exists $r > 0$ such that $r \leq \delta$ and for all $i = 1, \ldots, m$, $B(-a_i, r) \subseteq B(0, \epsilon)$. Since $-x_i \in \mathbb{Z}^n$, by Lemma 5 we obtain that for all $i = 1, \ldots, m$ there exists $b_i \in \mathcal{P}$ at distance less or equal than r from $-a_i$. Thus, we have $b_i \in B(-a_i, \delta) \cap V$ and therefore, by the selection of δ, we obtain that

$$0 \in \text{conv}(\{b_1, \ldots, b_m, a_1, \ldots, a_m\}) \subseteq \text{conv}(B(0, \epsilon) \cap \mathcal{P}).$$

□

Lemma 8 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set, let $u \in K \cap \mathbb{Z}^n$ and let $d = \{u + \lambda r \,|\, \lambda > 0\} \subseteq \text{int}(K)$. Then $\{u\} \cup d \subseteq \text{conv}(K \cap \mathbb{Z}^n)$.*

Proof If r is rational scalable, then the result is straightforward. Suppose therefore that r is not rational scalable. Also without loss of generality we may assume $\|r\| = 1$.

Observe that $u \in \text{conv}(K \cap \mathbb{Z}^n)$. Therefore it is sufficient to show that for $\gamma > 0$, $u + \gamma r \in \text{conv}(K \cap \mathbb{Z}^n)$. Note now that $d = \{u + \lambda r \,|\, \lambda > 0\} \subseteq \text{int}(K)$, is equivalent to $\exists \epsilon > 0$ such that $B(u + \lambda r, \epsilon) \subseteq K \;\forall \lambda \geq \gamma$. Without loss of generality we may assume that $u = 0$. We will show then that there exists $\mu \geq \gamma$ such that $\mu r \in \text{conv}(K \cap \mathbb{Z}^n)$. Let $\mathcal{P}$ be the projection of $\{x \in \mathbb{Z}^n \,|\, \langle r, x\rangle \geq \gamma\}$ on the linear subspace $\{x \in \mathbb{R}^n \,|\, \langle r, x\rangle = 0\}$.

By Lemma 7, we have that $0 = \sum_{i=1}^{p} \lambda_i v^i$ where $v^i \in \mathcal{P} \cap B(0, \epsilon)$, $0 < \lambda_i \leq 1$ for all $i = 1, \dots, p$, $\sum_{1=1}^{p} \lambda_i = 1$. Let $v^1, \dots, v^p$ be the projection of the integer points $u^1, \dots, u^p$ where, for all $i = 1, \dots, p$, we have $u^i = v^i + \mu_i r \in \mathbb{Z}^n$ and $\mu_i \geq \gamma$. For all $i = 1, \dots, p$, since the distance between v^i and the half-line $\{\lambda r \,|\, \lambda \geq 0\}$ is less than ϵ and $\mu_i \geq \gamma$, we obtain that $u^i \in \{\lambda r \,|\, \lambda \geq \gamma\} + B(0, \epsilon) \subseteq K$. Therefore $u^i \in K \cap \mathbb{Z}^n$.

Now observe that

$$\sum_{i=1}^{p} \lambda_i (v^i + \mu_i r) = \sum_{i=1}^{p} \lambda_i v^i + r \sum_{i=1}^{p} \lambda_i \mu_i = r \sum_{i=1}^{p} \lambda_i \mu_i.$$

Since $\sum_{i=1}^{p} \lambda_i = 1$, we obtain that $\sum_{i=1}^{p} \lambda_i \mu_i \geq \gamma$. Thus, a point of the form μr where $\mu \geq \gamma$ belongs to $\text{conv}(K \cap \mathbb{Z}^n)$, completing the proof. □

Now we have all the tools needed to verify Theorem 2.

Proof of Theorem 2. Let $u \in \text{int}(K) \cap \mathbb{Z}^n$. Then we claim that $u(K) = \text{rec.cone}(K)$. Observe first that since K is closed, we obtain that $u(K) \subseteq \text{rec.cone}(K)$. Let $r \in \text{rec.cone}(K)$. Now observe that since $u \in \text{int}(K) \cap \mathbb{Z}^n$, $\{u + \lambda r \,|\, \lambda > 0\} \subseteq \text{int}(K)$. Thus by Lemma 8, the half-line line $\{u + \lambda r \,|\, \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Thus, $r \in u(K)$, completing the proof of the claim.

Now observe Theorem 1 implies *(2.)* $\Rightarrow$ *(1.)* and the above claim together with Theorem 1 implies *(1.)* $\Rightarrow$ *(2.)*. We now verify *(1.)* $\iff$ *(3.)*.

Let us prove "$\Leftarrow$". Assume that every exposed face of K satisfies the condition. We will verify that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. By Theorem 1, it is sufficient to show that $u(K)$ is identical for every $u \in K \cap \mathbb{Z}^n$. Observe that we have verified that $u(K) = \text{rec.cone}(K) \;\forall u \in \text{int}(K) \cap \mathbb{Z}^n$. Therefore, it remains to be shown that $u(K) = \text{rec.cone}(K)$ for all $u \in \text{bd}(K)$. Consider any $u \in \text{bd}(K)$ and let $r \in \text{rec.cone}(K)$. Then either $u + \lambda r \in \text{int}(K)$ for all $\lambda > 0$ or $r \in \text{rec.cone}(F)$ for some proper exposed face F. In the first case by Lemma 8, the half-line line $\{u + \lambda r \,|\, \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. In the second case, by the condition, we have that $\{u + \lambda r \,|\, \lambda \geq 0\} \subseteq \text{conv}(F \cap \mathbb{Z}^n) \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Thus $u(K) = \text{rec.cone}(K)$, completing the proof.

Let us prove "$\Rightarrow$". Let $\text{conv}(K \cap \mathbb{Z}^n)$ be closed. Then by Theorem 1, we know that $u(K)$ is closed and identical for all $u \in K \cap \mathbb{Z}^n$. Since we have verified that $u(K) = \text{rec.cone}(K)$ for all $u \in \text{int}(K) \cap \mathbb{Z}^n$, we obtain that $u(K) = \text{rec.cone}(K)$ for all $u \in K \cap \mathbb{Z}^n$. Now examine any proper exposed face of F. If $F \cap \mathbb{Z}^n \neq \emptyset$, $u \in F \cap \mathbb{Z}^n$ and $r \in \text{rec.cone}(F)$, then we have that $r \in \text{rec.cone}(K)$ and thus $\{u + \lambda r \,|\, \lambda \geq 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Therefore, it remains to verify that $\text{conv}(K \cap \mathbb{Z}^n) \cap F = \text{conv}(F \cap \mathbb{Z}^n)$ to complete the proof. Clearly $\text{conv}(F \cap \mathbb{Z}^n) \subseteq \text{conv}(K \cap \mathbb{Z}^n) \cap F$. If $x \in \text{conv}(K \cap \mathbb{Z}^n) \cap F$, then x is a convex combination of $z^1, \ldots, z^p$ where $z^i \in K \cap \mathbb{Z}^n$ for $i \in \{1, \ldots, p\}$. However, since $x \in F$, $z^i \in F$ for all $i \in \{1, ..., p\}$. Thus, $x \in \text{conv}(F \cap \mathbb{Z}^n)$, completing the proof. □

3.3 Closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ where K is a strictly convex set

A set $K \subseteq \mathbb{R}^n$ is called a *strictly convex set*, if K is a convex set and for all $x, y \in K$, $\lambda x + (1 - \lambda)y \in \text{rel.int}(K)$ for $\lambda \in (0, 1)$.

Theorem 3 *If $K \subseteq \mathbb{R}^n$ is a full-dimensional closed strictly convex set, then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed.*

Proof First note that if K is bounded or if $K \cap \mathbb{Z}^n = \emptyset$, then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. Therefore we assume that K is unbounded and $K \cap \mathbb{Z}^n \neq \emptyset$.

We first verify that K does not contain a line. Assume by contradiction that K contains a line in the direction $r \neq 0$. Examine $x \in \text{bd}(K)$. Then points of the form $x + \lambda r$ and $x - \lambda r$ belong to K, where $\lambda > 0$. In particular, $x + \lambda r, x - \lambda r \in \text{bd}(K)$ since $x \in \text{bd}(K)$. However this contradicts the fact that K is strictly convex.

Consider a point $u \in K \cap \mathbb{Z}^n$. Let $r \in \text{rec.cone}(K)$. Since K is strictly convex, we obtain that that set $\{u + \lambda r \,|\, \lambda > 0\}$ is contained in the interior of K. Therefore, by Lemma 8 we obtain that the set $\{u + \lambda r \,|\, \lambda \geq 0\}$ is contained in $\text{conv}(K \cap \mathbb{Z}^n)$. Thus, $u(K) = \text{rec.cone}(K)$ for all $u \in K \cap \mathbb{Z}^n$. Therefore, by Theorem 1 we obtain that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. □

Thus in the case of full-dimensional closed strictly convex set K, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed independent of the recession cone. The sets K^3 and K^4 in Example 1 are examples of this fact.

It is easily verified that every face of K is zero-dimensional, i.e. a single point. Therefore in fact the statement of Theorem 3 follows straightforwardly from Theorem 2 in the case when K is not lattice-free. It turns out that if $K \subseteq \mathbb{R}^n$ is a full-dimensional unbounded closed strictly convex set and $K \cap \mathbb{Z}^n \neq \emptyset$, then K is not lattice-free. The proof would follow from Lemma 5.

3.4 Closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ where K is a full-dimensional pointed closed convex cone

In this section we prove the following result.

Theorem 4 *Let K be a full-dimensional pointed closed convex cone in $\mathbb{R}^n$. Then $\overline{\text{conv}}(K \cap \mathbb{Z}^n) = K$. In particular,* $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed if and only if every extreme ray of K is rational scalable.*

Proof We first verify that $\overline{\text{conv}}(K \cap \mathbb{Z}^n) = K$. By convexity of K, we obtain that $\text{conv}(K \cap \mathbb{Z}^n) \subseteq K$. Since K is also closed, we obtain that $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \subseteq K$. Now, let $r \in \text{int}(K)$. Clearly, we have $d = \{0 + \lambda r \,|\, \lambda > 0\} \subseteq \text{int}(K)$. So, by Lemma 8 , we obtain $\{0\} \cup d \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Hence, $\text{int}(K) \subseteq \text{conv}(K \cap \mathbb{Z}^n)$. Since K is a full-dimensional closed convex set, we have $K = \overline{\text{int}(K)}$. Thus, by taking the closure on both sides of the inclusion $\text{int}(K) \subseteq \text{conv}(K \cap \mathbb{Z}^n)$, we obtain $K \subseteq \overline{\text{conv}}(K \cap \mathbb{Z}^n)$.

We now verify that $\text{conv}(K \cap \mathbb{Z}^n)$ is closed if and only if all the extreme rays of K are rational scalable rays. Suppose $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. Then $\text{conv}(K \cap \mathbb{Z}^n) = K$. If r is any extreme ray of K, then observe that $K \setminus \{\lambda r \,|\, \lambda > 0\}$ is a convex set. Since $\{\lambda r \,|\, \lambda > 0\} \subseteq \text{conv}(K \cap \mathbb{Z}^n)$, there must be an integer point in the set $\{\lambda r \,|\, \lambda > 0\}$. In other words, r is rational scalable.

Now assume that every extreme ray of K is rational scalable. Let R be the set of all extreme rays of K. Then observe that

$$K = \text{cone}(R) \subseteq \text{conv}(K \cap \mathbb{Z}^n) \subseteq K,$$

where the first equality follows from Theorem 7. Thus, $\text{conv}(K \cap \mathbb{Z}^n) = K$ or equivalently $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. □

We note here that K^6 in Example 1 is an example for a non-polyhedral cone where each extreme ray is rational scalable. Therefore $\text{conv}(K^6 \cap \mathbb{Z}^3) = K^6$.

3.5 Closedness of $\text{conv}(K \cap \mathbb{Z}^n)$ where K contains lines

Given a set K and a half-line $d = \{u + \lambda r \,|\, \lambda \geq 0\}$ we say K is *coterminal* with d if

$$\sup\{\mu \,|\, \mu > 0, u + \mu r \in K\} = \infty.$$

This definition is originally presented in [6]. Given a closed convex set K, a face F of K is called extreme facial ray of K if F is a closed half-line.

In this section, we will verify the following result.

Theorem 5 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set such that the lineality space* $L = \text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$ *is not trivial. Then,* $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed if and only if the following two conditions hold:*

1. $\text{conv}(K \cap L^{\perp} \cap P_{L^{\perp}}(\mathbb{Z}^n))$ *is closed.*
2. *L is a rational linear subspace.*

Note that when L is a rational linear subspace (otherwise we already know that $\text{conv}(K \cap \mathbb{Z}^n)$ is not closed), we obtain that $P_{L^{\perp}}(\mathbb{Z}^n)$ is a lattice. Therefore, if the set $K \cap L^{\perp}$ does not contain any line, then we can characterize the closedness of $\text{conv}(K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n))$ using the properties we have for convex sets not containing lines (these results can be easily extended for a general lattice).

Before presenting the proof of Theorem 5, we describe some useful corollaries.

Corollary 1 *Let K be a closed convex set and let $\text{rec.cone}(K)$ be a rational polyhedral cone. Then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed.*

Proof Observe that if $K \cap \mathbb{Z}^n = \emptyset$, the result is trivial. We assume for the rest of the proof that $K \cap \mathbb{Z}^n \neq \emptyset$.

Let $L = \text{lin.space}(K)$. Since L is rational, $\text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) = L$. Therefore, by Theorem 5, we only need to verify that $\text{conv}(K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n))$ is closed.

Notice that the set $K \cap L^\perp$ does not contain any line. To simplify the proof, by using Lemma 4 we may assume without loss of generality that $L^\perp = \{x \in \mathbb{R}^n \,|\, x_i = 0 \ \forall i = k+1, \ldots, n\}$. Thus, it is sufficient (after projecting out the last $n-k$ components) to show that $\text{conv}(K' \cap \mathbb{Z}^k)$ is closed, where $K' \subseteq \mathbb{R}^k$ is a closed convex set not containing any line and $\text{rec.cone}(K')$ is a rational polyhedral cone. However, note now that $u(K') \supseteq \text{rec.cone}(K') \supseteq \text{rec.cone}(\overline{\text{conv}}(K' \cap \mathbb{Z}^k)) \supseteq u(K')$ for all $u \in K' \cap \mathbb{Z}^n$, where the first inclusion is due to the fact that $\text{rec.cone}(K')$ is a rational polyhedral cone, the second inclusion is due to the fact that K' is closed and the last inclusion is the same as (2). Thus $u(K')$ is closed and identical for all $u \in K' \cap \mathbb{Z}^k$. Therefore, by Theorem 1 we conclude that K' is closed which completes the proof. $\square$

Corollary 2 *If $\text{lin.space}(K)$ is not a rational linear subspace and $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$, then $\text{conv}(K \cap \mathbb{Z}^n)$ is not closed.*

Proof By Lemma 8, we conclude $\text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) = \text{lin.space}(K)$, which completes the proof. $\square$

Next we present some results needed to verify Theorem 5. The crucial results needed are the following theorems from [5].

Theorem 9 ([5]) *If $A \subseteq \mathbb{R}^n$ is a closed set not containing a line, then $\text{conv}(A)$ is closed if and only if A is coterminal with all the extreme facial rays of $\overline{\text{conv}}(A)$.*

Theorem 10 ([5]) *Let $A \subseteq \mathbb{R}^n$ such that $L = \text{lin.space}(\overline{\text{conv}}(A))$ is not trivial. Then, $\text{conv}(A)$ is closed if and only if*

1. *The set $P_{L^\perp}(A)$ is coterminal with every extreme facial ray of $\overline{\text{conv}}(A) \cap L^\perp$.*
2. *For every extreme point z of $\overline{\text{conv}}(A) \cap L^\perp$, $\text{conv}(A \cap (z + L)) = z + L$.*

The following straightforward lemmas, that we present without any proofs, show some properties of the projection operation.

Lemma 9 *Let $A, B \subseteq \mathbb{R}^n$ and denote $L \subseteq \text{lin.space}(\overline{\text{conv}}(A))$. We have the following:*

1. $P_{L^\perp}(\overline{B}) \subseteq \overline{P_{L^\perp}(B)}$.
2. $P_{L^\perp}(\text{conv}(B)) = \text{conv}(P_{L^\perp}(B))$.
3. $P_{L^\perp}(\overline{\text{conv}}(A)) = \overline{\text{conv}}(A) \cap L^\perp$.
4. $P_{L^\perp}(\overline{\text{conv}}(A)) = \overline{\text{conv}}(P_{L^\perp}(A))$.

Lemma 10 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. Denote $L = \text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$. Then $P_{L^\perp}(K \cap \mathbb{Z}^n) = P_{L^\perp}(K) \cap P_{L^\perp}(\mathbb{Z}^n)$. In particular, $P_{L^\perp}(K \cap \mathbb{Z}^n) = K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n)$.*

We now have all the tools for proving Theorem 5.

Proof of Theorem 5 Observe that if $K \cap \mathbb{Z}^n = \emptyset$, the result is trivial. We assume for the rest of the proof that $K \cap \mathbb{Z}^n \neq \emptyset$.

Claim 1: If L is a rational linear subspace, then (*1.*) of Theorem 5 is equivalent to (*1.*) of Theorem 10 with $A = K \cap \mathbb{Z}^n$. Since L is a rational linear subspace, $P_{L^\perp}(\mathbb{Z}^n)$ is a lattice and therefore $P_{L^\perp}(\mathbb{Z}^n)$ is a closed set. Hence $K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n)$ is a closed set. Moreover, by Lemma 10, we have that

$$K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n) = P_{L^\perp}(K \cap \mathbb{Z}^n). \tag{6}$$

Thus, $P_{L^\perp}(K \cap \mathbb{Z}^n)$ is closed and $\text{conv}(K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n))$ is closed if and only if $\text{conv}(P_{L^\perp}(K \cap \mathbb{Z}^n))$ is a closed set. Since $P_{L^\perp}(K \cap \mathbb{Z}^n)$ is closed, we may apply Theorem 9 to $P_{L^\perp}(K \cap \mathbb{Z}^n)$ to obtain that $\text{conv}(P_{L^\perp}(K \cap \mathbb{Z}^n))$ is closed if and only if the set $P_{L^\perp}(K \cap \mathbb{Z}^n)$ is coterminal with every extreme facial ray of $\overline{\text{conv}}(P_{L^\perp}(K \cap \mathbb{Z}^n))$. By Lemma 9, $\overline{\text{conv}}(P_{L^\perp}(K \cap \mathbb{Z}^n)) = P_{L^\perp}(\overline{\text{conv}}(K \cap \mathbb{Z}^n)) = \overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$. Therefore, $\text{conv}(K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n))$ is closed if and only if $P_{L^\perp}(K \cap \mathbb{Z}^n)$ is coterminal with every extreme facial ray of $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$.

Observe that since the set $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$ does not contain any lines, it must have at least one extreme point.

Let us prove "⇒". Suppose $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. Then, by Theorem 10 for every extreme point z of $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$ we have that $\text{conv}(K \cap \mathbb{Z}^n \cap (z + L)) = z + L$. Thus $z + L$ is the convex hull of some nonempty subset of integer points and therefore L is a rational linear subspace. This proves (*2.*) of Theorem 5. Moreover, since L is a rational linear subspace, we have (*1.*) of Theorem 5 by Claim 1.

Let us prove "⇐". Now suppose (*1.*) and (*2.*) of Theorem 5. Then by Claim 1, we have (*1.*) of Theorem 10. We will prove (*2.*) of Theorem 10, that is, for every extreme point z of $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp$, we have $\text{conv}(K \cap \mathbb{Z}^n \cap (z + L)) = z + L$. We first prove that $(z + L) \cap K \cap \mathbb{Z}^n \neq \emptyset$. Since $\overline{\text{conv}}(K \cap \mathbb{Z}^n) \cap L^\perp = \overline{\text{conv}}(P_{L^\perp}(K \cap \mathbb{Z}^n))$, by Lemma 3 we have that $z \in P_{L^\perp}(K \cap \mathbb{Z}^n)$, and therefore there exists $l \in L$ such that $z + l \in K \cap \mathbb{Z}^n$. Hence, $(z + L) \cap K \cap \mathbb{Z}^n \neq \emptyset$. Now let $\{l_1, \dots, l_p\} \subseteq \mathbb{Z}^n$ be a basis of L and let $w \in (z + L) \cap K \cap \mathbb{Z}^n$. Since $L \subseteq \text{lin.space}(K)$ for all $\lambda_1, \dots, \lambda_p \in \mathbb{Z}$, the points $w, w + \lambda_1 l_1, \dots, w + \lambda_p l_p$ belong to $(z + L) \cap K \cap \mathbb{Z}^n$. Thus, by convexity of $\text{conv}((z+L) \cap K \cap \mathbb{Z}^n)$, for all $\lambda_1, \dots, \lambda_p \in \mathbb{R}$, the points $w, w + \lambda_1 l_1, \dots, w + \lambda_p l_p$ belong to $\text{conv}((z + L) \cap K \cap \mathbb{Z}^n)$. Hence, $\text{conv}((z + L) \cap K \cap \mathbb{Z}^n)$ contains an affine subspace whose dimension is the same as dimension of $z + L$. Since $\text{conv}(K \cap \mathbb{Z}^n \cap (z + L)) \subseteq z + L$, we obtain that $\text{conv}(K \cap \mathbb{Z}^n \cap (z + L)) = z + L$. Thus we obtain (*2.*) of Theorem 10. Therefore, $\text{conv}(K \cap \mathbb{Z}^n)$ is closed. □

4 Polyhedrality of conv($K \cap \mathbb{Z}^n$)

We use the following notation in this section. Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then $\sigma_K : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined as $\sigma_K(a) = \sup\{\langle a, x\rangle \mid x \in K\}$ is the support function of K. Given a cone T, we represent its polar by T^*. In particular, $(\text{rec.cone}(K))^* = \{d \in \mathbb{R}^n \mid \langle d, u\rangle \leq 0 \ \forall u \in \text{rec.cone}(K)\}$.

Let us develop some intuition regarding the question of polyhedrality of $\text{conv}(K \cap \mathbb{Z}^n)$. Suppose for simplicity that K contains no lines, K is full-dimensional and $\text{int}(K) \cap \mathbb{Z}^n$ is non-empty. Then by Theorem 2, we obtain that a necessary condition for $\text{conv}(K \cap \mathbb{Z}^n)$ to be closed is that $\text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec.cone}(K)$. Therefore, in this setting, if we require $\text{conv}(K \cap \mathbb{Z}^n)$ to be a rational polyhedron, it is necessary that K has a rational polyhedral recession cone. However, this is not sufficient. Consider the case of the parabola K^3 presented in Example 1. It is easy to verify that $\text{conv}(K^3 \cap \mathbb{Z}^2)$ is not a polyhedron. To see what is 'going wrong', observe that $\min\{x_1 \mid x \in K^3\} = -\infty$, even though $(-1, 0)$ is orthogonal to the all vectors in the recession cone. Intuitively, this causes $\text{conv}(K^3 \cap \mathbb{Z}^2)$ to have an infinite number of extreme points. This motivates the definition of 'thin set' (see Definition 2, Section 2). In terms of its support function and the polar of its recession cone, a closed convex set K is thin if and only if the following holds for all $c \in \mathbb{R}^n$: $\sigma_K(c) < +\infty$ if and only if $c \in (\text{rec.cone}(K))^*$.

In this section we verify the following result.

Theorem 6 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. If K is thin and recession cone of K is a rational polyhedral cone, then* $\text{conv}(K \cap \mathbb{Z}^n)$ *is a polyhedron. Moreover, if* $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ *and* $\text{conv}(K \cap \mathbb{Z}^n)$ *is a polyhedron, then K is thin and* $\text{rec.cone}(K)$ *is a rational polyhedral cone.*

Since every polyhedron is a thin set, Theorem 6 generalizes the result in [8]. We present a simple example illustrating Theorem 6 when K is not a polyhedral set.

Example 3 Consider the set $K^8 = \{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\}$. It is straightforward to verify that K^8 is thin and $\text{rec.cone}(K^8) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0, y_2 \geq 0\}$ is a rational polyhedron. Thus, $\text{conv}(K^8 \cap \mathbb{Z}^2) = \{(x_1, x_2) \mid x_1 \geq 1, x_2 \geq 1\}$ is a polyhedron. On the other hand observe that while each of the sets K^1, K^2, K^3, K^4, K^6 in Example 1 contains integer points in its interior, none of them are both thin and have rational polyhedral recession cone. Thus by Theorem 6, the convex hull of integer points in all these sets is non-polyhedral.

4.1 Sufficient conditions for conv($K \cap \mathbb{Z}^n$) to be polyhedral

The following variant of Gordan-Dickson's Lemma is from [3].

Lemma 11 *Let $m \in \mathbb{N}$ and $X \subseteq \mathbb{Z}^m$ and assume there exists $x_0 \in \mathbb{Z}^m$ such that $x \geq x_0$ for every $x \in X$. Then there exists a finite set $Y \subseteq X$ such that for every $x \in X$ there exists $y \in Y$ satisfying $y \leq x$.*

Proposition 1 (Sufficient Condition) *If K is thin and recession cone of K is a rational polyhedral cone, then* $\text{conv}(K \cap \mathbb{Z}^n)$ *is a polyhedron.*

Proof Observe that if $K \cap \mathbb{Z}^n = \emptyset$, then $\text{conv}(K \cap \mathbb{Z}^n) = \emptyset$ is a polyhedron. Therefore, assume for the rest of the proof that $K \cap \mathbb{Z}^n \neq \emptyset$. Since $\text{rec.cone}(K)$ is a rational polyhedron, we obtain $\text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec.cone}(K)$. Choose an integer matrix $A \in \mathbb{Z}^{m \times n}$ such that

$$\text{rec.cone}(K) = \{x \in \mathbb{R}^n \mid Ax \geq 0\}.$$

For $i = 1, \ldots, m$ let $a_i \in \mathbb{R}^n$ denote the ith row of A. Since K is thin, for all $i = 1, \ldots, m$ we have $\inf\{\langle a_i, x\rangle \mid x \in K\} > -\infty$, so we obtain $\inf\{\langle a_i, x\rangle \mid x \in K \cap \mathbb{Z}^n\} > -\infty$. Thus, by defining the vector $x_0 \in \mathbb{Z}^m$ as follows

$$(x_0)_i = \inf\{\langle a_i, x\rangle \mid x \in K \cap \mathbb{Z}^n\}, \quad \forall\, i = 1, \ldots, m$$

we conclude that $Ax \geq x_0$ for every $x \in K \cap \mathbb{Z}^n$. Therefore, by applying Lemma 11 to the set $\{Ax \mid x \in K \cap \mathbb{Z}^n\} \subseteq \mathbb{Z}^m$, we obtain that there exists a finite set $Y \subseteq K \cap \mathbb{Z}^n$ such that for every $x \in K \cap \mathbb{Z}^n$ there exists $y \in Y$ satisfying $Ay \leq Ax$. To prove that $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron, it is sufficient to show that

$$\text{conv}(K \cap \mathbb{Z}^n) = \text{conv}(Y) + \text{rec.cone}(K).$$

Let $x \in K \cap \mathbb{Z}^n$ and $y \in Y$ as above, that is, such that $A(x - y) \geq 0$. We have $x - y \in \text{rec.cone}(K)$. This shows $K \cap \mathbb{Z}^n \subseteq Y + \text{rec.cone}(K)$ and yields the inclusion $\text{conv}(K \cap \mathbb{Z}^n) \subseteq \text{conv}(Y) + \text{rec.cone}(K)$. The reverse inclusion $\text{conv}(Y) + \text{rec.cone}(K) \subseteq \text{conv}(K \cap \mathbb{Z}^n)$ follows directly from the fact $Y \subseteq K \cap \mathbb{Z}^n$ and that $\text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec.cone}(K)$. □

We note here that the proof technique of Proposition 1 was suggested by an anonymous referee.

4.2 Necessary conditions for $\text{conv}(K \cap \mathbb{Z}^n)$ to be polyhedral

We begin with a few lemmas before presenting the 'necessary direction' of Theorem 6.

Lemma 12 *Let $Q \subseteq \mathbb{R}^n$ be a full-dimensional maximal lattice free convex set and let $\langle c, x\rangle \leq d$ be a valid inequality for Q. Then there exists $\delta > 0$ such that $\langle c, x\rangle \geq d - \delta$ is a valid inequality for Q.*

Proof Assume by contradiction that $\inf\{\langle c, x\rangle \mid x \in Q\} = -\infty$. Since Q is a polyhedron (by Theorem 8), we obtain that there exists a recession direction r of Q such that $\langle r, c\rangle < 0$. However, because $\text{rec.cone}(Q) = \text{lin.space}(Q)$, we have that $-r$ is a recession direction of Q. Then $\sup\{\langle c, x\rangle \mid x \in Q\} = +\infty$, contradicting the assumption that $\langle c, x\rangle \leq d$ is a valid inequality for Q. □

Lemma 13 *If $K \subseteq \mathbb{R}^n$ is thin and $T \subseteq \mathbb{R}^n$ is a closed subset of K such that* $\text{rec.cone}(T) = \text{rec.cone}(K)$*, then T is thin.*

Proof Suppose $\inf\{\langle c, x\rangle \,|\, x \in T\}$ is unbounded. Then, $\inf\{\langle c, x\rangle \,|\, x \in K\}$ is unbounded. Since K is thin, there exists $d \in \text{rec.cone}(K) = \text{rec.cone}(T)$ such that $\langle d, c\rangle < 0$. If $\inf\{\langle c, x\rangle \,|\, x \in T\}$ is bounded, then $\langle d, c\rangle \geq 0$ for all $d \in \text{rec.cone}(T)$. □

Proposition 2 (Necessary Condition) *Let $K \subseteq \mathbb{R}^n$ be a a closed convex set such that* $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$*. If* $\text{conv}(K \cap \mathbb{Z}^n)$ *is a polyhedron, then K is thin and* $\text{rec.cone}(K)$ *is a rational polyhedral cone.*

Proof Let $P = \{x \in \mathbb{R}^n \,|\, \langle a_i, x\rangle \leq b_i,\ i = 1, \ldots, m\}$ be a description of $\text{conv}(K \cap \mathbb{Z}^n)$. Note that P is a rational polyhedron. We will show first that for all $i = 1, \ldots, m$, we have $\sup\{\langle a_i, x\rangle \,|\, x \in K\} < \infty$. Let $i \in \{1, ..., m\}$ and assume by contradiction that $\sup\{\langle a_i, x\rangle \,|\, x \in K\} = \infty$. Consider the set $K_i = K \cap \{x \in \mathbb{R}^n \,|\, \langle a_i, x\rangle \geq b_i\}$. Notice that $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$, so K must be a full-dimensional set. Also, by assumption, we have $K \not\subseteq \{x \in \mathbb{R}^n \,|\, \langle a_i, x\rangle \leq b_i\}$. Therefore it can be verified that $\text{int}(K) \cap \{x \in \mathbb{R}^n \,|\, \langle a_i, x\rangle > b_i\} \neq \emptyset$. This implies $\text{int}(K_i) = \text{int}(K) \cap \{x \in \mathbb{R}^n \,|\, \langle a_i, x\rangle > b_i\} \neq \emptyset$ and thus K_i is of full dimension.

Moreover, we have $\text{int}(K_i) \cap \mathbb{Z}^n = (\text{int}(K_i) \cap K) \cap \mathbb{Z}^n \subseteq \text{int}(K_i) \cap P = \emptyset$, so K_i is a lattice-free set. Hence, there exists a full-dimensional maximal lattice-free polyhedron $Q = \{x \in \mathbb{R}^n \mid \langle c_j, x\rangle \leq d_j,\ j = 1, \ldots, q\}$ such that $K_i \subseteq Q$.

Since K is not lattice-free we obtain that $K \not\subseteq Q$. Therefore there exists $x_0 \in K \setminus Q$, that is, $x_0 \in K$, $\langle a_i, x_0\rangle < b_i$, and there exists $j \in \{1, \ldots, q\}$ such that $\langle c_j, x_0\rangle > d_j$. By Lemma 12, there exists $\delta > 0$ such that $x \in Q$ implies $\langle c_j, x\rangle \geq d_j - \delta$.

Let $\{x_n\}_{n\geq 1} \subseteq K_i$ such that $\lim_{n\to\infty}\langle a_i, x_n\rangle = \infty$ and $\lambda_n \in (0, 1)$ such that the point $y_n = (1-\lambda_n)x_0 + \lambda_n x_n$ satisfies $\langle a_i, y_n\rangle = b_i$. Since $x_0, x_n \in K$, by convexity of K, we have $y_n \in K$. Therefore we obtain that $y_n \in K_i$.

On the other hand,

$$\begin{aligned}\langle c_j, y_n\rangle - d_j &= (1 - \lambda_n)\langle c_j, x_0\rangle + \lambda_n\langle c_j, x_n\rangle - d_j \\ &\geq (1 - \lambda_n)(\langle c_j, x_0\rangle - d_j) - \lambda_n\delta \\ &= (\langle c_j, x_0\rangle - d_j) - \lambda_n[(\langle c_j, x_0\rangle - d_j) + \delta].\end{aligned} \tag{7}$$

where the inequality follows from the fact that $\{x_n\}_{n\geq 1} \subseteq K_i \subseteq Q \subseteq \{x \in \mathbb{R}^n :\ \langle c_j, x\rangle \geq d_j - \delta\}$.

Notice that, by definition, $\lambda_n = \frac{b_i - \langle a_i, x_0\rangle}{\langle a_i, x_n\rangle - \langle a_i, x_0\rangle}$ and thus $\lim_{n\to\infty}\lambda_n = 0$. Hence, by 7, for sufficiently large n, we have $\langle c_j, y_n\rangle > d_j$, a contradiction with the fact $y_n \in K_i \subseteq Q$. So, we must have $\sup\{\langle a_i, x\rangle \mid x \in K\} < \infty$, for all $i \in \{1, ..., m\}$.

We conclude that there exist numbers $\bar{b}_i$, for all $i = 1, \ldots, m$, with $b_i \leq \bar{b}_i < \infty$ such that $K \subseteq P' := \{x \,|\, \langle a_i, x\rangle \leq \bar{b}_i,\ i = 1, \ldots, m\}$. Hence, since $P \subseteq K \subseteq P'$, we have $\text{rec.cone}(K) = \{x \,|\, \langle a_i, x\rangle \leq 0,\ i = 1, \ldots, m\}$, so $\text{rec.cone}(K)$ is a rational polyhedral cone. Moreover, every polyhedron is thin, so by Lemma 13, we conclude K is also thin, as desired. □

We note here that the addition technical condition that $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ in Proposition 2 is not artificial. We illustrate this with examples next.

Example 4 1. Here is an example that shows that $\text{conv}(K \cap \mathbb{Z}^n)$ can be a polyhedron and yet it is not thin, since it is lattice-free. Consider the set

$$\begin{aligned} K^9 := \text{conv}(&\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_3 = 0, x_1 = 0, x_2 \geq 0\} \\ \cup\ &\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_3 = 0.5, x_2 \geq x_1^2\} \\ \cup\ &\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_3 = 1, x_1 = 0, x_2 \geq 0\}). \end{aligned}$$

Observe that $\text{conv}(K^9 \cap \mathbb{Z}^3) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_1 = 0, x_2 \geq 0, 0 \leq x_3 \leq 1\}$ is a polyhedron. However note that K^9 is not thin since $\text{rec.cone}(K^9) = \{\lambda(0,1,0) \,|\, \lambda \geq 0\}$ and $\inf\{\langle(-1,0,0),x\rangle \,|\, x \in K^9\} = -\infty$ but $\langle(0,1,0),(-1,0,0)\rangle = 0$. Finally note that K^9 is lattice-free.

2. Here is an example that shows that $\text{conv}(K \cap \mathbb{Z}^n)$ can be a polyhedron and yet $\text{rec.cone}(K)$ is not a rational polyhedral cone, since it is lattice-free. Consider the set

$$\begin{aligned} K^{10} := \text{conv}(&\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_2 - \sqrt{2}x_1 = 0, x_3 = 0\} \\ \cup\ &\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_2 - \sqrt{2}x_1 = 1, x_3 = 0.5\} \\ \cup\ &\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_2 - \sqrt{2}x_1 = -1, x_3 = 0.5\} \\ \cup\ &\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_2 - \sqrt{2}x_1 = 0, x_3 = 1\}). \end{aligned}$$

Then $K^{10} \cap \mathbb{Z}^3 = \{(0,0,0),(0,0,1)\}$ and thus $\text{conv}(K^{10} \cap \mathbb{Z}^3)$ is a polyhedron. However, note that $\text{rec.cone}(K^{10})$ is not a rational polyhedral cone. Also observe that K^{10} is lattice-free.

5 Remarks

We first remark that all the key results in this paper (Theorem 1, Theorem 2, Theorem 3, Theorem 4, Theorem 5, Theorem 6) hold if we replace $\mathbb{Z}^n$ by any general lattice $\Gamma \subseteq \mathbb{R}^n$ and investigate the closedness and polyhedrality of $\text{conv}(K \cap \Gamma)$.

It is possible to relax the requirement of $\text{conv}(K \cap \mathbb{Z}^n)$ being a polyhedron and ask the question when $\text{conv}(K \cap \mathbb{Z}^n)$ is *locally polyhedron*, i.e., the intersection of $\text{conv}(K \cap \mathbb{Z}^n)$ with any polytope is also a polytope. To the best of our knowledge the most general sufficient conditions known for $\text{conv}(K \cap \mathbb{Z}^n)$ to be locally polyhedral are presented in [9] for the case where K is general polyhedron (not necessary rational). Coming up with necessary and sufficient conditions for $\text{conv}(K \cap \mathbb{Z}^n)$ to be locally polyhedron in the case where K is a general convex set is an interesting open question.

Another important question is determining necessary and sufficient conditions for the following optimization problem

$$\begin{aligned} z^* = &\min \langle d, x \rangle \\ &s.t. \quad x \in K \cap \mathbb{Z}^n, \end{aligned} \tag{8}$$

to be solvable, i.e., if z^* is bounded and $K \cap \mathbb{Z}^n \neq \emptyset$ implies there exists $x^* \in K \cap \mathbb{Z}^n$ such that $\langle d, x^* \rangle \leq \langle d, x \rangle \ \forall x \in K \cap \mathbb{Z}^n$. Clearly if $\text{conv}(K \cap \mathbb{Z}^n)$ is a rational polyhedron, then the optimization problem is solvable for any d. Another sufficient condition that can be easily verified is that d is a rational vector. However, finding general necessary and sufficient conditions for (8) to be solvable is a challenging question.

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References

1. A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli, *Maximal lattice-free convex sets in linear subspaces*, Mathematics of Operations Research **35** (2010), 704–720.
2. M. Conforti, G. Cornuéjols, and G. Zambelli, *Polyhedral approaches to mixed integer linear programming*, 50 Years of Integer Programming 1958-2008 (M. Jünger, T. Liebling, D. Naddef, G. L. Nemhauser, W. Pulleyblank, G. Reinelt, G. Rinaldi, and L. A. Wolsey, eds.), Springer Berlin Heidelberg, 2010, pp. 343–385.
3. R. Hemmecke, *Representations of lattice point sets: Theory, Algorithms, Applications.*, Ph.D. thesis, University Magdeburg, October 2006, Habilitation thesis.
4. R. Hemmecke and R. Weismantel, *Representation of sets of lattice points*, SIAM Journal on Optimization **18** (2007), 133–137.
5. F. Hüsseinov, *A note on the closedness of the convex hull and its applications*, Journal of Convex Analysis **6** (1999), 387–393.
6. V. L. Klee, *Extremal structure of convex sets*, Archiv der Mathematik **8** (1957), 234–240.
7. L. Lovász, *Geometry of numbers and integer programming*, Mathematical Programming: Recent Developments and Applications (1989).
8. R. R. Meyer, *On the existance of optimal solutions of integer and mixed-integer programming problems*, Mathematical Programming **7** (1974), 223–225.
9. J.-O. Moussafir, *Convex hulls of integral points*, Journal of Mathematical Sciences **113** (2003), 647–665.
10. G. T. Rockafeller, *Convex analysis*, Princeton University Press, New Jersey, NJ, 1970.
11. A. Schrijver, *Theory of linear and integer programming*, John Wiley & Sons, Inc., New York, NY, USA, 1986.