# Sensitivity analysis for mixed binary quadratic programming

Diego Cifuentes, Santanu S. Dey, Jingye Xu

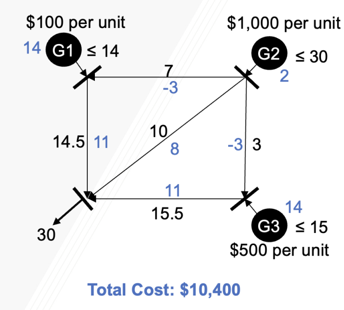
Aug, 2024

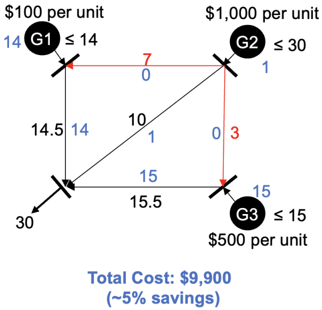
## Section 1

# Introduction: Sensitivity analysis for operations related IPs

# An example from power system: optimal transmission switching (OTS)

- Optimal transmission switching is an affordable way to mitigate congestion, allowing the dispatch of cheaper generators first and reducing the overall cost.



# Sensitivity analysis

$$\left.\begin{array}{rl} f(b) := \min & x^\top Q x + c^\top x \\ \text{s.t.} & Ax = b \\ & x \in \{0,1\}^{n_1} \times \mathbb{R}_+^{n_2} \end{array}\right\} \begin{array}{l} \text{Mixed binary} \\ \text{quadratic} \\ \text{program (MBQP)} \end{array}$$

# Sensitivity analysis

$$
\left.\begin{array}{lll} f(b) := \min & x^\top Q x + c^\top x \\ \text{s.t.} & Ax = b \\ & x \in \{0,1\}^{n_1} \times \mathbb{R}_+^{n_2} \end{array}\right\} \begin{array}{l}\text{Mixed binary} \\ \text{quadratic} \\ \text{program (MBQP)}\end{array}
$$

Sensitivity: If we have solved for $f(b)$, can we use this information to predict/obtain bounds on $f(b+\delta)$?

$$
\begin{array}{ll} f(b+\delta) := \min & x^\top Q x + c^\top x \\ \text{s.t.} & \underbrace{A}_{\text{remains same}} x = b + \delta \\ & \underbrace{x \in \{0,1\}^{n_1} \times \mathbb{R}_+^{n_2}.}_{\text{remains same}} \end{array}
$$

# Motivation for sensitivity analysis

$$\left.\begin{array}{lll} f(b) := \min & x^\top Q x + c^\top x \\ \text{s.t.} & Ax = b \\ & x \in \{0,1\}^{n_1} \times \mathbb{R}_+^{n_2} \end{array}\right\} \text{Mixed binary quadratic program (MBQP)}$$

Why we care about sensitivity analysis?

- Many operations related optimization tasks have this flavor. (Unit commitment [Lee, Leung, Margot (2004)], [Rajan, S Takriti (2005)], [Damcı-Kurt, Küçükyavuz,Atamtürk (2013)], [Knueven, J Ostrowski, JP Watson (2019)]; Various problems in supply chain last-mile delivery [Greening, Dahan, Erera (2021)])
- $A$ represents constraints regarding the physical/logical configuration, so remains the same
- $b$ represents instance specific information. Example: demand changes from instance to instance.

# Motivation for sensitivity analysis

$$
\left.\begin{array}{lll} f(b) := \min & x^\top Q x + c^\top x \\ \text{s.t.} & Ax = b \\ & x \in \{0,1\}^{n_1} \times \mathbb{R}^{n_2}_+ \end{array}\right\} \begin{array}{l}\text{Mixed binary} \\ \text{quadratic} \\ \text{program (MBQP)}\end{array}
$$

Why we care about sensitivity analysis?

- Many operations related optimization tasks have this flavor. (Unit commitment [Lee, Leung, Margot (2004)], [Rajan, S Takriti (2005)], [Damcı-Kurt, Küçükyavuz,Atamtürk (2013)], [Knueven, J Ostrowski, JP Watson (2019)]; Various problems in supply chain last-mile delivery [Greening, Dahan, Erera (2021)])
- $A$ represents constraints regarding the physical/logical configuration, so remains the same
- $b$ represents instance specific information. Example: demand changes from instance to instance.

| |
|---|
| Getting high quality dual-bound for similar instances without needing to start solving from scratch would be very useful in this setting. |

# Classical approach

- Classical result by [Cook, Gerards, Schrijver, Tardos (1992)] for MILPs.
- Nice improvements: [Eisenbrand, Weismantel (2019)], [Lee, Paat, Stallknecht, Xu (2020)][Celaya, Kuhlmann, Paat, Weismantel (2022)], [Del Pia, Ma (2022)].

| Opt Soln. MILP ($b$) | $\leftrightarrow$ | Opt Soln. LP relax ($b$) | $\leftrightarrow$ | Opt Soln. LP relax ($b+\delta$) | $\leftrightarrow$ | Opt Soln. MILP ($b+\delta$) |
|---|---|---|---|---|---|---|

# Classical approach

- Classical result by [Cook, Gerards, Schrijver, Tardos (1992)] for MILPs.
- Nice improvements: [Eisenbrand, Weismantel (2019)], [Lee, Paat, Stallknecht, Xu (2020)][Celaya, Kuhlmann, Paat, Weismantel (2022)], [Del Pia, Ma (2022)].

| Opt Soln. MILP ($b$) | $\leftrightarrow$ | Opt Soln. LP relax ($b$) | $\leftrightarrow$ | Opt Soln. LP relax ($b+\delta$) | $\leftrightarrow$ | Opt Soln. MILP ($b+\delta$) |
|---|---|---|---|---|---|---|

- *bound* $\infty -$ norm between opt. soln. of MILP($b$) and MILP($b+\delta$)

- This bound is useful for general integer case. In the binary case, such results are less useful, since the bounds are on the infinity norm of the integer solutions.

# This talk

- Complexity of sensitivity analysis.
- Sensitivity analysis via duality theorem.
- Some preliminary computational result.

Section 2

# Complexity of sensitivity analysis

# Some notation

$$
\begin{aligned}
f(b) = \quad \min \quad & x^\top Qx + c^\top x \\
\text{s.t.} \quad & Ax = b \\
& x \in \{0,1\}^{n_1} \times \mathbb{R}_+^{n_2}.
\end{aligned}
$$

- Sensitivity analysis is a computational task taking the following input
  - an MBQP instance $(A, b, c, Q, n_1)$
  - optimal value $f(b)$ and solution
  - Perturbation $\delta$

  Compute/approximate $\Delta f(\delta) := |f(b+\delta) - f(b)|$ as a function of $\delta$.

## Definition

An algorithm is called $(\alpha, \beta)$-approximation for some $\beta \geq 1 \geq \alpha > 0$ if it takes the above input and outputs $p$ satisfying:

$$\alpha \cdot \Delta f(\delta) \leq p \leq \beta \cdot \Delta f(\delta).$$

# Some notation

$$\begin{aligned} f(b) = \quad \min \quad & x^\top Q x + c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \in \{0,1\}^{n_1} \times \mathbb{R}_+^{n_2}. \end{aligned}$$

- Sensitivity analysis is a computational task taking the following input
  - an MBQP instance $(A, b, c, Q, n_1)$
  - optimal value $f(b)$ and solution
  - Perturbation $\delta$

  Compute/approximate $\Delta f(\delta) := |f(b+\delta) - f(b)|$ as a function of $\delta$.

## Definition

An algorithm is called $(\alpha, \beta)$-approximation for some $\beta \geq 1 \geq \alpha > 0$ if it takes the above input and outputs $p$ satisfying:

$$\alpha \cdot \Delta f(\delta) \leq p \leq \beta \cdot \Delta f(\delta).$$

Unlike classic approximation algorithm setting, two-sided bounds are required.

# NP-hardness of sensitivity analysis with respect to rhs changes

**Theorem (Cifuentes, D., Xu [2023])**

*It is NP-hard to achieve $(\alpha, \beta)$-approximation for any $\beta \geq 1 \geq \alpha > 0$ for general MBQPs, even if exactly one entry of b is changed by one.*

- The proof idea is to find some trivial IP, which becomes non-trivial after changing one entry of *b*.

Section 3

# Sensitivity analysis via duality

# Sensitivity analysis in convex conic programming

Given $\mathcal{K}$ a convex cone and $\mathcal{K}^*$ its dual cone where
$\mathcal{K}^* := \{x : \langle x, y\rangle \geq 0, \forall y \in \mathcal{K}\}$.

$$v_p(b) := \min_x c^\top x \qquad\qquad v_d := \max_y b^\top y$$

$$(P) \quad \text{s.t.} \quad Ax = b \qquad\qquad (D) \quad \text{s.t.} \quad A^* y + s = c$$

$$x \in \mathcal{K} \qquad\qquad s \in \mathcal{K}^*$$

# Sensitivity analysis in convex conic programming

Given $\mathcal{K}$ a convex cone and $\mathcal{K}^*$ its dual cone where
$\mathcal{K}^* := \{x : \langle x, y \rangle \geq 0, \forall y \in \mathcal{K}\}$.

$$v_p(b) := \min_x c^\top x \quad (P) \quad \text{s.t. } Ax = b, \; x \in \mathcal{K} \qquad \geq \qquad v_d := \max_y b^\top y \quad (D) \quad \text{s.t. } A^* y + s = c, \; s \in \mathcal{K}^*$$

Two nice properties:

- weak duality: $v_p \geq v_d$
- feasible region of (D) is independent of $b$

# Sensitivity analysis in convex conic programming

Given $\mathcal{K}$ a convex cone and $\mathcal{K}^*$ its dual cone where $\mathcal{K}^* := \{x : \langle x, y \rangle \geq 0, \forall y \in \mathcal{K}\}$.

$$
\begin{aligned}
v_p(b) := \min_x \; & c^\top x \\
(P) \quad \text{s.t.} \; & Ax = b \\
& x \in \mathcal{K}
\end{aligned}
\qquad \geq \qquad
\begin{aligned}
v_d := \max_y \; & b^\top y \\
(D) \quad \text{s.t.} \; & A^* y + s = c \\
& s \in \mathcal{K}^*
\end{aligned}
$$

Two nice properties:

- weak duality: $v_p \geq v_d$
- feasible region of (D) is independent of $b$

Find a good dual solution $y_0 \Rightarrow v_p(b + \delta) \geq (b + \delta)^\top y_0$

Sensitivity analysis is to find a good dual solution

# Sensitivity analysis in convex conic programming

Given $\mathcal{K}$ a convex cone and $\mathcal{K}^*$ its dual cone where
$\mathcal{K}^* := \{x : \langle x, y \rangle \geq 0, \forall y \in \mathcal{K}\}$.

$$
\begin{array}{rl}
v_p(b) := \min_x c^\top x & \\
(P) \;\; \text{s.t.} \;\; Ax = b & \\
x \in \mathcal{K} &
\end{array}
\quad \geq \quad
\begin{array}{rl}
v_d := \max_y b^\top y & \\
(D) \;\; \text{s.t.} \;\; A^* y + s = c & \\
s \in \mathcal{K}^* &
\end{array}
$$

Two nice properties:

- weak duality: $v_p \geq v_d$
- feasible region of (D) is independent of $b$

Find a good dual solution $y_0 \Rightarrow v_p(b + \delta) \geq (b + \delta)^\top y_0$

Sensitivity analysis is to find a good dual solution

# Sensitivity analysis in convex conic programming

Given $\mathcal{K}$ a convex cone and $\mathcal{K}^*$ its dual cone where
$\mathcal{K}^* := \{x : \langle x, y\rangle \geq 0, \forall y \in \mathcal{K}\}$.

$$
\begin{array}{lll}
v_p(b) := \min_x c^\top x & & v_d := \max_y b^\top y \\
(P) \quad \text{s.t. } Ax = b & \geq & (D) \quad \text{s.t. } A^* y + s = c \\
\qquad\qquad x \in \mathcal{K} & & \qquad\qquad s \in \mathcal{K}^*
\end{array}
$$

Two nice properties:

- weak duality: $v_p \geq v_d$
- feasible region of (D) is independent of $b$

Find a good dual solution $y_0 \Rightarrow v_p(b+\delta) \geq (b+\delta)^\top y_0$

Sensitivity analysis is to find a good dual solution

This leads to the following framework:

MBQP $\Longrightarrow$ convex conic relaxation $\Longrightarrow$ sensitivity analysis from dual

# Sensitivity analysis in convex conic programming

Given $\mathcal{K}$ a convex cone and $\mathcal{K}^*$ its dual cone where
$\mathcal{K}^* := \{x : \langle x, y \rangle \geq 0, \forall y \in \mathcal{K}\}$.

$$v_p(b) := \min_x c^\top x \qquad\qquad v_d := \max_y b^\top y$$
$$(P)\ \text{s.t. } Ax = b \quad \underbrace{=}_{?} \quad (D)\ \text{s.t. } A^* y + s = c$$
$$x \in \mathcal{K} \qquad\qquad s \in \mathcal{K}^*$$

Two nice properties:

- weak duality: $v_p \geq v_d$
- feasible region of (D) is independent of $b$

Find a good dual solution $y_0 \Rightarrow v_p(b+\delta) \geq (b+\delta)^\top y_0$

Sensitivity analysis is to find a good dual solution

This leads to the following framework:

MBQP $\underbrace{\Longrightarrow}_{\textit{may not be tight}}$ convex conic relaxation $\underbrace{\Longrightarrow}_{\text{not tight if no strong duality}}$ sensitivity analysis from dual

# $\mathcal{CP}$ and $\mathcal{COP}$

- The first question solved by [Burer (2009)].
- Considering the dual of Burer's conic reformulation was first suggested by [Guo, Bodur, Taylor (2021)].

# $\mathcal{CP}$ and $\mathcal{COP}$

- The first question solved by [Burer (2009)].
- Considering the dual of Burer's conic reformulation was first suggested by [Guo, Bodur, Taylor (2021)].

We require the following cones to describe the convex relaxation and it dual that we use:

- Completely positive cone
  $\mathcal{CP} = \{X \text{ is n-by-n symmetric matrix} : X = UU^\top \text{ for some } U \geq 0\}$

# $\mathcal{CP}$ and $\mathcal{COP}$

- The first question solved by [Burer (2009)].
- Considering the dual of Burer's conic reformulation was first suggested by [Guo, Bodur, Taylor (2021)].

We require the following cones to describe the convex relaxation and it dual that we use:

- Completely positive cone
  $\mathcal{CP} = \{X \text{ is n-by-n symmetric matrix} : X = UU^\top \text{ for some } U \geq 0\}$
- Its dual cone, copositive cone
  $\mathcal{COP} = \{X \text{ is n-by-n symmetric matrix} : y^\top Xy \geq 0, \forall y \geq 0\}$

# $\mathcal{CP}$ and $\mathcal{COP}$

- The first question solved by [Burer (2009)].
- Considering the dual of Burer's conic reformulation was first suggested by [Guo, Bodur, Taylor (2021)].

We require the following cones to describe the convex relaxation and it dual that we use:

- Completely positive cone
  $\mathcal{CP} = \{X \text{ is n-by-n symmetric matrix} : X = UU^\top \text{ for some } U \geq 0\}$
- Its dual cone, copositive cone
  $\mathcal{COP} = \{X \text{ is n-by-n symmetric matrix} : y^\top Xy \geq 0, \forall y \geq 0\}$

# Burer's result

$$f(b) = \begin{array}{ll} \min & x^\top Qx + 2c^\top x \\ \text{s.t.} & Ax = b \\ & x \in \{0,1\}^{n_1} \times \mathbb{R}_+^{n_2}. \end{array} \Bigg\} \text{ Mixed binary quadratic program (MBQP)}$$

# Burer's result

$$f(b) = \begin{array}{ll} \min & x^\top Qx + 2c^\top x \\ \text{s.t.} & Ax = b \\ & x \in \{0,1\}^{n_1} \times \mathbb{R}_+^{n_2}. \end{array}$$

Mixed binary quadratic program (MBQP)

$$f_{\mathcal{CP}}(b) = \begin{array}{ll} \inf & Q \cdot X + 2c^\top x \\ \text{s.t.} & a_i^\top x = b_i,\, i \in [m] \\ & a_i^\top X a_i = b_i^2,\, i \in [m] \\ & X_{jj} = x_j,\, \forall j \in [n_1] \\ & \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \in \mathcal{CP} \end{array}$$

Competely-positive program (CPP)

# Burer's result

$$f(b) = \begin{array}{ll} \min & x^\top Q x + 2c^\top x \\ \text{s.t.} & Ax = b \\ & x \in \{0,1\}^{n_1} \times \mathbb{R}^{n_2}_+ . \end{array} \quad \Bigg\} \text{ Mixed binary quadratic program (MBQP)}$$

It is convenient to work with the following notation:

$$\begin{array}{rll} f_{\mathcal{CP}}(b) := & \inf & \langle C, Y \rangle \\ & \text{s.t.} & \langle T, Y \rangle = 1 \\ \text{(Original constraints)} & & \langle A_i, Y \rangle = 2b_i, \forall i \in [m] \\ \text{(Square original constraint)} & & \langle AA_i, Y \rangle = b_i^2, \forall i \in [m] \\ \text{(implied by binary)} & & \langle N_j, Y \rangle = 0, \forall j \in [n_1] \\ & & Y \in \mathcal{CP} \end{array} \quad \Bigg\} \text{ Completely-positive program (CPP)}$$

where $C = \begin{bmatrix} 0 & c^\top \\ c & Q \end{bmatrix}$, $T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_i = \begin{bmatrix} 0 & a_i^\top \\ a_i & 0 \end{bmatrix}$,
$AA_i = \begin{bmatrix} 0 & 0 \\ 0 & a_i a_i^\top \end{bmatrix}$, $N_j = \begin{bmatrix} 0 & -e_j^\top \\ -e_j & 2e_j e_j^\top \end{bmatrix}$

# Burer's result

$$
\left.
\begin{array}{rll}
f_{\mathcal{CP}}(b) := & \inf & \langle C, Y \rangle \\
 & \text{s.t.} & \langle T, Y \rangle = 1 \\
\text{(Original constraints)} & & \langle A_i, Y \rangle = 2b_i, \forall i \in [m] \\
\text{(Square original constraint)} & & \langle AA_i, Y \rangle = b_i^2, \forall i \in [m] \\
\text{(implied by binary)} & & \langle N_j, Y \rangle = 0, \forall j \in [n_1] \\
 & & Y \in \mathcal{CP}
\end{array}
\right\}
\begin{array}{l}
\text{Completely-} \\
\text{positive} \\
\text{program} \\
\text{(CPP)}
\end{array}
$$

where

**Theorem ([Burer (2009)])**

$f(b) = f_{\mathcal{CP}}(b)$.

# Lets talk about the dual

$$\underbrace{\text{MBQP} \quad \equiv}_{\text{Burer[2009]}} \quad \underbrace{\text{Completely-positive reformulation}}_{\text{(CPP)}} \underbrace{\equiv}_{?} \underbrace{\text{Co-positive dual}}_{\text{(DUAL)}}$$

Completely-positive reformulation
(CPP)

$$\begin{aligned} f_{\mathcal{CP}} := \inf\ & \langle C, Y \rangle \\ \text{s.t.}\ & \langle T, Y \rangle = 1 \quad (\theta) \\ & \langle A_i, Y \rangle = 2b_i, \forall i \in [m] \quad (\alpha_i) \\ & \langle AA_i, Y \rangle = b_i^2, \forall i \in [m] \quad (\beta_i) \\ & \langle N_j, Y \rangle = 0, \forall j \in [n_1] \quad (\eta_i) \\ & Y \in \mathcal{CP} \end{aligned}$$

$$\underbrace{\equiv}_{?}$$

Co-positive dual
(DUAL)

$$\begin{aligned} f_{\text{DUAL}} := \sup\ & \theta + \\ & \textstyle\sum_{i=1}^{m}(\alpha_i \cdot 2b_i + \beta_i \cdot b_i^2) \\ \text{s.t.}\ & C - (\theta \cdot T + \\ & \textstyle\sum_{i=1}^{m} \alpha_i \cdot A_i + \\ & \textstyle\sum_{i=1}^{m} \beta_i \cdot AA_i + \\ & \textstyle\sum_{i=1}^{n_1} \eta_j \cdot N_j) \in \mathcal{COP} \end{aligned}$$

As discussed, weak duality always holds.
What about strong duality?

# Main result

## Theorem (Cifuentes, D., Xu [2023])

*Given a feasible and bounded MBQP, if*

- *Either the feasible region of MBQP is bounded (i.e., the continuous variables are bounded), or*
- *the objective function of the MBQP is convex,*

*then* $f_{\mathcal{CP}} = f_{\mathsf{DUAL}}$.

| | Obj. func. convex | Obj. func. not convex |
|---|---|---|
| Bounded Feas.R. | ✓ | ✓ |
| Unbounded Feas.R. | ✓ | |

# Main result

## Theorem (Cifuentes, D., Xu [2023])

*Given a feasible and bounded MBQP, if*

- *Either the feasible region of MBQP is bounded (i.e., the continuous variables are bounded), or*
- *the objective function of the MBQP is convex,*

*then* $f_{\mathcal{CP}} = f_{\mathsf{DUAL}}$.

| | Obj. func. convex | Obj. func. not convex |
|---|---|---|
| Bounded Feas.R. | ✓ | ✓ |
| Unbounded Feas.R. | ✓ | × |

# Main result

## Theorem (Cifuentes, D., Xu [2023])

*Given a feasible and bounded MBQP, if*

- *Either the feasible region of MBQP is bounded (i.e., the continuous variables are bounded), or*
- *the objective function of the MBQP is convex,*

*then* $f_{\mathcal{CP}} = f_{\mathsf{DUAL}}$.

| | Obj. func. convex | Obj. func. not convex |
|---|---|---|
| Bounded Feas.R. | ✓ | ✓ |
| Unbounded Feas.R. | ✓ | |

- [Brown, Bernal Neira, Venturelli, Pavone (2022)] Proved the bounded case. [non-constructive proof]
- [Guo, Bodur, Taylor (2021)] Empirically validated these results.
- [Linderoth, Raghunathan (2022)]

# Non-convex, unbounded feasible region

Consider the following example:

$$\begin{aligned} \min \quad & x_1^2 - x_2^2 \\ \text{s.t.} \quad & x_1 - x_2 = 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Opt value of above M(B)QP is 0, i.e. $f = 0$.
- By Burer's result, $f_{\mathcal{CP}} = f = 0$.

# Non-convex, unbounded feasible region

Consider the following example:

$$\begin{aligned} \min \quad & x_1^2 - x_2^2 \\ \text{s.t.} \quad & x_1 - x_2 = 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Opt value of above M(B)QP is 0, i.e. $f = 0$.
- By Burer's result, $f_{\mathcal{CP}} = f = 0$.
- The co-positive dual is infeasible! Proof:

# Non-convex, unbounded feasible region

Consider the following example:

$$\begin{aligned} \min \quad & x_1^2 - x_2^2 \\ \text{s.t.} \quad & x_1 - x_2 = 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Opt value of above M(B)QP is 0, i.e. $f = 0$.
- By Burer's result, $f_{\mathcal{CP}} = f = 0$.
- The co-positive dual is infeasible! Proof:

$$M := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$M \in \quad \mathcal{COP}$.

Consider $y = \begin{bmatrix} 0 \\ 1 \\ 1 + \epsilon \end{bmatrix}$,

$$\begin{aligned} y^\top M y &= 1 - (1+\epsilon)^2 + \beta(1 + (1+\epsilon)^2 - 2(1+\epsilon)) \\ &= -\epsilon^2 - 2\epsilon + \beta\epsilon^2 < 0 \text{ (for sufficiently small } \epsilon) \end{aligned}$$

# Strong duality result - our proof is constructive

## Theorem

*Consider a feasible and bounded instance of MBQP where either the feasible region is bounded or objective function is convex. Given:*

- *a valid low bound $l$ on the objective function value of MBQP, and*
- $\epsilon > 0$

*then there we can construct a feasible solution to the DUAL and its objective value is at least $l - \epsilon$.*

# Strong duality result - our proof is constructive

## Theorem

*Consider a feasible and bounded instance of MBQP where either the feasible region is bounded or objective function is convex. Given:*

- *a valid low bound $l$ on the objective function value of MBQP, and*
- $\epsilon > 0$

*then there we can construct a feasible solution to the DUAL and its objective value is at least $l - \epsilon$.*

- This gives an alternate proof of Burer's Theorem for the case of bounded feasible region or convex objective function.

# Strong duality result - our proof is constructive

**Theorem**

*Consider a feasible and bounded instance of MBQP where either the feasible region is bounded or objective function is convex. Given:*

- *a valid low bound $l$ on the objective function value of MBQP, and*
- $\epsilon > 0$

*then there we can construct a feasible solution to the DUAL and its objective value is at least $l - \epsilon$.*

- This gives an alternate proof of Burer's Theorem for the case of bounded feasible region or convex objective function.
- We can construct copositive dual solutions "easily" and start obtaining dual bounds for perturbed instances.

# Brief comments on proof techniques

- A key lemma in the construction of the COP solution is the following.

## Lemma (Stability)

$$f(b, \epsilon_0) = \begin{array}{ll} \min_{x,\epsilon^{(1)},\epsilon^{(2)}} & x^\top Qx + 2c^\top x \\ \text{s.t.} & Ax = b + \epsilon^{(1)} \\ & x_j + \epsilon_j^{(2)} \in \{0,1\} \ \forall j \in [n_1] \\ & x_j \geq 0 \ \forall j \in [n] \\ & |\epsilon^{(1)}|_\infty \leq \epsilon_0 \\ & |\epsilon^{(2)}|_\infty \leq \epsilon_0 \end{array} \left.\vphantom{\begin{array}{l} a\\a\\a\\a\\a\\a \end{array}}\right\} \begin{array}{l}\text{Perturbed} \\ \text{mixed} \\ \text{binary} \\ \text{quadratic} \\ \text{program}\end{array}$$

*If the feasible region of MBQP is bounded or* $Q \succeq 0$*, then there exists* $\epsilon^* > 0$ *and* $s \in \mathbb{R}$ *that only depend on* $A, b, c, Q$ *such that*

$$f(b, \epsilon_0) \geq f(b, 0) - s \cdot \epsilon_0,$$

*for all* $0 \leq \epsilon_0 < \epsilon^*$.

# When the stability lemma fails

Example where strong duality does not hold:

$$
\begin{aligned}
0 = f(0) := \quad \min \quad & x_1^2 - x_2^2 \\
\text{s.t.} \quad & x_1 - x_2 = 0 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

# When the stability lemma fails

Example where strong duality does not hold:

$$
\begin{aligned}
f(\epsilon_0) := \quad \min \quad & x_1^2 - x_2^2 \\
\text{s.t.} \quad & x_1 - x_2 = \epsilon \\
& x_1, x_2 \geq 0 \\
& |\epsilon| \leq \epsilon_0
\end{aligned}
$$

# When the stability lemma fails

Example where strong duality does not hold:

$$\begin{aligned} f(\epsilon_0) := \quad \min \quad & x_1^2 - x_2^2 \\ \text{s.t.} \quad & x_1 - x_2 = \epsilon \\ & x_1, x_2 \geq 0 \\ & |\epsilon| \leq \epsilon_0 \end{aligned}$$

For any fixed $\epsilon < 0$
$\lim_{x_1 \to \infty} x_1^2 - (x_1 - \epsilon)^2 = -\infty$

# When the stability lemma fails

Example where strong duality does not hold:

$$\left.\begin{array}{rll} -\infty = f(\epsilon_0) := & \min & x_1^2 - x_2^2 \\ & \text{s.t.} & x_1 - x_2 = \epsilon \\ & & x_1, x_2 \geq 0 \\ & & |\epsilon| \leq \epsilon_0 \end{array}\right\} \forall \epsilon_0 > 0$$

So the stability lemma does not hold here.

# Sketch of constructive proof

For simplicity, we consider pure binary case.

$$
\begin{array}{l}
f_{\mathrm{DUAL}} := \sup \ -\theta - \sum_{i=1}^{m} (\alpha_i \cdot 2b_i + \beta_i \cdot b_i^2) \\
\text{s.t. } C + (\theta \cdot T \ + \sum_{i=1}^{m} \alpha_i \cdot A_i + \sum_{i=1}^{m} \beta_i \cdot AA_i + \sum_{i=1}^{n_1} \eta_j \cdot N_j) \in \mathcal{COP}
\end{array}
$$

# Sketch of constructive proof

For simplicity, we consider pure binary case.

$$f_{\text{DUAL}} := \sup \; -\theta - \textstyle\sum_{i=1}^{m}(\alpha_i \cdot 2b_i + \beta_i \cdot b_i^2)$$
$$\text{s.t. } C + (\theta \cdot T \; + \textstyle\sum_{i=1}^{m} \alpha_i \cdot A_i + \sum_{i=1}^{m} \beta_i \cdot AA_i + \sum_{i=1}^{n_1} \eta_j \cdot N_j) \in \mathcal{COP}$$

Given any lower bound $l$ and $\epsilon$, we construct $M$ such that

- $M = C + \text{Block } 1 + \text{Block } 2 + \text{Block } 3$
- Each block $= \underbrace{\theta' \cdot T \; + \sum_{i=1}^{m} \alpha_i' \cdot A_i + \sum_{i=1}^{m} \beta_i' \cdot AA_i + \sum_{i=1}^{n_1} \eta_j' \cdot N_j}_{\text{some combination of dual variables}}$

# Sketch of constructive proof

For simplicity, we consider pure binary case.

$$f_{\text{DUAL}} := \sup \; -\theta - \sum_{i=1}^{m} (\alpha_i \cdot 2b_i + \beta_i \cdot b_i^2)$$
$$\text{s.t. } C + (\theta \cdot T + \sum_{i=1}^{m} \alpha_i \cdot A_i + \sum_{i=1}^{m} \beta_i \cdot AA_i + \sum_{i=1}^{n_1} \eta_j \cdot N_j) \in \mathcal{COP}$$

Given any lower bound $l$ and $\epsilon$, we construct $M$ such that

- $M = C + \text{Block 1} + \text{Block 2} + \text{Block 3}$
- Each block $= \underbrace{\theta' \cdot T + \sum_{i=1}^{m} \alpha_i' \cdot A_i + \sum_{i=1}^{m} \beta_i' \cdot AA_i + \sum_{i=1}^{n_1} \eta_j' \cdot N_j}_{\text{some combination of dual variables}}$

Goal:

- The total objective value is $l - \epsilon$
- $M$ is copositive: $y^\top M y \geq 0, \forall y = \begin{bmatrix} t \\ x \end{bmatrix} \geq 0.$

# Sketch of constructive proof

For simplicity, we consider pure binary case.

$$f_{\text{DUAL}} := \sup \ -\theta - \sum_{i=1}^{m}(\alpha_i \cdot 2b_i + \beta_i \cdot b_i^2)$$
$$\text{s.t.} \ C + (\theta \cdot T \ + \sum_{i=1}^{m} \alpha_i \cdot A_i + \sum_{i=1}^{m} \beta_i \cdot AA_i + \sum_{i=1}^{n_1} \eta_j \cdot N_j) \in \mathcal{COP}$$

Given any lower bound $l$ and $\epsilon$, we construct $M$ such that

- $M = C + \text{Block 1} + \text{Block 2} + \text{Block 3}$
- Each block $= \underbrace{\theta' \cdot T \ + \sum_{i=1}^{m} \alpha_i' \cdot A_i + \sum_{i=1}^{m} \beta_i' \cdot AA_i + \sum_{i=1}^{n_1} \eta_j' \cdot N_j}_{\text{some combination of dual variables}}$

Goal:

- The total objective value is $l - \epsilon$
- $M$ is copositive: $y^\top My \geq 0, \forall y = \begin{bmatrix} t \\ x \end{bmatrix} \geq 0.$

For our construction, $t = 0$ is easy to check. Sufficient to assume $t = 1$.

# Each Block

There exists some closed-form formula:
Block1

- PSD and has objective value 0
- $(\begin{bmatrix} 1 \\ x \end{bmatrix})^\top$Block 1$(\begin{bmatrix} 1 \\ x \end{bmatrix})$ is large positive number if $x$ significantly violates original linear constraints. (If $(a^i)^\top x = b_i + \epsilon$ and "$|\epsilon|$ is large")

# Each Block

There exists some closed-form formula:

Block1

- PSD and has objective value 0
- $(\begin{bmatrix} 1 \\ x \end{bmatrix})^\top \text{Block 1}(\begin{bmatrix} 1 \\ x \end{bmatrix})$ is large positive number if $x$ significantly violates original linear constraints. (If $(a^i)^\top x = b_i + \epsilon$ and "$|\epsilon|$ is large")

Block2

- Copositive and has objective value arbitrarily close to 0
- $(\begin{bmatrix} 1 \\ x \end{bmatrix})^\top \text{Block 2}(\begin{bmatrix} 1 \\ x \end{bmatrix})$ is large positive number if $x$ significantly violates being binary

Block3

# Each Block

There exists some closed-form formula:

Block1

- PSD and has objective value 0
- $(\begin{bmatrix}1\\x\end{bmatrix})^\top$Block 1$(\begin{bmatrix}1\\x\end{bmatrix})$ is large positive number if $x$ significantly violates original linear constraints. (If $(a^i)^\top x = b_i + \epsilon$ and "$|\epsilon|$ is large")

Block2

- Copositive and has objective value arbitrarily close to 0
- $(\begin{bmatrix}1\\x\end{bmatrix})^\top$Block 2$(\begin{bmatrix}1\\x\end{bmatrix})$ is large positive number if $x$ significantly violates being binary

Block3

- PSD and has objective value $l - \epsilon$
- $(\begin{bmatrix}1\\x\end{bmatrix})^\top$Block 3$(\begin{bmatrix}1\\x\end{bmatrix}) \geq -l + \hat{\epsilon}$

# Combining everything together

Remember $M = C + \underbrace{\text{Block 1}}_{\textit{copositive}} + \underbrace{\text{Block 2}}_{\textit{copositive}} + \underbrace{\text{Block 3}}_{\textit{copositive}}$.

# Combining everything together

Remember $M = C + \underbrace{\text{Block 1}}_{copositive} + \underbrace{\text{Block 2}}_{copositive} + \underbrace{\text{Block 3}}_{copositive}$.

For any $y = \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0$, we partition $x$ into three cases:

- $x$ "significantly" violates original linear constraints, then

$$y^\top My \geq y^\top Cy + \underbrace{y^\top Block1y}_{very\ \ large\ \ postive\ \ number} \geq 0$$

# Combining everything together

Remember $M = C + \underbrace{\text{Block 1}}_{copositive} + \underbrace{\text{Block 2}}_{copositive} + \underbrace{\text{Block 3}}_{copositive}$.

For any $y = \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0$, we partition $x$ into three cases:

- $x$ "significantly" violates original linear constraints, then

$$y^\top My \geq y^\top Cy + \underbrace{y^\top Block1y}_{very\ \ large\ \ postive\ \ number} \geq 0$$

- $x$ "significantly" violates being binary, then

$$y^\top My \geq y^\top Cy + \underbrace{y^\top Block2y}_{very\ \ large\ \ postive\ \ number} \geq 0$$

# Combining everything together

Remember $M = C + \underbrace{\text{Block 1}}_{copositive} + \underbrace{\text{Block 2}}_{copositive} + \underbrace{\text{Block 3}}_{copositive}$.

For any $y = \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0$, we partition $x$ into three cases:

- $x$ "significantly" violates original linear constraints, then

$$y^\top My \geq y^\top Cy + \underbrace{y^\top Block1y}_{very\ \ large\ \ postive\ \ number} \geq 0$$

- $x$ "significantly" violates being binary, then

$$y^\top My \geq y^\top Cy + \underbrace{y^\top Block2y}_{very\ \ large\ \ postive\ \ number} \geq 0$$

- $x$ "almost" satisfies original linear constraints and "almost" being binary, then stability lemma implies that

$$y^\top Cy \approx l$$

and

$$y^\top My \geq y^\top Cy + y^\top Block3y \geq l - l + \hat{\epsilon} \geq 0$$

# Existence of infinitely many solution

- Given $l, \epsilon$, one can construct nearly optimal dual solution without solving copositive programming.

# Existence of infinitely many solution

- Given $I, \epsilon$, one can construct nearly optimal dual solution without solving copositive programming.
- Remember Block 1 is PSD with zero objective
- Let $M$ be an $\epsilon$-optimal dual solution, then

$$M^* = M + r \underbrace{\text{Block 1}}_{\text{recession direction}}$$

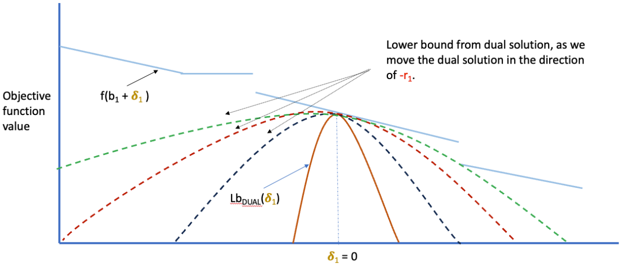
is also $\epsilon$-optimal dual solution for any $r > 0$.

# Existence of infinitely many solution

- Given $I, \epsilon$, one can construct nearly optimal dual solution without solving copositive programming.
- Remember Block 1 is PSD with zero objective
- Let $M$ be an $\epsilon$-optimal dual solution, then

$$M^* = M + r \underbrace{\text{Block 1}}_{\text{recession direction}}$$

  is also $\epsilon$-optimal dual solution for any $r > 0$.
- $M^*, M$ are dual solution with the same objective. Larger $r$ provides a weaker sensitivity analysis. Our constructed solution has too large $r$.
- Subtracting multiples of $r$ improves the quality of the sensitivity analysis.



# Select for the "best" optimal dual solution

We want DUAL optimal solution that have small contributions from $r$.

- $$\begin{array}{ll} \inf_\lambda & \lambda \\ \text{s.t.} & \underbrace{M}_{\text{our closed-form formula}} + \lambda \cdot \text{Block1} \in \mathcal{COP} \end{array}$$

# Select for the "best" optimal dual solution

We want DUAL optimal solution that have small contributions from $r$.

- $$\begin{array}{ll} \inf_{\lambda} & \lambda \\ \text{s.t.} & \underbrace{M}_{\text{our closed-form formula}} + \lambda \cdot \text{Block1} \in \mathcal{COP} \end{array}$$

- Actual problem solved in experiments involves solving a restriction of $\mathcal{COP}$ with some other 'engineering' tricks.

Section 4

# Preliminary computational results

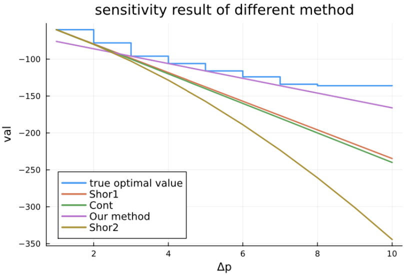
# Stable set with side cardinality constraint

Given a bipartite graph $G = (V_1 \cup V_2, E)$, we consider the following instances:

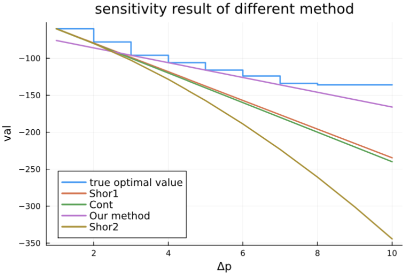
$$\begin{array}{lll} \min_x & -2c^\top x & \\ \text{s.t.} & e^\top x \le k & \text{(Cardinality constraint, Changing k)} \\ & x_i + x_j \le 1, \forall (i,j) \in E & \text{(Stable set)} \\ & x \in \{0,1\}^{|V_1|+|V_2|} & \end{array}$$

- $G$ is random bipartite graph with $|V_1| = |V_2| = 10$ and each edge $(i, j)$ is present in $E$ with probability $\{0.3, 0.5, 0.7\}$ and each entries of $c_i$ is uniformly sampled from $\{0, \ldots, 10\}$;
- Each entry of $a_i$ is uniformly sampled from $\{0, \ldots, 10\}$.
- For each setting, 20 instances are generated.

# Sensitivity with respect to rhs of Cardinality constraint (k)



# Sensitivity with respect to rhs of Cardinality constraint (k)




$$\text{Gap} = \frac{\text{IP}-\text{Dual Val}}{\text{IP}-\text{Shor}}$$

Table: Average relative gap

| $\Delta k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | avg time(s) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Shor1 (SDP) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 7.30 |
| Shor2 (SDP) | 1.33 | 2.04 | 2.7 | 2.71 | 2.76 | 2.79 | 2.87 | 2.88 | 2.94 | 3.03 | 10.17 |
| Our method | 0.83 | 0.02 | 0.00 | 0.11 | 0.19 | 0.26 | 0.32 | 0.38 | 0.41 | 0.44 | 8.35 |
| Cont (LP) | 0.97 | 1.07 | 1.09 | 1.08 | 1.07 | 1.06 | 1.05 | 1.05 | 1.04 | 1.04 | 0.00 |

# Fixed charge models

$$\begin{aligned} \min_{x,y} \; & -2c^\top x + 2d^\top y \\ \text{s.t. } & a_i^\top x \le b_i, \forall i \in [m] \quad \text{(LP constraints, Changing b)} \\ & x_i \le y_i, \forall i \in [n] \quad \text{(Fixed charge constraints)} \\ & x \ge 0, y \in \{0,1\}^n. \end{aligned}$$

- $n = 20, m = 5$
- Each entry of $c$ is uniformly sampled from $[0, 5]$ and $d$ is all ones vector.
- Each entry of $a_i$ is uniformly sampled from $\{0, \ldots, 10\}$

# Sensitivity with respect to rhs of $Ax \leq b$

$$
\begin{aligned}
\min_{x,y}\ & -2c^\top x + 2d^\top y \\
\text{s.t. } & a_i^\top x \leq b_i, \forall i \in [m] \quad \text{(LP constraints, Changing b)} \\
& x_i \leq y_i, \forall i \in [n] \quad \text{(Fixed charge constraints)} \\
& x \geq 0, y \in \{0,1\}^n.
\end{aligned}
$$

Table: Average relative gap for (SSLP) – all densities

| $\|\Delta b\|_\infty$ | $\leq 1$ | $\leq 2$ | $\leq 3$ | avg time(s) |
|---|---|---|---|---|
| Shor1 (SDP) | 1 | 1 | 1 | 3.63 |
| Shor2 (SDP) | 1.20 | 1.48 | 1.64 | 7.21 |
| Our method | 0.59 | 0.55 | 0.60 | 5.82 |
| Cont | 1.00 | 1.00 | 1.00 | 0.00 |

# Conclusions

- We formally studied the computational complexity of sensitivity analysis.
- On the dual side, we analyzed the COP-dual of Burer's CPP reformulation, its properties and use.

# Conclusions

- We formally studied the computational complexity of sensitivity analysis.
- On the dual side, we analyzed the COP-dual of Burer's CPP reformulation, its properties and use.

<u>Future directions:</u>

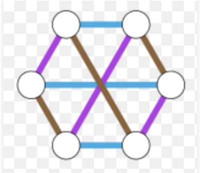
- Find faster ways to solve the *modified COP-duals.*
- More general problems than MBQPs, for example, general quadratically constrained quadratic programs.

*Thank you!*

https://arxiv.org/abs/2312.06714

# NP-hardness of sensitivity analysis with respect to rhs changes

Let $G = (V, E)$ be a simple graph. An edge coloring of $G$ is an assignment of colors to edges so that no incident edge will have the same color. The minimum number of colors required is called *edge chromatic number* and denoted by $\chi'(G)$.



## Theorem (Vizing theorem)

*For any simple graph $G$, $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$ where $\Delta(G)$ is the maximum degree of vertices in $G$.*

## Theorem

*It is NP-hard to determine the edge chromatic number of cubic graphs, which is to distinguish $\chi'(G) = 3$ or $\chi'(G) = 4$*

# NP-hardness of sensitivity analysis with respect to rhs changes

## Theorem

*For any simple graph $G$, $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$ where $\Delta(G)$ is the maximum degree of vertices in $G$.*

Let $G$ be a cubic graph.

$$
\begin{aligned}
z_1 := \min & \sum_{i \in [H]} w_i \\
\text{s.t. } & \sum_{i \in [H]} w_i \geq 4 \\
& \sum_{i \in [H]} x_{ei} = 1, \forall e \in E \\
& x_{ri} + x_{si} \leq w_i, \forall (r, s) \text{ adjacent} \\
& x \in \{0,1\}^{|E| \times [H]}, w \in \{0,1\}^{[H]}
\end{aligned}
$$

# NP-hardness of sensitivity analysis with respect to rhs changes

## Theorem

*For any simple graph $G$, $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$ where $\Delta(G)$ is the maximum degree of vertices in $G$.*

Let $G$ be a cubic graph.

$$
\begin{aligned}
z_1 := \min & \sum_{i \in [H]} w_i \\
\text{s.t.} & \sum_{i \in [H]} w_i \geq 4 \\
& \sum_{i \in [H]} x_{ei} = 1, \forall e \in E \\
& x_{ri} + x_{si} \leq w_i, \forall (r,s) \text{ adjacent} \\
& x \in \{0,1\}^{|E| \times [H]}, w \in \{0,1\}^{[H]}
\end{aligned}
$$

$$
\begin{aligned}
z_2 := \min & \sum_{i \in [H]} w_i \\
\text{s.t.} & \sum_{i \in [H]} w_i \geq 3 \\
& \sum_{i \in [H]} x_{ei} = 1, \forall e \in E \\
& x_{ri} + x_{si} \leq w_i, \forall (r,s) \text{ adjacent} \\
& x \in \{0,1\}^{|E| \times [H]}, w \in \{0,1\}^{[H]}
\end{aligned}
$$

# NP-hardness of sensitivity analysis with respect to rhs changes

## Theorem

*For any simple graph $G$, $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$ where $\Delta(G)$ is the maximum degree of vertices in $G$.*

Let $G$ be a cubic graph.

$$
\begin{aligned}
z_1 := \min & \sum_{i\in[H]} w_i \\
\text{s.t.} & \sum_{i\in[H]} w_i \geq 4 \\
& \sum_{i\in[H]} x_{ei} = 1, \forall e \in E \\
& x_{ri} + x_{si} \leq w_i, \forall (r,s) \text{ adjacent} \\
& x \in \{0,1\}^{|E|\times[H]}, w \in \{0,1\}^{[H]}
\end{aligned}
$$

$$
\begin{aligned}
z_2 := \min & \sum_{i\in[H]} w_i \\
\text{s.t.} & \sum_{i\in[H]} w_i \geq 3 \\
& \sum_{i\in[H]} x_{ei} = 1, \forall e \in E \\
& x_{ri} + x_{si} \leq w_i, \forall (r,s) \text{ adjacent} \\
& x \in \{0,1\}^{|E|\times[H]}, w \in \{0,1\}^{[H]}
\end{aligned}
$$

- Any $(\alpha, \beta)$-approximation to predict $|z_1 - z_2|$ is equivalent to deciding $\chi'(G) = 3$ or $4$

# Sketch of constructive proof - Block 1

For each constraints $i \in [m]$,

$$R_i := b_i^2 T - b_i A_i + AA_i = \begin{bmatrix} b_i^2 & -b_i a_i^\top \\ -b_i a_i & b_i^2 a_i a_i^\top \end{bmatrix}$$

- $R_i$ is psd and therefore copositive.
- $R_i$ has zero objective value.

# Sketch of constructive proof - Block 1

For each constraints $i \in [m]$,

$$R_i := b_i^2 T - b_i A_i + AA_i = \begin{bmatrix} b_i^2 & -b_i a_i^\top \\ -b_i a_i & b_i^2 a_i a_i^\top \end{bmatrix}$$

- $R_i$ is psd and therefore copositive.
- $R_i$ has zero objective value.
- If $y = \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0$, $y^\top R_i y = (b_i - a_i^\top x)^2$
- If $x$ significantly violates $b_i = a_i^\top x$, then $y^\top R_i y$ is a large positive number.

# Sketch of constructive proof - Block 1

For each constraints $i \in [m]$,

$$R_i := b_i^2 T - b_i A_i + AA_i = \begin{bmatrix} b_i^2 & -b_i a_i^\top \\ -b_i a_i & b_i^2 a_i a_i^\top \end{bmatrix}$$

- $R_i$ is psd and therefore copositive.
- $R_i$ has zero objective value.
- If $y = \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0$, $y^\top R_i y = (b_i - a_i^\top x)^2$
- If $x$ significantly violates $b_i = a_i^\top x$, then $y^\top R_i y$ is a large positive number.

$$\text{Block } 1 = \underbrace{t_1}_{a\ \ positive\ \ scalar} \sum_{i \in [m]} R_i$$

- $y^\top (Block1) y$ is large if $x$ significantly violates original linear constraints.

# Sketch of constructive proof - Block 2

For any $j \in [n1]$,

$$W_j := f_j \sum_{i \in m} R_i - N_j + r_j T$$

## Lemma

*For any positive $r_j > 0$, there exists some $f_j$ such that $W_j$ is copositive*

- $W_j$ is not psd. This distinguishes $\mathcal{COP}$ relaxation from SDP relaxation.
- $W_j$ has arbitrarily small objective value.

# Sketch of constructive proof - Block 2

For any $j \in [n1]$,

$$W_j := f_j \sum_{i \in m} R_i - N_j + r_j T$$

## Lemma

*For any positive $r_j > 0$, there exists some $f_j$ such that $W_j$ is copositive*

- $W_j$ is not psd. This distinguishes $\mathcal{COP}$ relaxation from SDP relaxation.
- $W_j$ has arbitrarily small objective value.
- If $y = \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0$, $y^\top W_j y \geq \max\{0, x_j(1 - x_j)\}$
- If $x_j$ significantly violates $x_j \in \{0, 1\}$, then $y^\top W_j y$ is a large positive number.

# Sketch of constructive proof - Block 2

For any $j \in [n1]$,

$$W_j := f_j \sum_{i \in m} R_i - N_j + r_j T$$

## Lemma

*For any positive $r_j > 0$, there exists some $f_j$ such that $W_j$ is copositive*

- $W_j$ is not psd. This distinguishes $\mathcal{COP}$ relaxation from SDP relaxation.
- $W_j$ has arbitrarily small objective value.
- If $y = \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0$, $y^\top W_j y \geq \max\{0, x_j(1 - x_j)\}$
- If $x_j$ significantly violates $x_j \in \{0, 1\}$, then $y^\top W_j y$ is a large positive number.

$$\text{Block } 2 = \underbrace{t_2}_{a\ \ positive\ \ scalar} \sum_{j \in [n1]} W_j$$

- $y^\top (Block2) y$ is large if $x$ significantly violates being binary

# Sketch of constructive proof - Block 3

$$\text{Block3} = \underbrace{t_3}_{a\ \ positive\ \ scalar} \sum_{i \in m} AA_i - IT = \begin{bmatrix} -I & 0 \\ 0 & t_3 \sum_{i \in m} a_i^\top a_i \end{bmatrix}$$

- $\text{Block3}_{x,x}$ is strictly copositive.
- Block3 has objective value $I - \epsilon$ by choosing $t_3$ properly

# Sketch of constructive proof - Block 3

$$\text{Block3} = \underbrace{t_3}_{a \ \ positive \ \ scalar} \sum_{i \in m} AA_i - IT = \begin{bmatrix} -I & 0 \\ 0 & t_3 \sum_{i \in m} a_i^\top a_i \end{bmatrix}$$

- $\text{Block3}_{x,x}$ is strictly copositive.
- Block3 has objective value $I - \epsilon$ by choosing $t_3$ properly
- For $y = \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0$, $y^\top (Block3) y \geq -I + \underbrace{\epsilon_1}_{small \ \ postive \ \ number}$