Regularized MIP Model for Optimal Power Flow with Energy Storage Systems and its Applications

Dahye Han*[1], Nan Jiang†[1], Santanu S. Dey‡[1], and Weijun Xie§[1]

[1]School of Industrial and Systems Engineering, Georgia Institute of Technology

Abstract

Incorporating energy storage systems (ESS) into power systems has been studied in many recent works, where binary variables are often introduced to model the complementary nature of battery charging and discharging. A conventional approach for these ESS optimization problems is to relax binary variables and convert the problem into a linear program. However, such linear programming relaxation models can yield unrealistic fractional solutions, such as simultaneous charging and discharging. In this paper, we develop a regularized Mixed-Integer Programming (MIP) model for the ESS optimal power flow (OPF) problem. We prove that under mild conditions, the proposed regularized model admits a zero integrality gap with its linear programming relaxation; hence, it can be solved efficiently. By studying the properties of the regularized MIP model, we show that its optimal solution is also near-optimal to the original ESS OPF problem, thereby providing a valid and tight upper bound for the ESS OPF problem. The use of the regularized MIP model allows us to solve two intractable problems: a two-stage stochastic ESS OPF problem and a trilevel min-max-min network contingency problem.

1 Introduction

Modern electrical grids have undergone significant transformations in the past few decades with increased integration of renewable energy resources and distributed energy resources. In spite of many benefits brought by these new entrants, power grids are also experiencing increased uncertainties due to inherent dependency on weather and short-term demand forecasts, which are challenging to accurately predict. To mitigate these challenges, many Independent System Operators (ISOs) are turning their attention to energy storage systems (ESS), also referred to as batteries for convenience in this paper. In their most recent annual study, the U.S. Energy Information Administration (2023) predicted 160 gigawatts of total installed battery storage capacity in the U.S. by the year 2050.

Adding batteries to the electrical grids raises many new optimization concerns, encompassing both the market and operational sides, that must be addressed (Gür 2018). Hoffman et al. (2011) pointed out the lack of analytical models to optimally incorporate batteries into the grid. Sioshansi et al. (2022) discussed the challenges in energy storage modeling and summarized several essential constraints to be added for energy storage models. These constraints form the basis of many

*dahye.han@gatech.edu
†nanjiang@gatech.edu
‡santanu.dey@isye.gatech.edu
§wxie@gatech.edu

works that embed ESS in their corresponding optimization models (see the details in Löhndorf and Wozabal 2023, Wu et al. 2023).

Some previous studies that consider ESS in optimization problems include unit commitment problems with hydro storage (see, e.g., Jiang et al. 2012), economic dispatch problems (see, e.g., Yan et al. 2016), optimal bidding strategy for battery operators (see, e.g., Jiang and Powell 2015), planning problems for wind farm and battery sitting (see, e.g., Qi et al. 2015), and control policy problems for optimizing the revenue of battery operators (see, e.g., Salas and Powell 2018).

Incorporating batteries in the optimal power flow (OPF) problems, which aim to reduce unmet demand or overproduction in the system while meeting physical constraints in the electrical grid, adds a unique complexity because a battery can act as both a demand and a generator. When a battery is being charged, it is viewed as a demand, whereas when it is discharged, it works as another generator in the system. This complementary bilinear constraint can be formulated as a linear constraint using binary variables to indicate whether the battery is charging or discharging at a certain time, making it a mixed-integer program (MIP). We refer to this problem as the optimal power flow problem with batteries or energy storage systems.

In this paper, we analyze and propose a new regularized MIP model for the optimal power flow problem with batteries.

1.1 Relevant Literature

The OPF problem itself is a long-studied topic within the domain of power and energy systems (Carpentier 1962), where an accurate solution to the power flow problem is obtained by using the AC power flow model. However, due to the computational intractability arising from its nonlinear and nonconvex nature, ongoing efforts have been made to reformulate or relax the AC power flow model into a convex one (see, for example, details in Jabr 2008, Lavaei and Low 2012, Kocuk et al. 2016, Coffrin et al. 2015). The DC power flow model, which ignores reactive power, is by far the most widely used model. ISOs also employ the DC power flow model in the current electricity market. Moreover, for long-term planning problems, the DC approximation appears to be reasonable (see, e.g., Cole et al. 2017). Therefore, we focus on the DC model of power flow equations.

The incorporation of ESS into DCOPF problems has been of interest for more than a decade. Chandy et al. (2010) studied the impact of batteries on generation schedules and showed that, for the case with a single generator and a single demand load, there is a pattern for the state-of-charge of a battery with the assumption that the battery is lossless. However, batteries are indeed loss-incurring, and a more accurate model of battery operations formulates the problem with binary variables to represent the complementary nature of charging and discharging for a battery with a round-trip efficiency of less than 1. While these models represent more realistic battery operations, they are more challenging to solve due to the nonconvex nature of the resulting formulations. To avoid this complexity due to binary variables, many papers use convex constraints to model the battery operations (see, e.g., Pozo et al. 2014, Lorca and Sun 2016, Kody et al. 2022). Specifically, Pozo (2022) summarized various linear programming (LP) formulations that are valid relaxations of the MIP. Despite the convenience of incorporating battery models into other optimization problems, these simple convex models may yield unrealistic solutions. This issue has been discussed in Arroyo et al. (2020), where counterexamples demonstrate that a battery charges and discharges simultaneously despite satisfying all conditions presented for strong convex relaxation models.

Active research is ongoing to propose improved solution methodologies or tighter formulations for related problems. For example, Kim and Powell (2011) formulated and derived an optimal dispatch policy for maximizing the profit of a wind farm with a co-located energy storage device. In Nascimento and Powell (2013), the authors used approximate dynamic programming for an economic dispatch problem with a single energy storage device. Recently, Baldick et al. (2023) developed valid inequalities for specific types of storage systems to maximize the profit of batteries based on price signals. However, in the context of DCOPF with ESS, there have been few attempts to bridge this gap between using a simple model that yields an infeasible fractional solution and using a complex model that is exact and yields feasible solutions.

1.2 Summary of Contributions

In this paper, we propose a regularized MIP model for battery operations in the DC optimal power flow problem. The main benefits of the regularized MIP model are summarized below.

- *Regularized MIP model with zero integrality gap.* In the exact battery model, one must enforce $p_t^c p_t^d = 0$, where p_t^c and p_t^d are the amount of power being used to charge and discharge a battery at time t, respectively. One can view this as enforcing a very specific sparsity condition. Often, sparsity is achieved by the addition of an ℓ_1 regularizer penalty (see, e.g., Tibshirani 1996, Dey et al. 2022). In the same spirit, we perturb the original objective function of DCOPF with batteries by adding ℓ_1 regularizers with respect to p_t^c and p_t^d for all times t. We prove that under mild conditions that are standard in most of the literature (see, e.g., Pozo 2022, Kody et al. 2022) and for a sufficiently large penalty, where the penalty value depends only on the efficiency of the battery (see details in Section 3), the regularized MIP has zero integrality gap with its LP relaxation. This regularized MIP model achieves the goal of being simple to solve yet produces feasible solutions for the actual battery operations. Moreover, the required penalty value is quite small for standard battery efficiencies, thus yielding near-optimal solutions in all our studies.

- *High-quality upper bound.* The optimal solution of the regularized MIP model is a feasible solution to the original MIP and provides a valid and tight upper bound. We formally study the structural difference between the optimal solution of the regularized problem and the original battery problem, provide an exactness condition, that is a condition under which we obtain the same solution, and prove a worst-case bound upper bound on the gap between their optimal objective values. We also evaluate our regularized MIP with a specific choice of regularizer parameter and empirically show that in practice the relative gap is small.

- *Application to long-term planning problems.* Leveraging the benefit of having no integrality gap for the regularized MIP, we examine two challenging applications.

  The first application is a planning problem under a stochastic demand. This problem is modeled as a two-stage stochastic optimization problem, where the first-stage decision is the placement of batteries in the network and the second-stage decision is the optimal operation scheme including power generations and battery operations under uncertain demand scenarios. Since the second stage can be equivalently formulated as an LP instead of a MIP, this significantly improves our ability to solve this class of problems. Our empirical studies on standard network instances with up to 2000 nodes show that we almost always recover the correct optimal solution to the first-stage decision variables in a significantly shorter time than solving the original problem.

  The second application is a trilevel min-max-min contingency problem involving binary decision variables at each level. At the outermost level, a network designer is planning the locations of the batteries. The middle level is an interdictor allowed to attack a budgeted amount of the network. The innermost minimization problem is the system operator solving DCOPF with batteries. Using our regularized model in the third level allows us to replace an integer program with its LP relaxation. We can then take the dual of this LP relaxation, thus allowing the problem to be reduced to a bilevel problem with bounded dual variables (see details in Section 4.5). The resulting bilevel problem can now be solved more efficiently with combinatorial algorithms. We empirically test on instances with up to 2000 nodes and are able to solve this class of challenging trilevel instances in less than 6 hours.

To the best of our knowledge, this is the first time that either the battery placement problem under stochastic demand or the $N-k$ contingency problem for battery operations has been studied. These models may be of independent interest to the power systems research community.

Organization. The remainder of the paper is organized as follows. Section 2 provides a detailed mathematical formulation of the DCOPF with ESS. Section 3 introduces the regularized MIP model, presents its structural properties, and provides a comparison against the original MIP model. Section 4 details the numerical experiments and demonstrates the power of using regularized formulation on two challenging optimization problems. Section 5 concludes the paper.

Notation. The following notation is used throughout the paper. We use bold letters (e.g., $\boldsymbol{x}, \boldsymbol{y}$) to denote vectors and matrices and use corresponding non-bold letters to denote their components. Given an integer n, we let $[n] := \{1, 2, \ldots, n\}$, and use $\mathbb{R}^n_+ := \{\boldsymbol{x} \in \mathbb{R}^n : x_i \geq 0, \forall i \in [n]\}$. We let $\mathbf{e}$ be the vector or matrix of all ones and let e_i be the i-th standard basis vector. Given a finite set I, we let $|I|$ denote its cardinality. The indicator function $\mathbb{1}(\boldsymbol{x} \in R) = 1$ if $\boldsymbol{x} \in R$, and 0, otherwise. Additional notations are introduced as needed.

2 Mathematical Formulation

In this section, we present the mathematical model for the DCOPF problem with batteries placed at a subset of network buses. Consider a network with a set of buses denoted as $\mathcal{N}$ and a set of transmission lines denoted as $\mathcal{L}$. The batteries are placed at a subset of buses $\mathcal{N}_b \subseteq \mathcal{N}$. For simplicity, we assume that all batteries within the network have the same initial state-or-charge and configuration of efficiency level, lower and upper bounds on state-of-charge, charging rate, and discharging rate. Moreover, we assume these parameters remain unchanged over time. Our results on the regularized formulation in the next section do not require these assumptions. Let set $\mathcal{T} = \{1, \ldots, T\}$ represent the finite time horizon with equal time intervals. At each time $t \in \mathcal{T}$, we decide the power output of each generator $p^g_{t,i}$ at bus $i \in \mathcal{N}$, taking into account the minimum and maximum generation limits, denoted $G^{\max}_i$ and $G^{\min}_i$, respectively, i.e.,

$$G^{\min}_i \leq p^g_{t,i} \leq G^{\max}_i, \qquad \forall t \in \mathcal{T},\ i \in \mathcal{N}. \tag{1a}$$

Notice that if there is no generator at a certain bus $i \in \mathcal{N}$, we set $G^{\min}_i = G^{\max}_i = 0$. We also decide the power flow $f_{t,ij}$ through transmission line $(i,\ j) \in \mathcal{L}$ subject to the limit for both directions F_{ij}, i.e.,

$$-F_{ij} \leq f_{t,ij} \leq F_{ij}, \qquad \forall t \in \mathcal{T},\ (i,j) \in \mathcal{L}. \tag{1b}$$

The flow on a transmission line is proportional to the difference in phase angles of the corresponding buses:

$$f_{t,ij} = B_{ij}(\theta_{t,i} - \theta_{t,j}), \qquad \forall t \in \mathcal{T},\ (i,j) \in \mathcal{L}, \tag{1c}$$

where B_{ij} is the susceptance of the transmission lines $(i,\ j) \in \mathcal{L}$.

For batteries placed at certain nodes $i \in \mathcal{N}_b$, we determine the operations of the batteries. Since a battery can only charge or discharge at a given time point, we use a binary variable $u_{t,i} \in \{0,1\}$ to denote charging ($u_{t,i} = 1$) and discharging ($u_{t,i} = 0$) states. When a battery is charging, we decide the charging amount $p^c_{t,i}$ subject to its upper and lower limits, denoted as $E^{\max}_c$ and $E^{\min}_c$, respectively. Similarly, when the battery is discharging, we decide the discharging amount $p^d_{t,i}$ subject to its upper limit $E^{\max}_d$ and lower limit $E^{\min}_d$, i.e.,

$$E^{\min}_c u_{t,i} \leq p^c_{t,i} \leq E^{\max}_c u_{t,i}, \qquad \forall t \in \mathcal{T},\ i \in \mathcal{N}_b, \tag{1d}$$

$$E^{\min}_d (1-u_{t,i}) \leq p^d_{t,i} \leq E^{\max}_d (1-u_{t,i}), \qquad \forall t \in \mathcal{T},\ i \in \mathcal{N}_b. \tag{1e}$$

We use $\eta \in (0,1]$ to denote the charging efficiency and $1/\eta$ to represent the discharging efficiency, accounting for energy losses incurred during imperfect round-trip energy conversions, which may result from factors like friction. The state-of-charge, represented by $p^s_{t,i}$, evolves based on the amount of battery charging and discharging. We assume an initial state-of-charge to be E_0, i.e.,

$$p^s_{t,i} = p^s_{t-1,i} + \eta \cdot p^c_{t,i} - 1/\eta \cdot p^d_{t,i}, \qquad \forall t \in \mathcal{T},\ i \in \mathcal{N}_b, \tag{1f}$$

$$p^s_{0,i} = E_0, \qquad \forall i \in \mathcal{N}_b. \tag{1g}$$

The state-of-charge of batteries is subject to upper and lower bounds for reliable operations, i.e.,

$$E^{\min} \leq p^s_{t,i} \leq E^{\max}, \qquad \forall t \in \mathcal{T},\ i \in \mathcal{N}_b. \tag{1h}$$

In case there are no batteries at certain nodes $i \in \mathcal{N} \setminus \mathcal{N}_b$, all battery-related variables are set to zero, i.e.,

$$p^s_{t,i} = p^s_{0,i} = p^c_{t,i} = p^d_{t,i} = u_{t,i} = 0, \qquad \forall t \in \mathcal{T},\ i \in \mathcal{N} \setminus \mathcal{N}_b. \tag{1i}$$

These operational decisions may lead to load shedding $p^{ls}_{t,i}$ if available power at bus $i \in \mathcal{N}$ is insufficient to meet the demand $D_{t,i}$. Conversely, the system may experience excess power $p^{ex}_{t,i}$ if available power exceeds demand. These variables act as slack variables and are always nonnegative:

$$\boldsymbol{p}^{ls},\ \boldsymbol{p}^{ex} \geq \mathbf{0}. \tag{1j}$$

It is important to note that in an optimal solution, only one of load shedding or excess power can occur at a given time at each bus, ensuring that $p^{ls}_{t,i} p^{ex}_{t,i} = 0$ for all $i \in \mathcal{N}$ and $t \in \mathcal{T}$.

Altogether, these decisions must satisfy the power balance equation:

$$\sum_{j \in \delta^+_i} f_{t,ij} - \sum_{j \in \delta^-_i} f_{t,ji} = p^g_{t,i} - D_{t,i} - p^c_{t,i} + p^d_{t,i} + p^{ls}_{t,i} - p^{ex}_{t,i}, \qquad \forall t \in \mathcal{T},\ i \in \mathcal{N}, \tag{1k}$$

where $\delta^+_i = \{j \in \mathcal{N} : (i,j) \in \mathcal{L}\}$ and $\delta^-_i = \{j \in \mathcal{N} : (j,i) \in \mathcal{L}\}$. Since both load shedding and excess power can be detrimental to the system, our goal is to minimize the total load shedding $p^{ls}_{t,i}$ and excess power $p^{ex}_{t,i}$ over the entire network buses $i \in \mathcal{N}$ and the time horizon $t \in \mathcal{T}$.

For the rest of the paper, for simplicity, we use $\boldsymbol{p} = (\boldsymbol{p}^g, \boldsymbol{p}^s, \boldsymbol{p}^c, \boldsymbol{p}^d, \boldsymbol{p}^{ls}, \boldsymbol{p}^{ex})$, the system-wide cost $c(\boldsymbol{p}) = \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} [p^{ls}_{t,i} + p^{ex}_{t,i}]$, $N = |\mathcal{N}|$, and $N_b = |\mathcal{N}_b|$. Using these notations, we are now ready to introduce the DCOPF problem with the battery, that is,

$$f^{\text{ori}} = \min_{\boldsymbol{\theta}, \boldsymbol{f}, \boldsymbol{p}, \boldsymbol{u}} \left\{ c(\boldsymbol{p}) \colon \text{(1a)} - \text{(1k)},\ \boldsymbol{u} \in \{0,1\}^{T \times N} \right\}. \tag{Battery}$$

Without loss of generality, we assume that $0 \leq F_{ij}$ for all $(i,j) \in \mathcal{L}$, $0 \leq G^{\min}_i \leq G^{\max}_i$ for each $i \in \mathcal{N}$, $0 \leq E^{\min} \leq E^{\max}$, $0 \leq E^{\min}_c \leq E^{\max}_c$, and $0 \leq E^{\min}_d \leq E^{\max}_d$. Hence, it follows immediately $\boldsymbol{p}^g,\ \boldsymbol{p}^s,\ \boldsymbol{p}^c,\ \boldsymbol{p}^d \geq \mathbf{0}$.

3 Regularized MIP Model

In this section, we first introduce a regularized MIP model and provide conditions such that this regularized MIP has the same optimal objective function value as its LP relaxation. To simplify the notation, we employ the function $g(\cdot)$ to map $\boldsymbol{p}$ to a two-dimensional vector: $g(\boldsymbol{p}) = (\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} p^c_{t,i},\ \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} p^d_{t,i})^\top$. Now, we introduce a regularization function aimed at penalizing $\boldsymbol{p}^c$ and $\boldsymbol{p}^d$ with a given $\boldsymbol{\lambda} = (\lambda_c,\ \lambda_d)^\top \in \mathbb{R}^2_+$, that is,

$$f^{\text{reg}}(\boldsymbol{\lambda}) = \min_{\boldsymbol{\theta},\boldsymbol{f},\boldsymbol{p},\boldsymbol{u}} \left\{ c(\boldsymbol{p}) + \boldsymbol{\lambda}^\top g(\boldsymbol{p}) \colon \text{(1a)} - \text{(1k)},\ \boldsymbol{u} \in \{0,1\}^{T\times N} \right\}. \tag{Reg-Battery}$$

In (Reg-Battery) problem, the only binary decision is $\boldsymbol{u} \in \{0,1\}^{T\times N}$. By relaxing this binary variable $\boldsymbol{u}$ to be continuous, we have the following convex relaxation for (Reg-Battery) problem:

$$f^{\text{reg}}_l(\boldsymbol{\lambda}) = \min_{\boldsymbol{\theta},\boldsymbol{f},\boldsymbol{p},\boldsymbol{u}} \left\{ c(\boldsymbol{p}) + \boldsymbol{\lambda}^\top g(\boldsymbol{p}) \colon \text{(1a)} - \text{(1k)},\ \boldsymbol{u} \in [0,1]^{T\times N} \right\}. \tag{LP-Reg-Battery}$$

One of our main results in this section is to provide nontrivial sufficient conditions such that (Reg-Battery) problem and (LP-Reg-Battery) problem have the same optimal objective function value.

Theorem 1. *Suppose that $E^{\min}_c = E^{\min}_d = 0$. If $\lambda_c + \eta^2\lambda_d \geq 1 - \eta^2$, then we have that $f^{\text{reg}}(\boldsymbol{\lambda}) = f^{\text{reg}}_l(\boldsymbol{\lambda})$.*

Proof. (LP-Reg-Battery) problem is a relaxation of (Reg-Battery) problem, so it remains to show that an optimal solution of (LP-Reg-Battery) problem is achieved with $u_{t,i} \in \{0,1\}$ for all $t \in \mathcal{T}$ and $i \in \mathcal{N}_b$, which is equivalent to showing that $p^c_{t,i} = 0$ or $p^d_{t,i} = 0$ for all $t \in \mathcal{T}$ and $i \in \mathcal{N}$. Let $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$ be an optimal solution of (LP-Reg-Battery) problem. Suppose that $\widehat{p}^c_{t^*,i^*} > 0$ and $\widehat{p}^d_{t^*,i^*} > 0$ for some $t^* \in \mathcal{T}$ and $i^* \in \mathcal{N}$. We show that we can always find another feasible solution $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{f}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{u}})$ such that at most one of $\tilde{p}^c_{t^*,i^*}$ and $\tilde{p}^d_{t^*,i^*}$ is positive for this given $t^* \in \mathcal{T}$ and $i^* \in \mathcal{N}$ and the corresponding objective value is at least as good as that of $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$. Such a solution can be constructed as follows:

$$\begin{aligned}
&\tilde{\boldsymbol{\theta}} = \widehat{\boldsymbol{\theta}},\ \tilde{\boldsymbol{f}} = \widehat{\boldsymbol{f}},\ \tilde{\boldsymbol{p}}^g = \widehat{\boldsymbol{p}}^g,\ \tilde{\boldsymbol{p}}^s = \widehat{\boldsymbol{p}}^s, \\
&\tilde{p}^c_{t,i} = \max\{\widehat{p}^c_{t,i} - 1/\eta^2 \cdot \widehat{p}^d_{t,i}, 0\}, && \forall t \in \mathcal{T},\ i \in \mathcal{N}, \quad \text{(2a)}\\
&\tilde{p}^d_{t,i} = \max\{\widehat{p}^d_{t,i} - \eta^2 \cdot \widehat{p}^c_{t,i}, 0\}, && \forall t \in \mathcal{T},\ i \in \mathcal{N}, \quad \text{(2b)}\\
&\tilde{p}^{ls}_{t,i} = \max\{-\widehat{p}^c_{t,i} + \widehat{p}^d_{t,i} + \widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i} + \tilde{p}^c_{t,i} - \tilde{p}^d_{t,i}, 0\}, && \forall t \in \mathcal{T},\ i \in \mathcal{N}, \quad \text{(2c)}\\
&\tilde{p}^{ex}_{t,i} = \max\{\widehat{p}^c_{t,i} - \widehat{p}^d_{t,i} - \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} - \tilde{p}^c_{t,i} + \tilde{p}^d_{t,i}, 0\}, && \forall t \in \mathcal{T},\ i \in \mathcal{N}, \quad \text{(2d)}\\
&\tilde{u}_{t,i} = \mathbb{1}\left\{\tilde{p}^c_{t,i} > 0\right\}, && \forall t \in \mathcal{T},\ i \in \mathcal{N}. \quad \text{(2e)}
\end{aligned}$$

Obviously, $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{f}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{u}})$ satisfies all the constraints of (LP-Reg-Battery) problem. In particular, for each $t \in \mathcal{T}, i \in \mathcal{N}$, the following equality is satisfied from the power balance equation (1k):

$$-\widehat{p}^c_{t,i} + \widehat{p}^d_{t,i} + \widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i} = -\tilde{p}^c_{t,i} + \tilde{p}^d_{t,i} + \tilde{p}^{ls}_{t,i} - \tilde{p}^{ex}_{t,i}. \tag{3}$$

Then, using the fact that $\tilde{p}^{ls}_{t,i}\tilde{p}^{ex}_{t,i} = 0$ and $\tilde{p}^c_{t,i}\tilde{p}^d_{t,i} = 0$ for each $t \in \mathcal{T}, i \in \mathcal{N}$, it is sufficient to consider the following four cases.

(Case 1) When $\tilde{p}^{d}_{t,i} = \tilde{p}^{ls}_{t,i} = 0$, the objective value of (LP-Reg-Battery) problem is

$$\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i} = \tilde{p}^{ex}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i}.$$

From condition (3), we have $\tilde{p}^{ex}_{t,i} = \widehat{p}^{c}_{t,i} - \widehat{p}^{d}_{t,i} - \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} - \tilde{p}^{c}_{t,i} + \tilde{p}^{d}_{t,i} + \tilde{p}^{ls}_{t,i}$. Then, we substitute it for the objective value of (LP-Reg-Battery) problem, that is,

$$\tilde{p}^{ex}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i} = \widehat{p}^{c}_{t,i} - \widehat{p}^{d}_{t,i} - \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} - \tilde{p}^{c}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i}.$$

Based on construction in (2a) and (2b), together with the presumption $\tilde{p}^{d}_{t,i} = 0$, we know $\widehat{p}^{d}_{t,i} - \eta^2 \cdot \widehat{p}^{c}_{t,i} \leq 0$, which implies that $\widehat{p}^{c}_{t,i} - \widehat{p}^{d}_{t,i}/\eta^2 \geq 0$ and $\tilde{p}^{c}_{t,i} = \widehat{p}^{c}_{t,i} - \widehat{p}^{d}_{t,i}/\eta^2$. Then, the objective value of (LP-Reg-Battery) problem can be simplified as

$$\begin{aligned} \widehat{p}^{c}_{t,i} - \widehat{p}^{d}_{t,i} - \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} - \tilde{p}^{c}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i} &= \widehat{p}^{c}_{t,i} - \widehat{p}^{d}_{t,i} - \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + (\lambda_c - 1)\left(\widehat{p}^{c}_{t,i} - \frac{1}{\eta^2} \cdot \widehat{p}^{d}_{t,i}\right) \\ &= -\widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \lambda_c \widehat{p}^{c}_{t,i} + \frac{1 - \lambda_c - \eta^2}{\eta^2} \widehat{p}^{d}_{t,i}. \end{aligned}$$

According to the assumption $\lambda_c + \eta^2 \lambda_d \geq 1 - \eta^2$ with $0 < \eta \leq 1$, together with the fact that $\widehat{p}^{ls}_{t,i}$ is nonnegative, we have

$$-\widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \lambda_c \widehat{p}^{c}_{t,i} + \frac{1 - \lambda_c - \eta^2}{\eta^2} \widehat{p}^{d}_{t,i} \leq \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \lambda_c \widehat{p}^{c}_{t,i} + \lambda_d \widehat{p}^{d}_{t,i}.$$

Thus, in this case, we have

$$\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i} \leq \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \lambda_c \widehat{p}^{c}_{t,i} + \lambda_d \widehat{p}^{d}_{t,i}.$$

(Case 2) When $\tilde{p}^{d}_{t,i} = \tilde{p}^{ex}_{t,i} = 0$, the objective value of (LP-Reg-Battery) problem is

$$\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i} = \tilde{p}^{ls}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i}.$$

From condition (3), we have

$$\tilde{p}^{ls}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i} = -\widehat{p}^{c}_{t,i} + \widehat{p}^{d}_{t,i} + \widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i} + \tilde{p}^{c}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i}.$$

Similarly, based on construction in (2a) and (2b), together with the presumption $\tilde{p}^{d}_{t,i} = 0$, we have $\tilde{p}^{c}_{t,i} = \widehat{p}^{c}_{t,i} - \widehat{p}^{d}_{t,i}/\eta^2$. Then, the objective value of (LP-Reg-Battery) problem can be simplified as

$$\begin{aligned} -\widehat{p}^{c}_{t,i} + \widehat{p}^{d}_{t,i} + \widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i} + \tilde{p}^{c}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i} &= -\widehat{p}^{c}_{t,i} + \widehat{p}^{d}_{t,i} + \widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i} + (\lambda_c + 1)\left(\widehat{p}^{c}_{t,i} - \frac{1}{\eta^2} \cdot \widehat{p}^{d}_{t,i}\right) \\ &= \widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i} + \lambda_c \widehat{p}^{c}_{t,i} + \left(1 - \frac{1}{\eta^2} - \frac{\lambda_c}{\eta^2}\right) \widehat{p}^{d}_{t,i}. \end{aligned}$$

According to the assumption $\lambda_c + \eta^2 \lambda_d \geq 1 - \eta^2$ with $0 < \eta \leq 1$, together with the fact that $\widehat{p}^{ex}_{t,i}$ is nonnegative, we have

$$\widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i} + \lambda_c \widehat{p}^{c}_{t,i} + \left(1 - \frac{1}{\eta^2} - \frac{\lambda_c}{\eta^2}\right) \widehat{p}^{d}_{t,i} \leq \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \lambda_c \widehat{p}^{c}_{t,i} + \lambda_d \widehat{p}^{d}_{t,i}.$$

Thus, in this case, we have

$$\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i} \leq \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \lambda_c \widehat{p}^{c}_{t,i} + \lambda_d \widehat{p}^{d}_{t,i}.$$

(Case 3) When $\tilde{p}^{c}_{t,i} = \tilde{p}^{ls}_{t,i} = 0$, the objective value of (LP-Reg-Battery) problem is

$$\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i} = \tilde{p}^{ex}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i}.$$

From condition (3), we have

$$\tilde{p}^{ex}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i} = \widehat{p}^{c}_{t,i} - \widehat{p}^{d}_{t,i} - \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \tilde{p}^{d}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i}.$$

Based on construction in (2a) and (2b), together with the presumption $\tilde{p}^{c}_{t,i} = 0$, we know $\widehat{p}^{c}_{t,i} - \widehat{p}^{d}_{t,i}/\eta^2 \leq 0$, which implies that $\widehat{p}^{d}_{t,i} - \eta^2 \cdot \widehat{p}^{c}_{t,i} \geq 0$ and $\tilde{p}^{d}_{t,i} = \widehat{p}^{d}_{t,i} - \eta^2 \cdot \widehat{p}^{c}_{t,i}$. Then, the objective value of (LP-Reg-Battery) problem can be simplified as

$$\begin{aligned}\widehat{p}^{c}_{t,i} - \widehat{p}^{d}_{t,i} - \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \tilde{p}^{d}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i} &= \widehat{p}^{c}_{t,i} - \widehat{p}^{d}_{t,i} - \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + (\lambda_d + 1)(\widehat{p}^{d}_{t,i} - \eta^2 \cdot \widehat{p}^{c}_{t,i}) \\ &= -\widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \left(1 - \eta^2 \lambda_d - \eta^2\right) \widehat{p}^{c}_{t,i} + \lambda_d \widehat{p}^{d}_{t,i}.\end{aligned}$$

According to the assumption $\lambda_c + \eta^2 \lambda_d \geq 1 - \eta^2$ with $0 < \eta \leq 1$, together with the fact that $\widehat{p}^{ls}_{t,i}$ is nonnegative, we have

$$-\widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \left(1 - \eta^2 \lambda_d - \eta^2\right) \widehat{p}^{c}_{t,i} + \lambda_d \widehat{p}^{d}_{t,i} \leq \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \lambda_c \widehat{p}^{c}_{t,i} + \lambda_d \widehat{p}^{d}_{t,i}.$$

Thus, in this case, we have

$$\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i} \leq \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \lambda_c \widehat{p}^{c}_{t,i} + \lambda_d \widehat{p}^{d}_{t,i}.$$

(Case 4) When $\tilde{p}^{c}_{t,i} = \tilde{p}^{ex}_{t,i} = 0$, the objective value of (LP-Reg-Battery) problem is

$$\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i} = \tilde{p}^{ls}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i}.$$

From condition (3), we have

$$\tilde{p}^{ls}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i} = -\widehat{p}^{c}_{t,i} + \widehat{p}^{d}_{t,i} + \widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i} - \tilde{p}^{d}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i}.$$

Similarly, based on construction in (2a) and (2b), together with the presumption $\tilde{p}^{c}_{t,i} = 0$, we have $\tilde{p}^{d}_{t,i} = \widehat{p}^{d}_{t,i} - \eta^2 \cdot \widehat{p}^{c}_{t,i}$. Then, the objective value of (LP-Reg-Battery) problem can be simplified as

$$\begin{aligned}-\widehat{p}^{c}_{t,i} + \widehat{p}^{d}_{t,i} + \widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i} - \tilde{p}^{d}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i} &= -\widehat{p}^{c}_{t,i} + \widehat{p}^{d}_{t,i} + \widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i} + (\lambda_d - 1)(\widehat{p}^{d}_{t,i} - \eta^2 \cdot \widehat{p}^{c}_{t,i}) \\ &= \widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i} + \left(\eta^2 - \eta^2 \lambda_d - 1\right) \widehat{p}^{c}_{t,i} + \lambda_d \widehat{p}^{d}_{t,i}.\end{aligned}$$

According to the assumption $\lambda_c + \eta^2 \lambda_d \geq 1 - \eta^2$ with $0 < \eta \leq 1$, together with the fact that $\widehat{p}^{ex}_{t,i}$ is nonnegative, we have

$$\widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i} + \left(\eta^2 - \eta^2 \lambda_d - 1\right) \widehat{p}^{c}_{t,i} + \lambda_d \widehat{p}^{d}_{t,i} \leq \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \lambda_c \widehat{p}^{c}_{t,i} + \lambda_d \widehat{p}^{d}_{t,i}.$$

Thus, in this case, we have

$$\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} + \lambda_c \tilde{p}^{c}_{t,i} + \lambda_d \tilde{p}^{d}_{t,i} \leq \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \lambda_c \widehat{p}^{c}_{t,i} + \lambda_d \widehat{p}^{d}_{t,i}.$$

Since above four cases hold for each $t \in \mathcal{T}$ and $i \in \mathcal{N}$, we have

$$\begin{aligned} c(\tilde{\boldsymbol{p}}) + \boldsymbol{\lambda}^\top g(\tilde{\boldsymbol{p}}) &= \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} \left[\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i}\right] + \lambda_c \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} \tilde{p}^{c}_{t,i} + \lambda_d \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} \tilde{p}^{d}_{t,i} \\ &\leq \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} \left[\widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i}\right] + \lambda_c \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} \widehat{p}^{c}_{t,i} + \lambda_d \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} \widehat{p}^{d}_{t,i} \\ &= c(\widehat{\boldsymbol{p}}) + \boldsymbol{\lambda}^\top g(\widehat{\boldsymbol{p}}). \end{aligned}$$

This completes the proof. □

Theorem 1 demonstrates the equivalence between the optimal objective function value of (Reg-Battery) problem and (LP-Reg-Battery) problem under regularization (for sufficiently high penalty) and the assumption that $E_c^{\min} = E_d^{\min} = 0$. Note that this assumption is standard and appears in many recent works, such as Pozo (2022) and Kody et al. (2022). The technique to devise an integral solution by perturbing the charge and discharge levels has been used in the context of optimizing for a single solar-battery storage system in Singh and Knueven (2021).

Theorem 1 easily leads to the following Corollary.

Corollary 1. *When $\eta = 1$, for any $\boldsymbol{\lambda} \geq \mathbf{0}$,* (Reg-Battery) *problem and* (LP-Reg-Battery) *problem are equivalent. In particular, when $\boldsymbol{\lambda} = \mathbf{0}$,* (Battery) *problem,* (Reg-Battery) *problem, and* (LP-Reg-Battery) *problem are all equivalent, i.e., $f^{\text{ori}} = f^{\text{reg}}(\mathbf{0}) = f_l^{\text{reg}}(\mathbf{0})$.*

Therefore, when efficiency $\eta = 1$ (i.e., the battery is lossless), relaxing the integrality of battery operations is exact. However, as $\eta = 1$ does not occur in practice, most literature concerning battery operation bases numerical experiments with efficiency $\eta < 1$.

Finally, we remark that the two assumptions of Theorem 1, (i.e., $E_c^{\min} = E_d^{\min} = 0$ and $\lambda_c + \eta^2\lambda_d \geq 1-\eta^2$) are the best that we may expect for the equivalence of the optimal objective function value of the regularized MIP model and its LP relaxation. The following two examples illustrate that (Reg-Battery) problem and (LP-Reg-Battery) problem do not have the same optimal objective function value if either of these two assumptions in Theorem 1 is violated.

Example 1. *($f^{\text{reg}}(\boldsymbol{\lambda}) \neq f_l^{\text{reg}}(\boldsymbol{\lambda})$ when $E_c^{\min},\ E^{\max} > 0$)* Consider a simple network with $\mathcal{N} = \{1,2\}$, $\mathcal{T} = \{1,2\}$, $\mathcal{L} = \{(1,2)\}$. Suppose one battery is placed at node 2 (there is no battery placed at node 1) with $\mathcal{N}_b = \{2\}$ and $E_c^{\min} = E_d^{\min} = \tau$, $E_c^{\max} = E_d^{\max} = 2$, $E^{\min} = 0$, $E^{\max} = 4$, $E_0 = 0$, and $\eta = 1/2$. Assume each node has one generator with $G_1^{\min} = G_2^{\min} = 2$ and $G_1^{\max} = G_2^{\max} = 4$. We further assume $-4 \leq f_{12} \leq 4$ and the demand is $D_{1,1} = 2,\ D_{1,2} = 4,\ D_{2,1} = 6,\ D_{2,2} = 4$. Without loss of generality, we assume that the Ohm's law constraint (1c) is satisfied. When $\boldsymbol{\lambda} = (3/5,\ 3/5)^\top$, for any $\tau \in (0, 1/2]$, an optimal solution of (Reg-Battery) problem is

$$\begin{aligned} &\widehat{p}^{c}_{1,1} = 0,\ \widehat{p}^{c}_{1,2} = 4\tau,\ \widehat{p}^{c}_{2,1} = 0,\ \widehat{p}^{c}_{2,2} = 0, \\ &\widehat{p}^{d}_{1,1} = 0,\ \widehat{p}^{d}_{1,2} = 0,\ \widehat{p}^{d}_{2,1} = 0,\ \widehat{p}^{d}_{2,2} = \tau, \\ &\widehat{p}^{ls}_{1,1} = 0,\ \widehat{p}^{ls}_{1,2} = 0,\ \widehat{p}^{ls}_{2,1} = 0,\ \widehat{p}^{ls}_{2,2} = 2-\tau, \\ &\widehat{p}^{ex}_{1,1} = 0,\ \widehat{p}^{ex}_{1,2} = 0,\ \widehat{p}^{ex}_{2,1} = 0,\ \widehat{p}^{ex}_{2,2} = 0, \\ &\widehat{u}_{1,1} = 0,\ \widehat{u}_{1,2} = 1,\ \widehat{u}_{2,1} = 0,\ \widehat{u}_{2,2} = 0, \end{aligned}$$

with the optimal objective value $\widehat{v} = 2 + 2\tau$.

While an optimal solution of the corresponding (LP-Reg-Battery) problem is

$$\begin{aligned}
&\tilde{p}^{c}_{1,1}=0,\ \tilde{p}^{c}_{1,2}=\tau,\ \tilde{p}^{c}_{2,1}=0,\ \tilde{p}^{c}_{2,2}=\frac{3\tau}{5},\\
&\tilde{p}^{d}_{1,1}=0,\ \tilde{p}^{d}_{1,2}=0,\ \tilde{p}^{d}_{2,1}=0,\ \tilde{p}^{d}_{2,2}=\frac{2\tau}{5},\\
&\tilde{p}^{ls}_{1,1}=0,\ \tilde{p}^{ls}_{1,2}=0,\ \tilde{p}^{ls}_{2,1}=0,\ \tilde{p}^{ls}_{2,2}=2+\frac{\tau}{5},\\
&\tilde{p}^{ex}_{1,1}=0,\ \tilde{p}^{ex}_{1,2}=0,\ \tilde{p}^{ex}_{2,1}=0,\ \tilde{p}^{ex}_{2,2}=0,\\
&\tilde{u}_{1,1}=0, \tilde{u}_{1,2}=1, \tilde{u}_{2,1}=0, \tilde{u}_{2,2}=\frac{3}{5},
\end{aligned}$$

with the optimal objective value $\tilde{v}=2+7\tau/5$. Therefore, $\tilde{v}<\widehat{v}$ for all $\tau\in(0,1/2]$. Hence, (Reg-Battery) problem and (LP-Reg-Battery) problem are not equivalent. ⋄

Example 2. *($f^{\text{reg}}(\boldsymbol{\lambda})\neq f^{\text{reg}}_{l}(\boldsymbol{\lambda})$ when $\lambda_c+\eta^2\lambda_d<1-\eta^2$)* Consider the same network as that in Example 1 but with different battery configurations and demands. One battery is placed at node 2 (i.e., $\mathcal{N}_b=\{2\}$) with $E^{\min}_{c}=E^{\min}_{d}=0, E^{\max}_{c}=E^{\max}_{d}=1$, $E^{\min}=0$, $E^{\max}=6$, $E_0=6$, and $\eta^2=1/3$. Demand is $D_{1,1}=1,\ D_{1,2}=2,\ D_{2,1}=4,\ D_{2,2}=8$. Other parameters remain the same. We assume that the Ohm's law constraint (1c) is satisfied. Let $\boldsymbol{\lambda}=(\tau,\ \tau)^{\top}$ for any $\tau\in[0,1/2)$. An optimal solution of (Reg-Battery) problem is

$$\begin{aligned}
&\widehat{p}^{c}_{1,1}=0,\ \widehat{p}^{c}_{1,2}=0,\ \widehat{p}^{c}_{2,1}=0,\ \widehat{p}^{c}_{2,2}=0,\\
&\widehat{p}^{d}_{1,1}=0,\ \widehat{p}^{d}_{1,2}=0,\ \widehat{p}^{d}_{2,1}=0,\ \widehat{p}^{d}_{2,2}=1,\\
&\widehat{p}^{ls}_{1,1}=0,\ \widehat{p}^{ls}_{1,2}=0,\ \widehat{p}^{ls}_{2,1}=0,\ \widehat{p}^{ls}_{2,2}=3,\\
&\widehat{p}^{ex}_{1,1}=1,\ \widehat{p}^{ex}_{1,2}=0,\ \widehat{p}^{ex}_{2,1}=0,\ \widehat{p}^{ex}_{2,2}=0,\\
&\widehat{u}_{1,1}=0,\ \widehat{u}_{1,2}=0,\ \widehat{u}_{2,1}=0,\ \widehat{u}_{2,2}=0,
\end{aligned}$$

with the optimal objective value $\widehat{v}=4+\tau$.

While an optimal solution of the corresponding (LP-Reg-Battery) problem is

$$\begin{aligned}
&\tilde{p}^{c}_{1,1}=0, \tilde{p}^{c}_{1,2}=3/4, \tilde{p}^{c}_{2,1}=0, \tilde{p}^{c}_{2,2}=0,\\
&\tilde{p}^{d}_{1,1}=0, \tilde{p}^{d}_{1,2}=1/4, \tilde{p}^{d}_{2,1}=0, \tilde{p}^{d}_{2,2}=1,\\
&\tilde{p}^{ls}_{1,1}=0, \tilde{p}^{ls}_{1,2}=0, \tilde{p}^{ls}_{2,1}=3, \tilde{p}^{ls}_{2,2}=0,\\
&\tilde{p}^{ex}_{1,1}=0, \tilde{p}^{ex}_{1,2}=1/2, \tilde{p}^{ex}_{2,1}=0, \tilde{p}^{ex}_{2,2}=0,\\
&\tilde{u}_{1,1}=0, \tilde{u}_{1,2}=3/4, \tilde{u}_{2,1}=0, \tilde{u}_{2,2}=0,
\end{aligned}$$

with the optimal objective value $\tilde{v}=7/2+2\tau$. Therefore, $\tilde{v}<\widehat{v}$ for all $\tau\in[0,1/2)$. Hence, (Reg-Battery) problem and (LP-Reg-Battery) problem are not equivalent. ⋄

In this section, we have shown that (Reg-Battery) problem is easy to solve, since there is no integrality gap between this problem and its linear programming relaxation. In the next two subsections, we begin to analyze the relationship between (Reg-Battery) problem and (Battery) problem. We would like to understand the differences between the optimal solutions and corresponding objective function values of these two problems both qualitatively and quantitatively.

3.1 Structural Properties of the Regularized MIP Model

In this section, we provide some structural properties of (Reg-Battery) problem. First, we baseline the value of $\boldsymbol{\lambda}$. Next, we present a two-part result on the structure of an optimal solution of the (Reg-Battery) problem as a function of the penalty coefficients $\boldsymbol{\lambda}$ that distinguishes (Battery) problem and (Reg-Battery) problem.

3.1.1 Baselining the value of λ

We begin with the following standard observation from linear programming applied to the convex hull of the feasible region of the (Reg-Battery).

Remark 1. *Function $f^{\text{reg}}(\boldsymbol{\lambda})$ is concave and monotone nondecreasing with respect to $\boldsymbol{\lambda} \in \mathbb{R}^2_+$.*

As $\boldsymbol{\lambda}$ gets larger, (Reg-Battery) problem gets more-and-more "different" from (Battery) problem. In particular, we expect that the battery to be used less, since it now costs more to charge or discharge. However, what is a "reasonable" value of $\boldsymbol{\lambda}$? Our first result below allows us to baseline the value of $\boldsymbol{\lambda}$, by showing that if both components of $\boldsymbol{\lambda}$ are equal to 1 or higher, then (Reg-Battery) problem effectively solves the problem with no batteries placed in the network.

Proposition 1. *For any $\boldsymbol{\lambda} \geq \mathbf{e}$, we have $f^{\text{reg}}(\boldsymbol{\lambda}) = f^{\text{nb}}$ where*

$$f^{\text{nb}} = \min_{\boldsymbol{\theta},\boldsymbol{f},\boldsymbol{p},\boldsymbol{u}} \left\{ c(\boldsymbol{p})\colon \text{(1a)} - \text{(1k)},\ \boldsymbol{u} \in \{0,1\}^{T\times N},\ \boldsymbol{p}^c = \boldsymbol{p}^d = \mathbf{0} \right\}. \tag{4}$$

Proof. Observe that the problem corresponding to f^{nb} is obtained by restricting the feasible region of $f^{\text{reg}}(\boldsymbol{\lambda})$ to $\boldsymbol{p}^c = \boldsymbol{p}^d = \mathbf{0}$. Thus, we have

$$f^{\text{nb}} \geq f^{\text{reg}}(\boldsymbol{\lambda}), \quad \forall \boldsymbol{\lambda} \geq \mathbf{e}.$$

To show the opposite inequality, it is sufficient to show that there is an optimal solution for $f^{\text{reg}}(\mathbf{e})$ such that $\boldsymbol{p}^c = \boldsymbol{p}^d = \mathbf{0}$, since we have $f^{\text{reg}}(\boldsymbol{\lambda}) \geq f^{\text{reg}}(\mathbf{e})$ for all $\boldsymbol{\lambda} \geq \mathbf{e}$ from Remark 1. We use the power balance equation (1k) written in the following form

$$\sum_{j\in\delta_i^+} f_{t,ij} - \sum_{j\in\delta_i^-} f_{t,ji} - p^g_{t,i} + D_{t,i} = -p^c_{t,i} + p^d_{t,i} + p^{ls}_{t,i} - p^{ex}_{t,i}, \qquad \forall t \in \mathcal{T}, i \in \mathcal{N}.$$

To construct such a feasible solution from the current optimal solution. Let $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$ be an optimal solution of (Reg-Battery) problem. We can construct another feasible solution $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{f}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{u}})$ as follows:

$$\begin{aligned}
&\tilde{\boldsymbol{\theta}} = \widehat{\boldsymbol{\theta}},\ \tilde{\boldsymbol{f}} = \widehat{\boldsymbol{f}},\ \tilde{\boldsymbol{p}}^g = \widehat{\boldsymbol{p}}^g,\ \tilde{\boldsymbol{p}}^s = E^0\mathbf{e},\ \tilde{\boldsymbol{p}}^c = \mathbf{0},\ \tilde{\boldsymbol{p}}^d = \mathbf{0}, \\
&\tilde{p}^{ls}_{t,i} = \max\{-\widehat{p}^c_{t,i} + \widehat{p}^d_{t,i} + \widehat{p}^{ls}_{t,i} - \widehat{p}^{ex}_{t,i}, 0\}, && \forall t \in \mathcal{T}, i \in \mathcal{N}, \\
&\tilde{p}^{ex}_{t,i} = \max\{\widehat{p}^c_{t,i} - \widehat{p}^d_{t,i} - \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i}, 0\}, && \forall t \in \mathcal{T},\ i \in \mathcal{N}.
\end{aligned}$$

Then, using the fact that $p^{ls}_{t,i}p^{ex}_{t,i} = 0$ and $p^c_{t,i}p^d_{t,i} = 0$ for each $t \in \mathcal{T}, i \in \mathcal{N}$, we consider the following four cases.

(Case 1) When $\widehat{p}^d_{t,i} = \widehat{p}^{ls}_{t,i} = 0$, $\tilde{p}^{ls}_{t,i} = 0$ and $\tilde{p}^{ex}_{t,i} = \widehat{p}^c_{t,i} + \widehat{p}^{ex}_{t,i}$. Hence, we have

$$\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} = \widehat{p}^c_{t,i} + \widehat{p}^{ex}_{t,i} = \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \widehat{p}^c_{t,i} + \widehat{p}^d_{t,i}.$$

(Case 2) When $\widehat{p}^{d}_{t,i} = \widehat{p}^{ex}_{t,i} = 0$, $\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} = |\widehat{p}^{ls}_{t,i} - \widehat{p}^{c}_{t,i}|$ and we have

$$\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} = |\widehat{p}^{ls}_{t,i} - \widehat{p}^{c}_{t,i}| \leq \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \widehat{p}^{c}_{t,i} + \widehat{p}^{d}_{t,i}.$$

(Case 3) When $\widehat{p}^{c}_{t,i} = \widehat{p}^{ls}_{t,i} = 0$, $\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} = |\widehat{p}^{ex}_{t,i} - \widehat{p}^{d}_{t,i}|$ and we have

$$\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} = |\widehat{p}^{ex}_{t,i} - \widehat{p}^{d}_{t,i}| \leq \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \widehat{p}^{c}_{t,i} + \widehat{p}^{d}_{t,i}.$$

(Case 4) When $\widehat{p}^{c}_{t,i} = \widehat{p}^{ex}_{t,i} = 0$, $\tilde{p}^{ex}_{t,i} = 0$ and $\tilde{p}^{ls}_{t,i} = \widehat{p}^{d}_{t,i} + \widehat{p}^{ls}_{t,i}$. Hence, we have

$$\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} = \widehat{p}^{d}_{t,i} + \widehat{p}^{ls}_{t,i} = \widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \widehat{p}^{c}_{t,i} + \widehat{p}^{d}_{t,i}.$$

Since the above four cases hold for all $t \in \mathcal{T}, i \in \mathcal{N}$, we have

$$c(\tilde{\boldsymbol{p}}) + g(\tilde{\boldsymbol{p}}) = \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{N}} \left[\tilde{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i}\right] \leq \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{N}} \left[\widehat{p}^{ls}_{t,i} + \widehat{p}^{ex}_{t,i} + \widehat{p}^{c}_{t,i} + \widehat{p}^{d}_{t,i}\right] = c(\widehat{\boldsymbol{p}}) + g(\widehat{\boldsymbol{p}}).$$

Therefore, $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{f}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{u}})$ is an optimal solution of (Reg-Battery) problem. This completes the proof. □

We remark that for general values of $\boldsymbol{\lambda}$, the result of Proposition 1, that is $f^{nb} = f^{reg}(\boldsymbol{\lambda})$ may not hold. The following example shows that the battery may always be used when $\boldsymbol{\lambda} \in (0,1)^2$.

Example 3. *($f^{\mathrm{reg}}(\boldsymbol{\lambda}) \neq f^{\mathrm{nb}}$ when $\boldsymbol{\lambda} \in (0,1)^2$)* We consider the same network as that in Example 1 but with different battery configurations and demands. One battery is placed at node 2 (i.e., $\mathcal{N}_b = \{2\}$) with $E_c^{\min} = E_d^{\min} = 0, E_c^{\max} = E_d^{\max} = 2$, $E^{\min} = 0$, $E^{\max} = 20$, $E_0 = 20$ and $\eta = 1$. Demand is $D_{1,1} = D_{1,2} = D_{2,1} = D_{2,2} = 5$. Other parameters remain the same. When $\boldsymbol{\lambda} \in (0,1)^2$, at optimality of (Reg-Battery) problem, we always have

$$p^d_{1,1} = p^d_{2,1} = 2 > 0.$$

This demonstrates that when violating the condition in Proposition 1, i.e., $\boldsymbol{\lambda} \in (0,1)^2$, it is possible that either $\boldsymbol{p}^c$ or $\boldsymbol{p}^d$ is always positive. ⋄

3.1.2 Structural properties of optimal solutions of Regularized MIP

The next two-part result on the structure of an optimal solution of (Reg-Battery) problem as a function of the penalty coefficients $\boldsymbol{\lambda}$ shows the distinction between (Battery) problem and (Reg-Battery) problem as a function of $\boldsymbol{\lambda}$.

In the first part, we show that $p^c_{t,i} p^{ls}_{t,i} = 0$ for all $t \in \mathcal{T},\ i \in \mathcal{N}$ holds for optimal solutions for all values of $\boldsymbol{\lambda}$, that is, this property is true for both (Battery) problem and (Reg-Battery) problem. Intuitively, this property holds because when the system is incurring load shedding ($p^{ls}_{t,i} > 0$), it would not create additional load shedding by charging a battery ($p^c_{t,i} > 0$). Similar to the result above, we may expect that when there is excess power ($p^{ex}_{t,i} > 0$), the amount of discharge would not be positive ($p^d_{t,i} = 0$), as a positive discharge amount would further increase excess power. It is reasonable to expect $p^d_{t,i} p^{ex}_{t,i} = 0$ for all $t \in \mathcal{T},\ i \in \mathcal{N}$. In the second part of our result, we demonstrate that this condition only holds when $\boldsymbol{\lambda}$ is sufficiently large. Indeed, it turns out that when $\boldsymbol{\lambda} = \boldsymbol{0}$, specifically in considering the (Battery) problem, the aforementioned condition might be violated, i.e., $p^d_{t,i} p^{ex}_{t,i} > 0$ for some $t \in \mathcal{T},\ i \in \mathcal{N}$.

Theorem 2. *Suppose $E_c^{\min} = E_d^{\min} = 0$. Let $\boldsymbol{p}$ be an optimal solution to* (Reg-Battery) *problem. Then:*

(i) For all $\boldsymbol{\lambda} \in \mathbb{R}_+^2$, we have $p_{t,i}^c p_{t,i}^{ls} = 0$ for all $t \in \mathcal{T},\ i \in \mathcal{N}$.

(ii) If $\lambda_c + \eta^2 \cdot \lambda_d > 1 - \eta^2$, then we have $p_{t,i}^d p_{t,i}^{ex} = 0$ for all $t \in \mathcal{T},\ i \in \mathcal{N}$.

Proof. See Appendix A. □

Notice that the observations in Theorem 2 can be found in standard IEEE networks with reasonable efficiency levels ($\eta \geq 0.8$), wherein batteries placed at specific nodes may discharge while the system may incur excess power simultaneously. Below, we also present an example illustrating this phenomenon with a low-efficiency level η, thus establishing structural differences in the optimal solutions of (Battery) problem and (Reg-Battery) problem for sufficiently large $\boldsymbol{\lambda}$.

Example 4. *(Condition (ii) in Theorem 2)* Consider a simple network with $\mathcal{N} = \{1, 2\}$, $\mathcal{T} = \{1, 2, 3\}$, $\mathcal{L} = \{(1, 2)\}$. Suppose one battery is placed at node 2 (there is no battery placed at node 1) with $\mathcal{N}_b = \{2\}$ and $E_c^{\min} = E_d^{\min} = 0$, $E_c^{\max} = E_d^{\max} = 2$, $E^{\min} = 0$, $E^{\max} = 4$, $E_0 = 4$, and $\eta = 0.1$. Assume each node has one generator with $G_1^{\min} = G_2^{\min} = 2$ and $G_1^{\max} = G_2^{\max} = 4$. We further assume $-4 \leq f_{12} \leq 4$ and the demand is $D_{1,1} = 2,\ D_{1,2} = 2,\ D_{2,1} = 1,\ D_{2,2} = 1,\ D_{3,1} = 2,\ D_{3,2} = 1$. Without loss of generality, we assume that the Ohm's law constraint (1c) is satisfied. An optimal solution of (Battery) problem (i.e., $\boldsymbol{\lambda} = \boldsymbol{0}$ in (Reg-Battery) problem), denoted as $(\boldsymbol{p}^*, \boldsymbol{u}^*)$, is

$$
\begin{aligned}
&{p^{c*}}_{1,1} = 0,\ {p^{c*}}_{1,2} = 0,\ {p^{c*}}_{2,1} = 0,\ {p^{c*}}_{2,2} = 2,\ {p^{c*}}_{3,1} = 0,\ {p^{c*}}_{3,2} = 1,\\
&{p^{d*}}_{1,1} = 0,\ {p^{d*}}_{1,2} = 0.03,\ {p^{d*}}_{2,1} = 0,\ {p^{d*}}_{2,2} = 0,\ {p^{d*}}_{2,1} = 0,\ {p^{d*}}_{2,2} = 0,\\
&{p^{ls*}}_{1,1} = 0,\ {p^{ls*}}_{1,2} = 0,\ {p^{ls*}}_{2,1} = 0,\ {p^{ls*}}_{2,2} = 0,\ {p^{ls*}}_{3,1} = 0,\ {p^{ls*}}_{3,2} = 0,\\
&{p^{ex*}}_{1,1} = 0,\ {p^{ex*}}_{1,2} = 0.03,\ {p^{ex*}}_{2,1} = 0,\ {p^{ex*}}_{2,2} = 0,\ {p^{ex*}}_{3,1} = 0,\ {p^{ex*}}_{3,2} = 0,\\
&u^*_{1,1} = 0,\ u^*_{1,2} = 0,\ u^*_{2,1} = 0,\ u^*_{2,2} = 1,\ u^*_{3,1} = 0,\ u^*_{3,2} = 1.
\end{aligned}
$$

Clearly, in this example, ${p^{d*}}_{1,2}{p^{ex*}}_{1,2} > 0$. However, we can avoid this situation after considering regularization. When $\boldsymbol{\lambda} = (0.99,\ 0.99)^\top$, i.e., this particular choice of $\boldsymbol{\lambda}$ satisfies Condition (ii) in Theorem 2, an optimal solution of (Reg-Battery) problem, denoted as $(\widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$, is

$$
\begin{aligned}
&\widehat{p}^c_{1,1} = 0,\ \widehat{p}^c_{1,2} = 0,\ \widehat{p}^c_{2,1} = 0,\ \widehat{p}^c_{2,2} = 0,\ \widehat{p}^c_{3,1} = 0,\ \widehat{p}^c_{3,2} = 0,\\
&\widehat{p}^d_{1,1} = 0,\ \widehat{p}^d_{1,2} = 0,\ \widehat{p}^d_{2,1} = 0,\ \widehat{p}^d_{2,2} = 0,\ \widehat{p}^d_{2,1} = 0,\ \widehat{p}^d_{2,2} = 0,\\
&\widehat{p}^{ls}_{1,1} = 0,\ \widehat{p}^{ls}_{1,2} = 0,\ \widehat{p}^{ls}_{2,1} = 0,\ \widehat{p}^{ls}_{2,2} = 0,\ \widehat{p}^{ls}_{3,1} = 0,\ \widehat{p}^{ls}_{3,2} = 0,\\
&\widehat{p}^{ex}_{1,1} = 0,\ \widehat{p}^{ex}_{1,2} = 0,\ \widehat{p}^{ex}_{2,1} = 2,\ \widehat{p}^{ex}_{2,2} = 0,\ \widehat{p}^{ex}_{3,1} = 0,\ \widehat{p}^{ex}_{3,2} = 1,\\
&\widehat{u}_{1,1} = 0,\ \widehat{u}_{1,2} = 0,\ \widehat{u}_{2,1} = 0,\ \widehat{u}_{2,2} = 0,\ \widehat{u}_{3,1} = 0,\ \widehat{u}_{3,2} = 0.
\end{aligned}
$$

In this example, Condition (i) in Theorem 2 is satisfied for both (Battery) problem and (Reg-Battery) problem whereas Condition (ii) is only satisfied for (Reg-Battery) problem. ⋄

To conclude the discussions in this subsection, we provide a summary of the choice of $\boldsymbol{\lambda}$ in Figure 1.

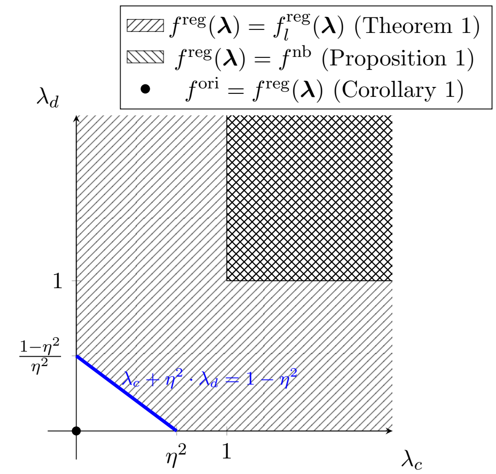


Figure 1: Model comparisons with different choices of λ_c and λ_d.

3.2 Error Quantification of the Solution from the Regularized MIP Model

As we have seen in Theorem 1, in (Reg-Battery) problem, the penalty required to have zero integrality gap with the LP relaxation decreases when efficiency η gets closer to 1. Nevertheless, we still aim for a better understanding of the quality of objective function change when we adjust the regularizer $\boldsymbol{\lambda}$. Hence, the goal of this subsection is to discuss analytical differences in comparison to the original model with respect to the solution quality.

First, we present a sufficient condition under which the optimal solution of (Reg-Battery) can be used to recover the optimal battery operation schedule for the original problem.

Proposition 2. *Let* $\mathcal{U}^* = \{\boldsymbol{u} : \exists\, \boldsymbol{\theta},\ \boldsymbol{f},\ \boldsymbol{p}$ *such that together* $(\boldsymbol{\theta}, \boldsymbol{f}, \boldsymbol{p}, \boldsymbol{u})$ *is an optimal solution of* (Battery) *problem*$\}$ *and* $\mathcal{P}^* = \{\boldsymbol{p} : \exists\, \boldsymbol{\theta},\ \boldsymbol{f},\ \boldsymbol{u}$ *such that together* $(\boldsymbol{\theta}, \boldsymbol{f}, \boldsymbol{p}, \boldsymbol{u})$ *is an optimal solution of* (Battery) *problem*$\}$. *Define the second-best optimal objective value of* (Battery) *problem as* $f^{\text{ori}}(\mathcal{U}^*) = \min_{\boldsymbol{\theta},\boldsymbol{f},\boldsymbol{p},\boldsymbol{u}} \left\{ c(\boldsymbol{p})\colon \text{(1a)} - \text{(1k)},\ \boldsymbol{u} \in \{0,1\}^{T\times N} \setminus \mathcal{U}^* \right\}$. *Define the difference between the best optimal objective value and the second best optimal objective value as* $\delta = |f^{\text{ori}}(\mathcal{U}^*) - f^{\text{ori}}| > 0$. *Suppose* $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$ *is an optimal solution to* (Reg-Battery) *problem. If* $\boldsymbol{\lambda}^\top g(\boldsymbol{p}^*) < \delta$ *for some* $\boldsymbol{p}^* \in \mathcal{P}^*$, *then* $\widehat{\boldsymbol{u}} \in \mathcal{U}^*$.

Proof. We prove this by contradiction. Suppose that the presumptions hold and $\widehat{\boldsymbol{u}} \notin \mathcal{U}^*$. Then, $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$ is a feasible but not an optimal solution to (Battery) problem. Since $\widehat{\boldsymbol{u}} \in \{0,1\}^{T\times N} \setminus \mathcal{U}^*$, we have $c(\widehat{\boldsymbol{p}}) \geq f^{\text{ori}}(\mathcal{U}^*)$. Let $(\boldsymbol{\theta}^*, \boldsymbol{f}^*, \boldsymbol{p}^*, \boldsymbol{u}^*)$ be an optimal solution to (Battery) problem such that $\boldsymbol{\lambda}^\top g(\boldsymbol{p}^*) = \min_{\boldsymbol{p}\in\mathcal{P}^*} \boldsymbol{\lambda}^\top g(\boldsymbol{p})$. Then by the definition of δ, we have:

$$c(\widehat{\boldsymbol{p}}) - c(\boldsymbol{p}^*) \geq |f^{\text{ori}}(\mathcal{U}^*) - f^{\text{ori}}| = \delta.$$

According to the optimality condition from (Reg-Battery) problem, we have

$$c(\widehat{\boldsymbol{p}}) + \boldsymbol{\lambda}^\top g(\widehat{\boldsymbol{p}}) \leq c(\boldsymbol{p}^*) + \boldsymbol{\lambda}^\top g(\boldsymbol{p}^*).$$

Rearranging the terms, we have

$$c(\widehat{\boldsymbol{p}}) - c(\boldsymbol{p}^*) \leq \boldsymbol{\lambda}^\top g(\boldsymbol{p}^*) - \boldsymbol{\lambda}^\top g(\widehat{\boldsymbol{p}}) \leq \boldsymbol{\lambda}^\top g(\boldsymbol{p}^*) < \delta.$$

Clearly, $c(\widehat{\boldsymbol{p}}) - c(\boldsymbol{p}^*) \geq \delta$ and $c(\widehat{\boldsymbol{p}}) - c(\boldsymbol{p}^*) < \delta$ cannot hold simultaneously. Hence, this is a contradiction. ∎

Notice that when $\boldsymbol{\lambda}$ is small (when η is close to 1), the sufficient condition of Propostion 2 is easy to satisfy. In our computational experiments, we often see this behavior. We provide an example to illustrate the exactness condition in Proposition 2.

Example 5. *(Exactness Condition of Proposition 2)* Consider a simple network with $\mathcal{N} = \{1, 2\}$, $\mathcal{T} = \{1, 2\}$, $\mathcal{L} = \{(1, 2)\}$. Suppose one battery is placed at node 2 (there is no battery placed at node 1) with $\mathcal{N}_b = \{2\}$ and $E_c^{\min} = E_d^{\min} = 0$, $E_c^{\max} = E_d^{\max} = 2$, $E^{\min} = 0$, $E^{\max} = 4$, $E_0 = 2$, and $\eta = 0.9$. Assume each node has one generator with $G_1^{\min} = G_2^{\min} = 2$ and $G_1^{\max} = G_2^{\max} = 4$. We further assume $-4 \leq f_{12} \leq 4$ and the demand is $D_{1,1} = 10,\ D_{1,2} = 4,\ D_{2,1} = 4,\ D_{2,2} = 4$. Without loss of generality, we assume that the Ohm's law constraint (1c) is satisfied. The optimal objective value of the (Battery) problem $f^{\text{ori}} = 4.2$. We enumerate all optimal solutions $\boldsymbol{u}$ that achieve this value and find that $\mathcal{U}^* = \{\widehat{\boldsymbol{u}}, \bar{\boldsymbol{u}}\}$ with

$$\bar{u}_{1,1} = 0,\ \bar{u}_{1,2} = 0,\ \bar{u}_{2,1} = 0,\ \bar{u}_{2,2} = 0,$$
$$\widehat{u}_{1,1} = 0,\ \widehat{u}_{1,2} = 0,\ \widehat{u}_{2,1} = 0,\ \widehat{u}_{2,2} = 1.$$

Excluding the solutions in $\mathcal{U}^*$, the second-best optimal objective value $f^{\text{ori}}(\mathcal{U}^*) = 6$. Hence, the difference $\delta = 1.8$. When $\boldsymbol{\lambda} = \left(1 - \eta^2/1 + \eta^2, 1 - \eta^2/1 + \eta^2\right)^\top = (19/181, 19/181)^\top$, $\arg\min\{g(\boldsymbol{p}^*) : p^* \in \mathcal{P}^*\} = [0, 1.8]^\top$. Notice that $\boldsymbol{\lambda}^\top g(\boldsymbol{p}^*) < \delta$. Therefore, the optimal solution of (Reg-Battery) problem should be exactly the (Battery) problem. We check this condition by solving (Reg-Battery) problem and we confirm that the solution from (Reg-Battery) problem is indeed exact. $\diamond$

In general, we may not be able to show that the solution of (Reg-Battery) problem recovers a solution to (Battery) problem. Instead, we next provide a bound that quantifies the difference between the objective function value $c(\boldsymbol{p})$ obtained from the optimal solution of (Battery) problem and that obtained from the optimal solution of (Reg-Battery) problem.

Theorem 3. *Let* $(\boldsymbol{\theta}^*, \boldsymbol{f}^*, \boldsymbol{p}^*, \boldsymbol{u}^*)$ *be the optimal solution of* (Battery) *problem and* $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$ *the optimal solution of* (Reg-Battery) *problem with regularizer* $(\lambda_c,\ \lambda_d)$*. The gap between* (Battery) *problem and* (Reg-Battery) *problem is*

$$c(\widehat{\boldsymbol{p}}) - c(\boldsymbol{p}^*) \leq T N_b \max\{E_c^{\max}\lambda_c, E_d^{\max}\lambda_d\}. \tag{5}$$

Proof. Notice that the feasible regions are the same in (Battery) problem and (Reg-Battery) problem. Hence, $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$ is a feasible solution to (Battery) problem and $(\boldsymbol{\theta}^*, \boldsymbol{f}^*, \boldsymbol{p}^*, \boldsymbol{u}^*)$ is a feasible solution to (Reg-Battery) problem. By optimality, we have

$$c(\widehat{\boldsymbol{p}}) + \boldsymbol{\lambda}^\top g(\widehat{\boldsymbol{p}}) \leq c(\boldsymbol{p}^*) + \boldsymbol{\lambda}^\top g(\boldsymbol{p}^*),$$

which implies that

$$c(\widehat{\boldsymbol{p}}) - c(\boldsymbol{p}^*) \leq \boldsymbol{\lambda}^\top g(\boldsymbol{p}^*) - \boldsymbol{\lambda}^\top g(\widehat{\boldsymbol{p}}) \leq \boldsymbol{\lambda}^\top g(\boldsymbol{p}^*).$$

Recall that $\boldsymbol{\lambda}^\top g(\boldsymbol{p}^*) = \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{N}} \left[\lambda_c p_{t,i}^{c*} + \lambda_d p_{t,i}^{d*}\right]$. Since $p_{t,i}^{c*} p_{t,i}^{d*} = 0$ for all $t \in \mathcal{T}, i \in \mathcal{N}$, it follows $\lambda_c p_{t,i}^{c*} + \lambda_d p_{t,i}^{d*} \leq \max\{E_c^{\max}\lambda_c, E_d^{\max}\lambda_d\}$ for all $t \in \mathcal{T}, i \in \mathcal{N}$. From (1i), $p_{t,i}^{c*} = p_{t,i}^{d*} = 0$ for all $t \in \mathcal{T},\ i \notin N_b$, we obtain the result in (5). $\square$

Our next goal is then to minimize the worst-case bound predicted by Theorem 3 by selecting specific values for $\boldsymbol{\lambda}$.

Proposition 3. *The best worst-case bound is acheieved at* $\boldsymbol{\lambda} = \left(\frac{E_d^{\max} - \eta^2 E_d^{\max}}{E_d^{\max} + \eta^2 E_c^{\max}}, \frac{E_c^{\max} - \eta^2 E_c^{\max}}{E_d^{\max} + \eta^2 E_c^{\max}}\right)^\top$.

Proof. Since T and N_b are fixed, minimizing the worst-case bound in Theorem 3 reduces to the following minimization problem:

$$\min_{\lambda_c \geq 0, \lambda_d \geq 0} \left\{ \max\{E_c^{\max}\lambda_c, E_d^{\max}\lambda_d\} \colon \lambda_c + \eta^2\lambda_d \geq 1 - \eta^2 \right\},$$

where the optimal objective value is achieved when $E_c^{\max}\lambda_c = E_d^{\max}\lambda_d$. Hence, an optimal solution $(\lambda_c^*, \lambda_d^*)$ is

$$(\lambda_c^*, \lambda_d^*) = \left(\frac{E_d^{\max} - \eta^2 E_d^{\max}}{E_d^{\max} + \eta^2 E_c^{\max}}, \frac{E_c^{\max} - \eta^2 E_c^{\max}}{E_d^{\max} + \eta^2 E_c^{\max}} \right).$$

This concludes the proof. ☐

We remark that when $E_c^{\max} = E_d^{\max}$, an optimal choice of (λ_c, λ_d) reduces to

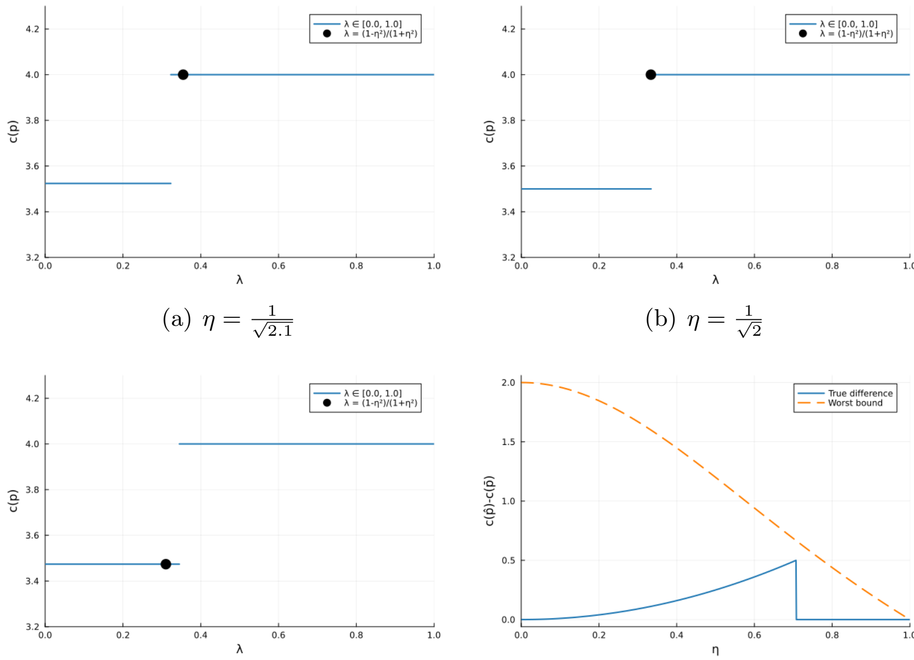
$$(\lambda_c, \lambda_d) = \left(\frac{1-\eta^2}{1+\eta^2}, \frac{1-\eta^2}{1+\eta^2} \right),$$

which implies that for this specific case, the best worst-case bound in (5) is achieved when $\lambda_c = \lambda_d$.

Theorem 3 and Proposition 3 provide the worst-case analysis of $c(\widehat{\boldsymbol{p}}) - c(\boldsymbol{p}^*)$, where $\boldsymbol{p}^*$ is an optimal solution of (Battery) problem and $\widehat{\boldsymbol{p}}$ is an optimal solution of (Reg-Battery) problem. However, we expect the actual difference between the objective function value of the problem to be much smaller due to the following reasons: (i) In the proof of Theorem 3, we drop the charging and discharging values of the regularized MIP solution by taking the minimum level of zero for all times considered. Even though we expect that the amounts of charging and discharging become smaller when $\boldsymbol{\lambda}$ increases, assuming them to be completely not charging or discharging may be an underestimation (Proposition 1). Indeed for high-efficiency values, we expect the amount of charging and discharging in the regularized MIP model to be close to the amount of charging and discharging in the original MIP model; and (ii) Also note that we upper bound amount of charging and discharging by $E_c^{\max}$ and $E_d^{\max}$, respectively, which in many cases can be a significant overestimation. Clearly, as shown in Theorem 3, the worst-case bound depends on the values of $E_c^{\max}$ and $E_d^{\max}$. Our empirical results in Section 4 show that the true difference is much less than the theoretical worst-case bound in the above result.

To summarize the results of this section, in the following example, we show that when the efficiency level increases, the optimal solution from (Reg-Battery) problem is an optimal solution to (Battery) problem. We also illustrate how the empirical gap is much smaller than the theoretical gap.

Example 6. *(Gap between solutions)* Consider a simple network with $\mathcal{N} = \{1, 2\}$, $\mathcal{T} = \{1, 2\}$, $\mathcal{L} = \{(1, 2)\}$. Suppose one battery is placed at node 2 (i.e., $\mathcal{N}_b = \{2\}$) with $E_c^{\min} = E_d^{\min} = 0$, $E_c^{\max} = E_d^{\max} = 1$, $E^{\min} = 0$, $E^{\max} = 6$, and $E_0 = 0$. Assume each node has one generator with $G_1^{\min} = G_2^{\min} = 2$ and $G_1^{\max} = G_2^{\max} = 4$. We further assume $-1 \leq f_{12} \leq 1$, the demand is $D_{1,1} = 5$, $D_{1,2} = 1$, $D_{2,1} = 8$, $D_{2,2} = 4$, and the values of the regularizers λ_c and λ_d are the same, i.e., $\lambda = \lambda_c = \lambda_d$. In Figures 2(a)-2(c), we numerically illustrate how the values of the regularizer affect the objective function values $c(\boldsymbol{p})$, where the vertical axis represents the objective function values $c(\boldsymbol{p})$, and the horizontal axis represents the regularize values λ. Three small incremental efficiency levels $\eta \in \{1/\sqrt{2.1}, 1/\sqrt{2}, 1/\sqrt{1.9}\}$ are considered. Based on Proposition 3, we also plot the objective function values with $\lambda = 1 - \eta^2/1 + \eta^2$. From Figure 2(a), we see that when $\eta = 1/\sqrt{2.1} \approx 0.69$, the optimal solution from the regularized MIP, with the choice of $(\lambda_c,\ \lambda_d)$ such that $\lambda_c + \eta^2\lambda_d \geq 1 - \eta^2$, is indeed not an optimal solution to the original MIP. In Figure 2(c), we observe that, with a higher efficiency level $\eta = 1/\sqrt{1.9} \approx 0.72$, the optimal solution from the regularized MIP is the optimal solution to the original MIP, while the choice of $(\lambda_c,\ \lambda_d)$ satisfying the condition that the LP relaxation of the regularized MIP model is without the integrality gap. In Figure 2(d), we show how the actual difference is much less than the theoretical worst-case bound for this example. ◇



(a) $\eta = \frac{1}{\sqrt{2.1}}$ (b) $\eta = \frac{1}{\sqrt{2}}$

(c) $\eta = \frac{1}{\sqrt{1.9}}$ (d) Gap Comparisons

Figure 2: Change in the objective value with respect to λ, where $\lambda_c = \lambda_d = \lambda$ in Proposition 3. The point denotes the value with $\lambda = (1-\eta^2)/(1+\eta^2)$ for Figures (a), (b), and (c). For Figure (d), the dashed curve (the upper one) represents the theoretical worst-case bound whereas the solid curve (the lower one) shows the empirical difference.

4 Numerical Experiments

In this section, we demonstrate the strength of the regularized MIP model in three aspects through extensive computational experiments:

1. *DCOPF with Battery:* We show that the solution retrieved from (Reg-Battery) problem is either optimal or near-optimal and better than that of (LP-Battery) problem.

2. *Long-term Planning with Stochastic Demands:* Next, we solve a battery placement problem under stochastic demand, which is modeled as a two-stage stochastic programming. In this problem, we show that the proposed method converges faster and scales to larger networks with improvements in the solution quality.

3. *Long-term Planning with $N-k$ Contingency:* Finally, we solve an $N-k$ contingency problem, which is modeled as a min-max-min problem with binary variables in all three levels. To the

best of our knowledge, there is no known efficient algorithm to solve such a trilevel problem with binary variables at all three levels. We show that (Reg-Battery) problem provides provably high-quality solutions.

For all experiments, we use standard IEEE instances available from MATPOWER (Zimmerman et al. 2010) and use the PowerModels package to read the network data (Coffrin et al. 2018). All numerical instances are implemented on Julia version 1.7 (Bezanson et al. 2017) using Gurobi version 10.0 as the optimization solver (Gurobi Optimization, LLC 2021) on a Linux x86 machine with a 64-bit operating system with 2.3GHz processor on 64GB RAM.

4.1 Experimental Setting

4.1.1 Networks

The network instance contains generator information, demand load information, and branch information, which is sufficient to model the DCOPF problem with batteries except for the multi-period demand load profiles and battery parameters. In PEGASE and RTE networks, the minimum output of a generator is nonnegative, whereas the minimum output is 0 in IEEE networks. For IEEE networks, the generator minimum output is scaled to be 1/3 of the maximum output, which is similar to the level of minimum output in PEGASE and RTE networks.

4.1.2 Hourly Load Scenarios

Since network information provides a single nominal load demand, we expand the given load demand to the time horizon considered on an hourly basis for one day, i.e., $T = 24$. We benchmark the hourly demand load of power in the U.S. lower region reported from U.S. Energy Information Administration (2022) and shape the demand load in each network data to create one $T = 24$ hourly demand load at each demand bus for a demand load profile in one day. Specifically, suppose that the benchmark demand is denoted as $\boldsymbol{D}^0 \in \mathbb{R}_+^T$. Let $D_i \in \mathbb{R}_+$ denote the nominal load demand at a demand node $i \in \mathcal{N}$. In order to have the optimal solutions with nontrivial battery operations, we rescale D_i so that $D_i \approx 0.8G_i^{\max}$. Then, the demand at time $t \in \mathcal{T}$ for node $i \in \mathcal{N}$ is given by $D_{t,i} = D_i D_t^0 / D_1^0$. When considering the stochastic demand, for each random demand $\tilde{\boldsymbol{D}}$, we add a Gaussian noise with a standard deviation, which is a certain fraction $\widehat{\sigma}$ of the demand, to obtain the stochastic demand $\tilde{\boldsymbol{D}}$. Formally, that is, with some finite number of demand scenarios considered, $\tilde{D}_{t,i}^j = D_{t,i} + r_{t,i,j}$ where $r_{t,i,j} \sim \mathcal{N}(0, \sigma_{t,i}^2)$ and $\sigma_{t,i} = \widehat{\sigma} \cdot D_{t,i}$ for all scenarios j.

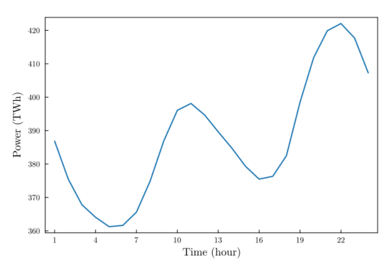


Figure 3: Benchmark Load Demand $\boldsymbol{D}^0$.

4.1.3 Battery Parameters

Battery parameters are largely adopted from Kody et al. (2022) and modified to account for the size of the network. Table 1 summarizes the battery parameters used for networks including IEEE 73, PEGASE 89, IEEE 118, and IEEE 162-bus systems. For smaller networks, the maximum storage limit, charge rate, and discharge rate are divided by 5 for the IEEE 14-bus system. For a slightly larger network IEEE 300-bus system, maximum limit/rates are multiplied by 2.5. Similarly, for large networks including PEGASE 1354 and RTE 1888, multiplication is by 5. Unless otherwise stated, these standard battery parameters are used for all numerical experiments in the paper.

Table 1: Battery Parameters for a Medium-Size Network

	Parameter	Value
Minimum storage limit	$\underline{p}^s$	0.00 p.u.
Maximum storage limit	$\bar{p}^s$	1.00 p.u.
Efficiency	η	0.95
Minimum charge rate	$\underline{p}^c$	0.00 p.u./hour
Maximum charge rate	$\bar{p}^c$	0.95 p.u./hour
Minimum discharge rate	$\underline{p}^d$	0.00 p.u./hour
Maximum discharge rate	$\bar{p}^d$	0.95 p.u./hour

4.2 Methods

We report the results with respect to the following three models:

- (Battery): The original MIP formulation.
- (LP-Battery): An LP relaxation of the original MIP formulation. We formally define the LP relaxation as:

$$f_l^{\mathrm{ori}} = \min_{\boldsymbol{\theta},\boldsymbol{f},\boldsymbol{p},\boldsymbol{u}} \left\{ c(\boldsymbol{p})\colon \text{(1a)} - \text{(1k)},\ \boldsymbol{u} \in [0,1]^{T\times N} \right\}. \qquad \text{(LP-Battery)}$$

- (Reg-Battery): The proposed regularized MIP formulation with the value of the regularizer determined based on Proposition 3.

Heuristic for Recovery of Feasible Solution From LP Relaxation. When relevant, to restrict the comparison to the primal solution, for any partial $\boldsymbol{u}$ resulting from (LP-Battery), we restore integrality by setting $u_{t,i} = 1$ if $p^s_{t,i} > p^s_{t-1,i}$ and $u_{t,i} = 0$ otherwise for each $t \in \mathcal{T}, i \in \mathcal{N}$. Charging levels ($p^c_{t,i}$) and discharging levels ($p^d_{t,i}$) are adjusted accordingly by maintaining the state-of-charge ($p^s_{t,i}$) and setting either of $p^c_{t,i}$ and $p^d_{t,i}$ to zero depending on $u_{t,i}$. We remark that this solution restoration process from LP solution uses the result from Theorem 1, and such a process is not easy to incorporate when the problem is nested in larger planning problems.

4.3 Results on DCOPF with Battery

We have proved in Section 3.2 that there exists an exactness condition and the best worst-case bound for the feasible solution produced by the regularized MIP model. In this subsection, we first empirically show that the exactness condition is satisfied in many cases, or the regularized MIP produces a near-optimal feasible solution to the DCOPF problem with batteries. We solve (Battery) to find the true optimal solution and evaluate the solution from the regularized MIP model, i.e., (Reg-Battery). We also compare the feasible solution restored from the LP relaxation, i.e., (LP-Battery) against the true optimal solution. Table 2 shows the average relative gap (computed as $(z_m - z*)/z^*$ where z_m is the objective value of the feasible solution from the method used and z^* is the optimal objective value of (Battery)) when the problem is solved with 40 random demand scenarios on networks. In this setting, b number of batteries are placed on selected buses with the largest power outputs. Since the actual efficiencies of grid-scale batteries are known to be higher than 80%, we consider different efficiency levels $\eta \in \{0.85, 0.9, 0.95\}$.

The experiment demonstrates that the regularized MIP yields solutions close to the optimal solution. The average across networks is limited to less than 0.5% for different efficiency levels considered. When compared to the solutions restored from LP relaxation, regularized MIP performs almost always better. LP relaxation performs better for certain instances with some specific efficiency η, such as in the IEEE 73-bus system. This loss, however, is limited and the maximum loss is still less than 0.5% Moreover, LP relaxation tends to have larger variances and unpredictable proximity to the optimal solution. For the RTE 1888-bus system with $\eta = 0.95$, despite the average relative gap being less than 2%, the largest gap is 10.25%, which can be detrimental to a large network system.

Table 2: The average relative gap with respect to the optimal objective value

		(Reg-Battery)			(LP-Battery)		
Network	b	$\eta = 0.85$	$\eta = 0.90$	$\eta = 0.95$	$\eta = 0.85$	$\eta = 0.90$	$\eta = 0.95$
IEEE 14	2	0.00%	0.00%	0.00%	11.88%	0.00%	0.00%
IEEE 73	2	0.33%	0.23%	0.11%	0.30%	0.17%	0.07%
PEGASE 89	2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
IEEE 118	2	0.00%	0.00%	0.00%	4.18%	3.96%	3.73%
IEEE 162	3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
IEEE 300	3	0.08%	0.05%	0.02%	0.81%	0.43%	0.10%
PEGASE 1354	5	0.00%	0.00%	0.00%	4.51%	0.26%	0.26%
RTE 1888	5	0.05%	0.02%	0.03%	8.16%	5.43%	1.74%
Average		0.06%	0.04%	0.02%	3.73%	1.28%	0.74%

4.4 Results on Long-Term Planning Problem with Stochastic Demands

In this section, instead of having a fixed location of batteries, the placement of batteries is a set of decisions. Since the placement can be considered as a long-term decision, this is modeled as a two-stage stochastic programming, where the first-stage decision is to determine the battery locations and the second-stage problem is the DCOPF problem with batteries placed and stochastic demand.

Formulation and Methodology. See Appendix B for details of the formulation and methodology.

Performance Metric. We calculate the relative gap from the best objective value achieved. Specifically, we let z_m be the objective function value from a feasible solution obtained by model m where m is one of three models. Let $\widehat{z} = \min\{z_m\}$, then, the relative gap is $(z_m - \widehat{z})/\widehat{z}$. Note that when a problem can be solved with (Battery), this achieves the optimal solution hence $\widehat{z}$ is the objective value from (Battery).

Computational Results. We simulate different values of noise level $\widehat{\sigma}$ over 10 times. Specifically, for a given network and a choice of model, this particular optimization problem is solved 10 times with different sets of stochastic load demand scenarios for the specified noise level. We set a time limit of 6 hours for each optimization problem.

Since the budget to install a fixed number of batteries is constant, we are interested in the expected value of the second-stage cost. In general, we observe that using (Reg-Battery) consistently provides high-quality solutions while reducing the computational time. Note that not all optimization problems for this experiment are solved to optimality or find feasible primal solutions. Therefore, Table 3, reports the average solution time and the primal solution quality of the first-stage solution, measured by the relative gap with respect to the best solution found against the three methods. The main benefits of using (Reg-Battery) are summarized below:

1. Within the time limit, in most cases, using (Reg-Battery) often yields solutions that are optimal or near-optimal compared to those obtained using (Battery) for smaller networks. Note that in all instances, the average relative gaps are at most 1%.

2. For larger networks, such as PEGASE 1354 and RTE 1888-bus systems, (Reg-Battery) consistently yields better performance than (Battery) and (LP-Battery). Although all three models may find feasible solutions, (Battery) and (LP-Battery) models produce significantly sub-optimal solutions compared to those obtained with (Reg-Battery). For example, from Table 3, the average relative gap in PEGASE 1354-bus system can be as high as 1.27% for noise level $\widehat{\sigma} = 0.1$ and in RTE 1888-bus system as high as 1.18% for noise level $\widehat{\sigma} = 0.15$ when using (LP-Battery). This difference can indeed mean significant loss for large systems.

3. Using (Reg-Battery) speeds up the solution process. Although the solution times are relatively similar for smaller networks, the differences grow in the larger networks. One might expect that modeling the second stage with (LP-Battery) and modeling it with (Reg-Battery) would have similar computational times, as the second stages are both linear and continuous, but this is not the case. Notably, this difference is particularly evident in larger networks.

All instances are available in the Dropbox via the link http://tinyurl.com/regularizedbattery.

4.5 Results on Long-Term Planning Under Contingency

In order to improve transmission reliability by strategically siting batteries, we consider a $N-k$ contingency problem with a deterministic demand. The $N-k$ contingency problem, addressing disruptions or attacks within the system, is an important question that has long been studied, but its importance has grown notably in recent years due to increased uncertainties across various aspects of the system (see Birge et al. (2023) for a supply chain example). Contingencies can be due to some cyber-security attacks (see Garifi et al. 2021) as well as unforeseen transmission line failures due to various physical threats including a higher frequency of wildfire incidents which are growing over the years. While Yang and Nagarajan (2022) studied optimal power flow under $N-1$ contingency in power systems, to the best of our knowledge, there is no prior work studying the DCOPF problem with $N-k$ contingency with the battery placement problem.

Formulation. This problem can also be understood as a trilevel min-max-min problem (or a defender-attacker-defender problem). A network designer makes a long-term decision on whether to install an energy storage system for node $i \in \mathcal{N}$, represented by variable x_i, to enhance the robustness of the power system. The system operator has a budget of b batteries to add. The second level is an interdictor who can disrupt up to k transmission lines with the goal of maximizing load shedding or excess power. The lowest level is a system operator solving DCOPF. The overall trilevel problem is presented in Figure 4. For simplicity, following the theoretical and computational results in Johnson and Dey (2022), we remove the Ohm's law constraint from the DCOPF in the third level. The detailed formulation is provided in Appendix B.

Table 3: The average solution time in seconds and the average relative gap against the best solution

	$\widehat{\sigma} = 0.10$					
	(Reg-Battery)		(LP-Battery)		(Battery)	
Network	Time (sec)	Rel. Gap	Time (sec)	Rel. Gap	Time (sec)	Rel. Gap
IEEE 14	760	0.00%	584	0.09%	13	0.00%
IEEE 73	907	0.00%	519	2.50%	524	0.00%
PEGASE 89	3262	0.05%	5655	0.14%	2719	0.00%
IEEE 118	5719	0.00%	9371	0.41%	19197	0.03%
IEEE 162	4213	0.08%	10577	0.02%	8723	0.00%
IEEE 300	4037	0.12%	20010	0.24%	16541	0.03%
PEGASE 1354	19537	0.09%	20573	1.27%	21600	0.89%
RTE 1888	21600	0.08%	21600	0.41%	21600	0.65%
	$\widehat{\sigma} = 0.15$					
	(Reg-Battery)		(LP-Battery)		(Battery)	
Network	Time (sec)	Rel. Gap	Time (sec)	Rel. Gap	Time (sec)	Rel. Gap
IEEE 14	724	0.00%	446	0.00%	15	0.00%
IEEE 73	932	0.13%	604	0.56%	606	0.00%
PEGASE 89	2339	0.08%	3743	0.20%	2535	0.00%
IEEE 118	2398	0.15%	6728	0.99%	16701	0.36%
IEEE 162	5016	0.09%	13593	0.02%	11743	0.00%
IEEE 300	5104	0.12%	20623	0.16%	12960	0.07%
PEGASE 1354	19601	0.04%	20546	1.17%	21600	1.00%
RTE 1888	19739	0.08%	21600	1.18%	19440	0.91%
	$\widehat{\sigma} = 0.20$					
	(Reg-Battery)		(LP-Battery)		(Battery)	
Network	Time (sec)	Rel. Gap	Time (sec)	Rel. Gap	Time (sec)	Rel. Gap
IEEE 14	639	0.00%	281	0.00%	14	0.00%
IEEE 73	703	0.06%	373	0.37%	396	0.00%
PEGASE 89	7245	0.07%	4128	0.12%	4142	0.00%
IEEE 118	7789	0.01%	6817	0.14%	10800	0.03%
IEEE 162	4169	0.08%	7652	0.79%	5558	0.00%
IEEE 300	7937	0.14%	20497	0.26%	19050	0.04%
PEGASE 1354	21600	0.05%	21600	0.83%	21600	0.78%
RTE 1888	21600	0.10%	21600	1.15%	21600	0.59%

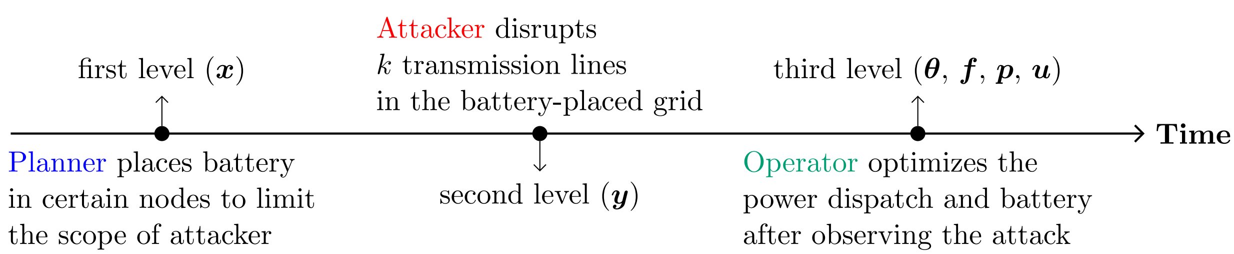


Figure 4: Long-term planning under $N - k$ contingency problem is a trilevel min-max-min problem with binary decision variables in each level.

Solution Methodology. There are a few algorithms proposed to solve bilevel or trilevel problems (see, e.g., Johnson et al. 2021, Zeng and An 2014, Bienstock and Özbay 2008). However, for this particular trilevel problem with binary variables in all three stages, applying existing methods is computationally intractable. Even for the smallest instance of the IEEE 14-bus system and with the first-stage decision $\boldsymbol{x}$ fixed, solving the bilevel max - min problem using the state-of-the-art algorithm proposed by Zeng and An (2014) does not converge in 6 hours. Hence, solving the original trilevel problem is intractable using the (Battery) formulation. However, using the (Reg-Battery) in the third level has the advantage that the third level becomes an LP per Theorem 1. Then, by taking the dual of the third level, the trilevel problem can be reduced to a bilevel problem. Moreover, we can make the following remark.

Remark 2. *Explicit bounds can be found for all dual variables of the third level problem that appear in bilinear terms together with the second level variables.*

This allows the exact linearization for bilinear terms that appear in the objective function using McCormick inequalities. We then apply the algorithm by Bienstock and Özbay (2008) to solve the resulting bilevel optimization problem. The details of the boundedness result (Proposition 5) and the algorithm are presented in Appendix B.

Using the regularized MIP model in the third level has another crucial advantage in that it can provide a feasible solution with an upper bound, along with the optimality gap derived using the LP relaxation of the third level to obtain lower bounds. In particular, the general form of our trilevel problem is the following:

$$z^{OPT} = \min_{\boldsymbol{x}\in\mathcal{X}} \left\{ \max_{\boldsymbol{y}\in\mathcal{Y}} \left\{ \min_{\boldsymbol{\theta},\boldsymbol{f},\boldsymbol{p},\boldsymbol{u}\in\mathcal{F}(\boldsymbol{x},\boldsymbol{y})} c(\boldsymbol{p}) \right\} \right\}. \tag{6}$$

The set $\mathcal{Y}$ is finite as there are a finite number of edges and only finite possible attack strategies are available. Let $\mathcal{Y} = \{\boldsymbol{y}^1, ..., \boldsymbol{y}^K\}$. Then (6) can be equivalently reformulated as the following:

$$\begin{aligned} z^{OPT} = \min_{\boldsymbol{x}\in\mathcal{X},\xi} \quad & \xi \\ \text{s.t.} \quad & \xi \geq \min\{c(\boldsymbol{p}) : \boldsymbol{\theta}, \boldsymbol{f}, \boldsymbol{p}, \boldsymbol{u} \in \mathcal{F}(\boldsymbol{x}, \boldsymbol{y}^i)\} \quad \forall i \in [K]. \end{aligned} \tag{7}$$

Then, due to Theorem 1, using the regularized MIP model in the third level is equivalent to solving the following:

$$\begin{aligned} z^{REG} = \min_{\boldsymbol{x}\in\mathcal{X},\xi} \quad & \xi \\ \text{s.t.} \quad & \xi \geq \min\{c(\boldsymbol{p}) + \boldsymbol{\lambda}^\top g(\boldsymbol{p}) : \boldsymbol{\theta}, \boldsymbol{f}, \boldsymbol{p}, \boldsymbol{u} \in \mathcal{F}(\boldsymbol{x}, \boldsymbol{y}^i)\} \quad \forall i \in [K]. \end{aligned} \tag{8}$$

Similarly, using the LP relaxation can be formulated as:

$$\begin{aligned} z^{LP} = \min_{\boldsymbol{x}\in\mathcal{X},\xi} \quad & \xi \\ \text{s.t.} \quad & \xi \geq \min\{c(\boldsymbol{p}) : \boldsymbol{\theta}, \boldsymbol{f}, \boldsymbol{p}, \boldsymbol{u} \in \mathcal{R}(\boldsymbol{x}, \boldsymbol{y}^i)\} \quad \forall i \in [K] \end{aligned} \tag{9}$$

where $\mathcal{R}(\boldsymbol{x}, \boldsymbol{y})$ is a linear relaxation of $\mathcal{F}(\boldsymbol{x}, \boldsymbol{y})$. It is then easy to verify that $z^{LP} \leq z^{OPT} \leq z^{REG}$. For example, let $\eta(\boldsymbol{x}, \boldsymbol{y}) = \min\{c(\boldsymbol{p}) : \boldsymbol{\theta}, \boldsymbol{f}, \boldsymbol{p}, \boldsymbol{u} \in \mathcal{F}(\boldsymbol{x}, \boldsymbol{y})\}$ and $\gamma(\boldsymbol{x}, \boldsymbol{y}) = \min\{c(\boldsymbol{p}) + \boldsymbol{\lambda}^\top g(\boldsymbol{p}) : \boldsymbol{\theta}, \boldsymbol{f}, \boldsymbol{p}, \boldsymbol{u} \in \mathcal{F}(\boldsymbol{x}, \boldsymbol{y})\}$. Let $(\xi^*, \boldsymbol{x}^*, \boldsymbol{y}^*)$ and $(\widehat{\xi}, \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ be optimal solutions corresponding to z^{OPT} and z^{REG} respectively. Let $\check{\boldsymbol{y}} \in \arg\max_{\boldsymbol{y}\in\mathcal{Y}} \eta(\widehat{\boldsymbol{x}}, \boldsymbol{y})$. Then we have $z^{REG} = \widehat{\boldsymbol{\xi}} = \gamma(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}) \geq \gamma(\widehat{\boldsymbol{x}}, \check{\boldsymbol{y}}) \geq \eta(\widehat{\boldsymbol{x}}, \check{\boldsymbol{y}}) \geq \eta(\boldsymbol{x}^*, \boldsymbol{y}^*) = \boldsymbol{\xi}^* = z^{OPT}$, where the first inequality follows from (8) and the optimality of $\widehat{y}$ for the second-level max optimization problem when $\boldsymbol{x}$ is fixed to $\widehat{\boldsymbol{x}}$, the second inequality follows from the definition of $\eta(\cdot)$ and $\gamma(\cdot)$, and the last inequality follows from the optimality of $\boldsymbol{x}^*$ for (7). A similar proof can be used to verify that $z^{LP} \leq z^{OPT}$.

Also note that if $\widehat{\boldsymbol{x}}$ is the optimal solution of z^{REG}, then using the notation from the previous paragraph, we have that $\max_{\boldsymbol{y}\in\mathcal{Y}} \left\{\min_{\boldsymbol{\theta},\boldsymbol{f},\boldsymbol{p},\boldsymbol{u}\in\mathcal{F}(\widehat{\boldsymbol{x}},\boldsymbol{y})} c(\boldsymbol{p})\right\} = \eta(\widehat{\boldsymbol{x}}, \check{\boldsymbol{y}}) \leq \gamma(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}) = z^{REG}$. Thus, we arrive at the following conclusion.

Proposition 4. *Let $\widehat{\boldsymbol{x}}$ be an optimal solution of (8). Then this solution is a solution for the trilevel problem (7) with an optimality gap of at most $(z^{REG} - z^{LP})/z^{REG}$.*

Computational Results. In Table 4, we report the minimum, maximum, and average of the optimality gap of 10 simulations for each network instance and different combinations of the maximum number of batteries placed and the maximum number of contingencies. Since (8) and (9) are not always solved to optimality, we use the upper bound of z^{REG} and lower bound of z^{LP} to compute the gap. We observe that for smaller network systems, the regularized MIP generates provably near-optimal solutions. For PEGASE 1354 and RTE 1888-bus systems, the gap is larger, but most likely this is due to having a poor lower bound on z^{LP} as (9) is not solved to optimality within the given time limit of 6 hours. Note that the gap in solving (8) and (9) is reported in Tables 5-8 in Appendix C. We observe that solving the trilevel problem with the regularized formulation is efficient. For most of the instances, (8) is solved to optimality well within the time limit. Only the larger network instances take the entire 6-hour time limit with a small optimality gap. Hence, using the regularized formulation not only gives a quality upper bound but can be used to solve such trilevel problems efficiently (see Tables 5-8 in Appendix C). Despite some gaps in larger instances, using the regularized MIP model provides a solution methodology with a guarantee that was not possible previously with existing methods.

Table 4: Solution Quality for $N-k$ Contingency Problem

	Optimality Gap											
	$b=2, k=3$			$b=2, k=5$			$b=3, k=5$			$b=5, k=10$		
Network	Min	Max	Avg	Min	Max	Avg	Min	Max	Avg	Min	Max	Avg
IEEE 14	0.62%	0.63%	0.62%	0.57%	0.58%	0.57%	0.57%	0.58%	0.57%	0.57%	0.58%	0.57%
IEEE 73	0.00%	0.07%	0.03%	0.00%	0.49%	0.06%	0.00%	0.49%	0.06%	0.00%	0.02%	0.01%
PEGASE 89	0.15%	0.24%	0.16%	0.12%	0.12%	0.12%	0.18%	0.18%	0.18%	0.27%	1.16%	1.04%
IEEE 118	1.05%	1.78%	1.25%	1.38%	2.47%	1.65%	2.06%	2.88%	2.40%	2.07%	4.78%	3.38%
IEEE 162	0.23%	0.24%	0.24%	0.22%	0.22%	0.22%	2.34%	4.50%	2.60%	2.42%	2.79%	2.63%
IEEE 300	3.69%	4.08%	3.81%	3.41%	3.64%	3.48%	3.69%	6.73%	4.09%	6.21%	8.24%	7.63%
PEGASE 1354	9.81%	19.43%	14.25%	7.86%	14.98%	13.53%	15.00%	22.33%	18.68%	16.98%	27.45%	19.26%
RTE 1888	6.73%	13.07%	8.12%	6.32%	12.50%	8.30%	12.11%	18.33%	14.84%	17.66%	26.87%	25.59%

5 Conclusion

In this paper, we proposed a new model to solve the DCOPF problem with battery operations. We regularized the objective function by penalizing the charge and discharge of batteries. In Theorem 1, we present a sufficient condition on the regularizers so that there is no integrality gap between the regularized MIP problem and its LP relaxation. When the efficiency of the battery is relatively high, this penalty is very small. Empirical results show that the optimal solution from this regularized model is often a true optimal solution to the original model or close to the optimal solution, performing much better than the theoretical guarantees verified in Theorem 3. Moreover, we prove in Theorem 2 that the optimal solution from the regularized model is more reasonable in that the battery operation does not contribute to further load-shedding or excess power depending on the state of the system at the time. This property may be of interest to the system operator.

For a simpler problem that only considers the battery operation with $\eta = 1$, a polynomial algorithm has been proposed in Bakhshi and Ostrowski (2023). However, only a few studies have focused on the complexity associated with the general efficiency level with $0 < \eta < 1$. Bansal and Günlük (2023) proved an NP-hardness for a similar problem where the storage level varies over time based on two complementary variables. The proof of NP-hardness relies on time-varying bounds and their result shows that (Battery) problem with time-varying bounds on $\boldsymbol{p}^c$ and $\boldsymbol{p}^d$ is NP-hard to solve. However, it remains an open problem that for a fixed bound on charge and discharge levels with a loss-incurring battery system ($0 < \eta < 1$), the problem is NP-hard.

We introduce two optimization problems that are related to solving the battery siting problem; one with a stochastic demand and the other one with $N - k$ contingency. Both these problems are intractable to solve using the exact battery formulation. We use the main benefit of the regularized formulation model to reformulate these challenging problems, and show that the regularized formulation solves large-scale instances of these problems efficiently and yields near-optimal solutions in most cases.

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References

G. Angulo, S. Ahmed, and S. S. Dey. Improving the integer l-shaped method. *INFORMS J. on Computing*, 28(3):483–499, aug 2016. ISSN 1526-5528.

J. M. Arroyo, L. Baringo, A. Baringo, R. Bolaños, N. Alguacil, and N. G. Cobos. On the use of a convex model for bulk storage in mip-based power system operation and planning. *IEEE Transactions on Power Systems*, 35(6):4964–4967, 2020. doi: 10.1109/TPWRS.2020.3020730.

A. Atamtürk and M. Zhang. Two-stage robust network flow and design under demand uncertainty. *Operations Research*, 55(4):662–673, 2007.

M. Bakhshi and J. Ostrowski. A polynomial algorithm for the lossless battery charging problem, 2023. URL `https://optimization-online.org/2023/07/a-polynomial-algorithm-for-the-lossless-battery-charging-problem/`.

R. Baldick, Y. Chen, and B. Huang. Optimization Formulations for Storage Devices with Disjoint Operating Modes. *Operations Research*, 2023.

I. Bansal and O. Günlük. Warehouse problem with bounds, fixed costs and complementarity constraints. *arXiv preprint*, 2023.

O. Ben-Ayed and C. E. Blair. Computational difficulties of bilevel linear programming. *Operations Research*, 38(3):556–560, 1990. ISSN 0030364X, 15265463. URL `http://www.jstor.org/stable/171369`.

J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah. Julia: A fresh approach to numerical computing. *SIAM Review*, 59(1):65–98, 2017. doi: 10.1137/141000671. URL `https://epubs.siam.org/doi/10.1137/141000671`.

D. Bienstock and N. Özbay. Computing robust basestock levels. *Discrete Optimization*, 5(2):389–414, 2008.

J. R. Birge, A. Capponi, and P.-C. Chen. Disruption and rerouting in supply chain networks. *Operations Research*, 71(2):750–767, 2023. doi: 10.1287/opre.2022.2409. URL `https://doi.org/10.1287/opre.2022.2409`.

J. Carpentier. Contribution to the economic dispatch problem. *Bulletin de la Societe Francoise des Electriciens*, 3(8):431–447, 1962.

K. M. Chandy, S. H. Low, U. Topcu, and H. Xu. A simple optimal power flow model with energy storage. *49th IEEE Conference on Decision and Control (CDC)*, pages 1051–1057, 2010. doi: 10.1109/CDC.2010.5718193.

C. Coffrin, H. L. Hijazi, and P. Van Hentenryck. The qc relaxation: A theoretical and computational study on optimal power flow. *IEEE Transactions on Power Systems*, 31(4):3008–3018, 2015.

C. Coffrin, R. Bent, K. Sundar, Y. Ng, and M. Lubin. Powermodels.jl: An open-source framework for exploring power flow formulations. In *2018 Power Systems Computation Conference (PSCC)*, pages 1–8, June 2018. doi: 10.23919/PSCC.2018.8442948.

W. Cole, B. Frew, T. Mai, Y. Sun, J. Bistline, G. Blanford, D. Young, C. Marcy, C. Namovicz, R. Edelman, et al. Variable renewable energy in long-term planning models: a multi-model perspective. Technical report, National Renewable Energy Lab.(NREL), Golden, CO (United States), 2017.

S. S. Dey, R. Mazumder, and G. Wang. Using ℓ 1-relaxation and integer programming to obtain dual bounds for sparse PCA. *Operations Research*, 70(3):1914–1932, 2022.

K. Garifi, E. S. Johnson, B. Arguello, and B. J. Pierre. Transmission grid resiliency investment optimization model with socp recovery planning. *IEEE Transactions on Power Systems*, 37(1):26–37, 2021.

T. M. Gür. Review of electrical energy storage technologies, materials and systems: challenges and prospects for large-scale grid storage. *Energy & Environmental Science*, 11(10):2696–2767, 2018.

Gurobi Optimization, LLC. Gurobi Optimizer Reference Manual, 2021. URL `https://www.gurobi.com`.

M. Hoffman, M. Kintner-Meyer, J. DeSteese, and A. Sadovsky. Analysis tools for sizing and placement of energy storage in grid applications, 01 2011.

R. Jabr. Optimal power flow using an extended conic quadratic formulation. *IEEE Transactions on Power Systems*, 23:1000 – 1008, 09 2008. doi: 10.1109/TPWRS.2008.926439.

R. G. Jeroslow. The polynomial hierarchy and a simple model for competitive analysis. *Mathematical Programming*, 32:146–164, 1985.

D. R. Jiang and W. B. Powell. Optimal hour-ahead bidding in the real-time electricity market with battery storage using approximate dynamic programming. *INFORMS Journal on Computing*, 27(3):525–543, 2015.

R. Jiang, J. Wang, and Y. Guan. Robust unit commitment with wind power and pumped storage hydro. *IEEE Transactions on Power Systems*, 27(2):800–810, 2012. doi: 10.1109/TPWRS.2011.2169817.

R. Jiang, M. Zhang, G. Li, and Y. Guan. Two-stage network constrained robust unit commitment problem. *European Journal of Operational Research*, 234(3):751–762, 2014.

E. S. Johnson and S. S. Dey. A scalable lower bound for the worst-case relay attack problem on the transmission grid. *INFORMS Journal on Computing*, 2022.

E. S. Johnson, S. S. Dey, J. Eckstein, C. A. Phillips, and J. D. Siirola. A covering decomposition algorithm for power grid cyber-network segmentation, 2021. URL `https://optimization-online.org/?p=18421`.

J. H. Kim and W. B. Powell. Optimal energy commitments with storage and intermittent supply. *Operations research*, 59(6):1347–1360, 2011.

B. Kocuk, S. S. Dey, and X. A. Sun. Strong SOCP Relaxations for the Optimal Power Flow Problem. *Operations Research*, 64(6):1177–1196, 2016. doi: 10.1287/opre.2016.1489. URL `https://doi.org/10.1287/opre.2016.1489`.

A. Kody, R. Piansky, and D. K. Molzahn. Optimizing transmission infrastructure investments to support line de-energization for mitigating wildfire ignition risk, 2022.

G. Laporte and F. V. Louveaux. The integer L-shaped method for stochastic integer programs with complete recourse. *Operations Research Letters*, 13(3):133–142, 1993. ISSN 0167-6377. doi: https://doi.org/10.1016/0167-6377(93)90002-X. URL `https://www.sciencedirect.com/science/article/pii/016763779390002X`.

J. Lavaei and S. H. Low. Zero duality gap in optimal power flow problem. *IEEE Transactions on Power Systems*, 27(1):92–107, 2012. doi: 10.1109/TPWRS.2011.2160974.

N. Löhndorf and D. Wozabal. The value of coordination in multimarket bidding of grid energy storage. *Operations Research*, 71(1):1–22, 2023.

A. Lorca and X. A. Sun. Multistage robust unit commitment with dynamic uncertainty sets and energy storage. *IEEE Transactions on Power Systems*, 32, 04 2016. doi: 10.1109/TPWRS.2016.2593422.

T. L. Magnanti and R. T. Wong. Accelerating Benders decomposition: Algorithmic enhancement and model selection criteria. *Operations research*, 29(3):464–484, 1981.

S. Mattia, F. Rossi, M. Servilio, and S. Smriglio. Staffing and scheduling flexible call centers by two-stage robust optimization. *Omega*, 72:25–37, 2017.

G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part i—convex underestimating problems. *Mathematical programming*, 10(1):147–175, 1976.

J. Nascimento and W. B. Powell. An optimal approximate dynamic programming algorithm for concave, scalar storage problems with vector-valued controls. *IEEE Transactions on Automatic Control*, 58(12): 2995–3010, 2013. doi: 10.1109/TAC.2013.2272973.

D. Pozo. Linear battery models for power systems analysis. *Electric Power Systems Research*, 212:108565, nov 2022. doi: 10.1016/j.epsr.2022.108565. URL `https://doi.org/10.1016%2Fj.epsr.2022.108565`.

D. Pozo, J. Contreras, and E. E. Sauma. Unit commitment with ideal and generic energy storage units. *IEEE Transactions on Power Systems*, 29(6):2974–2984, 2014. doi: 10.1109/TPWRS.2014.2313513.

W. Qi, Y. Liang, and Z.-J. M. Shen. Joint planning of energy storage and transmission for wind energy generation. *Operations Research*, 63(6):1280–1293, 2015.

D. F. Salas and W. B. Powell. Benchmarking a scalable approximate dynamic programming algorithm for stochastic control of grid-level energy storage. *INFORMS Journal on Computing*, 30(1):106–123, 2018.

B. Singh and B. Knueven. Lagrangian relaxation based heuristics for a chance-constrained optimization model of a hybrid solar-battery storage system. *Journal of Global Optimization*, 80(4):965–989, 2021.

R. Sioshansi, P. Denholm, J. Arteaga, S. Awara, S. Bhattacharjee, A. Botterud, W. Cole, A. Cortés, A. d. Queiroz, J. DeCarolis, Z. Ding, N. DiOrio, Y. Dvorkin, U. Helman, J. X. Johnson, I. Konstantelos, T. Mai, H. Pandžić, D. Sodano, G. Stephen, A. Svoboda, H. Zareipour, and Z. Zhang. Energy-storage modeling: State-of-the-art and future research directions. *IEEE Transactions on Power Systems*, 37 (2):860–875, 2022. doi: 10.1109/TPWRS.2021.3104768.

R. Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 58(1):267–288, 1996.

U.S. Energy Information Administration. Hourly electric grid monitor, 2022. URL `https://www.eia.gov/electricity/gridmonitor/dashboard/electric_overview/US48/US48`.

U.S. Energy Information Administration. Annual energy outlook 2023, 2023. URL `https://www.eia.gov/outlooks/aeo/narrative/index.php`.

D. van Hulst, D. Den Hertog, and W. Nuijten. Robust shift generation in workforce planning. *Computational Management Science*, 14(1):115–134, 2017.

O. Q. Wu, R. Kapuscinski, and S. Suresh. On the Distributed Energy Storage Investment and Operations. *Manufacturing and Service Operations Management*, 2023.

N. Yan, Z. X. Xing, W. Li, and B. Zhang. Economic dispatch application of power system with energy storage systems. *IEEE Transactions on Applied Superconductivity*, 26(7):1–5, 2016. doi: 10.1109/TASC.2016. 2598963.

H. Yang and H. Nagarajan. Optimal power flow in distribution networks under n − 1 disruptions: A multistage stochastic programming approach. *INFORMS Journal on Computing*, 34(2):690–709, 2022. doi: 10.1287/ijoc.2021.1080. URL `https://doi.org/10.1287/ijoc.2021.1080`.

B. Zeng and Y. An. Solving bilevel mixed integer program by reformulations and decomposition. *Optimization online*, pages 1–34, 2014.

R. D. Zimmerman, C. E. Murillo-Sánchez, and R. J. Thomas. Matpower: Steady-state operations, planning, and analysis tools for power systems research and education. *IEEE Transactions on power systems*, 26 (1):12–19, 2010.

Appendix A Proof of Theorem 2

Theorem 2. *Suppose $E_c^{\min} = E_d^{\min} = 0$. Let $\boldsymbol{p}$ be an optimal solution to* (Reg-Battery) *problem. Then:*

(i) For all $\boldsymbol{\lambda} \in \mathbb{R}_+^2$, we have $p_{t,i}^c p_{t,i}^{ls} = 0$ for all $t \in \mathcal{T},\ i \in \mathcal{N}$.

(ii) If $\lambda_c + \eta^2 \cdot \lambda_d > 1 - \eta^2$, then we have $p_{t,i}^d p_{t,i}^{ex} = 0$ for all $t \in \mathcal{T},\ i \in \mathcal{N}$.

The proof of Theorem 2 is divided into two parts via Lemma 1 and Lemma 2.

Lemma 1. *Suppose $E_c^{\min} = E_d^{\min} = 0$. For any $\boldsymbol{\lambda} \in \mathbb{R}_+^2$, let $\boldsymbol{p}$ be an optimal solution to* (Reg-Battery) *problem. Then, $p_{t,i}^c p_{t,i}^{ls} = 0$ for all $t \in \mathcal{T},\ i \in \mathcal{N}$.*

Proof. By contradiction, suppose there exists an optimal solution $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$ of (Reg-Battery) problem such that $\widehat{p}_{t,i}^c > 0$ and $\widehat{p}_{t,i}^{ls} > 0$ for at least one $(t,i) \in \mathcal{T} \times \mathcal{N}$.

Without loss of generality, we show the proof for one such $i \in \mathcal{N}$ as the proof can be extended for any multiple nodes. Let $\tau_0 \in \mathcal{T}$ be the first time period such that $\widehat{p}_{\tau_0,i}^c > 0$ and $\widehat{p}_{\tau_0,i}^{ls} > 0$. Let $\tau_1, \ldots, \tau_k \in \{\tau_0 + 1, \ldots, T\}$ such that $\widehat{p}_{\tau_j,i}^d > 0$ for $j \in [k]$ and $\widehat{p}_{t.i}^d = 0$ for $t \in \{\tau_0 + 1, \ldots, T\} \setminus \{\tau_1, \ldots, \tau_k\}$. We define adjustments to the state-of-charge as the following:

$$\delta_{\tau_j,i} = \begin{cases} -\eta \cdot \min\{\widehat{p}_{\tau_0,i}^c, \widehat{p}_{\tau_0,i}^{ls}\} & j = 0, \\ \delta_{\tau_{j-1},i} + \max\left\{E^{\min} - \widehat{p}_{\tau_j,i}^s - \delta_{\tau_{j-1},i}, 0\right\} & \forall j \in [k]. \end{cases}$$

Note that $\delta_{\tau_0,i} < 0$. We proceed to construct a solution $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{f}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{u}})$ from the current optimal solution $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$, where $\tilde{\boldsymbol{\theta}} = \widehat{\boldsymbol{\theta}}$, $\tilde{\boldsymbol{f}} = \widehat{\boldsymbol{f}}$, $\tilde{\boldsymbol{u}} = \widehat{\boldsymbol{u}}$, $\tilde{\boldsymbol{p}}^{ex} = \widehat{\boldsymbol{p}}^{ex}$, $\tilde{\boldsymbol{p}}^g = \widehat{\boldsymbol{p}}^g$, and changing values only corresponding to node i as follows:

$$\tilde{p}_{t,i}^c = \begin{cases} \widehat{p}_{t,i}^c, & \forall t \in \mathcal{T} \setminus \{\tau_0\}, \\ \widehat{p}_{\tau_0,i}^c - \min\{\widehat{p}_{\tau_0,i}^c,\ \widehat{p}_{\tau_0,i}^{ls}\}, & t = \tau_0, \end{cases}$$

$$\tilde{p}_{t,i}^d = \begin{cases} \widehat{p}_{t,i}^d, & \forall t \in \mathcal{T} \setminus \{\tau_1, \ldots, \tau_k\}, \\ \widehat{p}_{t,i}^d - \eta \cdot \max\left\{E^{\min} - \widehat{p}_{\tau_j,i}^s - \delta_{\tau_{j-1},i}, 0\right\}, & t = \tau_j,\ \forall j \in [k], \end{cases}$$

$$\tilde{p}_{t,i}^{ls} = \begin{cases} \widehat{p}_{t,i}^{ls}, & \forall t \in \mathcal{T} \setminus \{\tau_0, \tau_1, \ldots, \tau_k\}, \\ \widehat{p}_{t,i}^{ls} - \min\{\widehat{p}_{\tau_0,i}^c,\ \widehat{p}_{\tau_0,i}^{ls}\}, & t = \tau_0, \\ \widehat{p}_{t,i}^{ls} + \eta \cdot \max\left\{E^{\min} - \widehat{p}_{\tau_j,i}^s - \delta_{\tau_{j-1},i}, 0\right\}, & t = \tau_j,\ \forall j \in [k], \end{cases}$$

$$\tilde{p}_{t,i}^s = \begin{cases} \widehat{p}_{t,i}^s, & \forall t \in [\tau_0 - 1], \\ \widehat{p}_{t,i}^s + \delta_{\tau_j,i}, & \forall t \in \{\tau_j, \ldots, \tau_{j+1} - 1\},\ \forall j \in \{0, \ldots, k\}, \end{cases}$$

where $\tau_{k+1} - 1 = T$.

The new solution $\tilde{\boldsymbol{p}}$ above is created in the following fashion: We first reduce both $\widehat{p}_{\tau_0,i}^c$ and $\widehat{p}_{\tau_0,i}^{ls}$, which causes $\widehat{p}_{t,i}^s$ to reduce in time periods following τ_0. In particular, it may fall below $E^{\min}$. In order to fix this, we need to modify the discharging levels (and loss values of corresponding time periods) to ensure that the storage levels meet the minimum requirement $E^{\min}$. We carefully decrease the values of $\widehat{p}_{t,i}^d$ so that the minimum discharge level is satisfied and the state-of-charge level is never below $E^{\min}$ at the same time.

Claim 1. *$(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{f}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{u}})$ is a feasible solution to* (Reg-Battery) *with given $\boldsymbol{\lambda}$.*

Proof. It suffices to show that $\tilde{\boldsymbol{p}}$ satisfies (1d) – (1f), (1h) and (1j) – (1k).

- (1d): It is straightforward to verify that $\tilde{p}^c_{t,i} \geq 0 = E^{\min}_c$ for all $t \in \mathcal{T}$.
- (1e): We want to show that $\tilde{p}^d_{t,i} \geq 0 = E^{\min}_d$ for all $t \in \mathcal{T}$. It is sufficient to prove this for $t = \{\tau_1, \ldots, \tau_k\}$. There are two cases:
  - (i) If $E^{\min} - \widehat{p}^s_{\tau_j,i} - \delta_{\tau_{j-1},i} \leq 0$: This case is straightforward as $\tilde{p}^d_{t,i} = \widehat{p}^d_{t,i} \geq 0$.
  - (ii) If $E^{\min} - \widehat{p}^s_{\tau_j,i} - \delta_{\tau_{j-1},i} > 0$: In this case, note that from the proof above of $\tilde{p}^s_{t,i} \geq E^{\min}$, we have that $\tilde{p}^s_{\tau_j,i} = E^{\min}$. Also $\tilde{p}^s_{\tau_j-1,i} \geq E^{\min}$. Therefore, $\tilde{p}^d_{\tau_j,i} = \eta \cdot (\tilde{p}^s_{\tau_j-1,i} - \tilde{p}^s_{\tau_j,i}) \geq 0$.
- (1f): It is straightforward to verify that $\tilde{p}^s_{t,i} = \tilde{p}^s_{t-1,i} + \eta \cdot \tilde{p}^c_{t,i} - 1/\eta \cdot \tilde{p}^d_{t,i}$ for all $t \in \mathcal{T}$.
- (1h): We want to show that $\tilde{p}^s_{t,i} \geq E^{\min}$ for all $t \in \mathcal{T}$. Clearly $\tilde{p}^s_{t,i} = \widehat{p}^s_{t,i} \geq E^{\min}$ for all $t \in [\tau_0 - 1]$. For $t \geq \tau_0$, we show this in three parts:
  - (i) For $t = \tau_0$, from (1f) and the constructions above, we have $\widehat{p}^s_{\tau_0,i} = \widehat{p}^s_{\tau_0-1,i} + \eta \cdot \widehat{p}^c_{\tau_0,i} \geq E^{\min} + \eta \cdot \min\{\widehat{p}^c_{\tau_0,i},\ \tilde{p}^{ls}_{\tau_0,i}\}$. Therefore, $\tilde{p}^s_{\tau_0,i} = \widehat{p}^s_{\tau_0,i} - \eta \cdot \min\{\widehat{p}^c_{\tau_0,i},\ \tilde{p}^{ls}_{\tau_0,i}\} \geq E^{\min}$.
  - (ii) For $t = \tau_j$ for all $j \in [k]$, we have $\tilde{p}^s_{\tau_j,i} = \widehat{p}^s_{\tau_j,i} + \delta_{\tau_j,i} = \widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} + \max\{E^{\min} - \widehat{p}^s_{\tau_j,i} - \delta_{\tau_{j-1},i}, 0\} \geq E^{\min}$.
  - (iii) Finally, for any $t \in \{\tau_j + 1, \ldots, \tau_{j+1} - 1\}$ for $j \in \{0, \ldots, k\}$, observe that since $\widehat{p}^d_{\tau_j+1,i} = \cdots = \widehat{p}^d_{\tau_{j+1}-1,i} = 0$, we have $\tilde{p}^s_{t,i} = \widehat{p}^s_{t,i} + \delta_{\tau_j,i} \geq \widehat{p}^s_{\tau_j,i} + \delta_{\tau_j,i} = \tilde{p}_{\tau_j,i} \geq E^{\min}$, where the last inequality follows from the above.
- (1j): It is straightforward to verify that $\tilde{p}^{ls}_{t,i} \geq 0$ for all $t \in \mathcal{T}$.
- (1k): It is straightforward to verify that $\widehat{p}^c_{t,i} + \widehat{p}^{ex}_{t,i} - \widehat{p}^d_{t,i} - \widehat{p}^{ls}_{t,i} = \tilde{p}^c_{t,i} + \tilde{p}^{ex}_{t,i} - \tilde{p}^d_{t,i} - \tilde{p}^{ls}_{t,i}$ for all $t \in \mathcal{T}$.

□

Observe that $\tilde{p}^c_{\tau_0,i} \cdot \tilde{p}^{ls}_{\tau_0,i} = 0$. Next, we claim that the total additional adjustment made to δ over time is upper bounded.

Claim 2. $\sum_{j=1}^k \max\{E^{\min} - \widehat{p}^s_{\tau_j,i} - \delta_{\tau_{j-1},i}, 0\} \leq |\delta_{\tau_0,i}|$.

Proof. Suppose that $S \subseteq [k]$ such that $\max\{E^{\min} - \widehat{p}^s_{\tau_j,i} - \delta_{\tau_{j-1},i}, 0\} = E^{\min} - \widehat{p}^s_{\tau_j,i} - \delta_{\tau_{j-1},i}$. Let $S = \{j_1, j_2, \ldots, j_m\}$. If $S = \emptyset$, then there is nothing to verify. Otherwise, it is straightforward to verify that:

$$\begin{aligned}
&\sum_{j=1}^k \max\{E^{\min} - \widehat{p}^s_{\tau_j,i} - \delta_{\tau_{j-1},i}, 0\} \\
&= (E^{\min} - \widehat{p}^s_{\tau_{j_1},i} - \delta_{\tau_0,i}) + (E^{\min} - \widehat{p}^s_{\tau_{j_2},i} - \delta_{\tau_{j_2}-1,i}) + \cdots + (E^{\min} - \widehat{p}^s_{\tau_{j_m},i} - \delta_{\tau_{j_m-1},i}) \\
&= (E^{\min} - \widehat{p}^s_{\tau_{j_1},i} - \delta_{\tau_0,i}) + (\widehat{p}^s_{\tau_{j_1},i} - \widehat{p}^s_{\tau_{j_2},i}) + \cdots + (\widehat{p}^s_{\tau_{j_{m-1}},i} - \widehat{p}^s_{\tau_{j_m},i}) \\
&= E^{\min} - \widehat{p}^s_{\tau_{j_m},i} - \delta_{\tau_0,i} \leq |\delta_{\tau_0,i}|.
\end{aligned}$$

The second equality is due to the fact that $\delta_{\tau_{j_l},i} = E^{\min} - \widehat{p}^s_{\tau_{j_l},i}$ and $\delta_{\tau_{j_{l+1}}-1,i} = \delta_{\tau_{j_l},i}$ for all $l = 1, ..., m$. □

Finally, we are ready to compute the difference in objective function value of the two solutions:

$$\begin{aligned}
& c(\tilde{\boldsymbol{p}}) + \boldsymbol{\lambda}^\top g(\tilde{\boldsymbol{p}}) - c(\widehat{\boldsymbol{p}}) - \boldsymbol{\lambda}^\top g(\widehat{\boldsymbol{p}}) \\
=& \sum_{t\in\mathcal{T}}\sum_{i\in N}\left(\tilde{p}^{ls}_{t,i} - \widehat{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} - \widehat{p}^{ex}_{t,i} + \lambda_c(\tilde{p}^{c}_{t,i} - \widehat{p}^{c}_{t,i}) + \lambda_d(\tilde{p}^{d}_{t,i} - \widehat{p}^{d}_{t,i})\right) \\
=& - \min\{\widehat{p}^{c}_{\tau_0,i},\ \widehat{p}^{ls}_{\tau_0,i}\} + \sum_{j\in[k]} \eta \cdot \max\left\{E^{\min} - \widehat{p}^{s}_{\tau_j,i} - \delta_{\tau_{j-1},i}, 0\right\} \\
& - \lambda_c \cdot \min\{\widehat{p}^{c}_{\tau_0,i},\ \widehat{p}^{ls}_{\tau_0,i}\} - \lambda_d \cdot \sum_{j\in[k]} \eta \cdot \max\left\{E^{\min} - \widehat{p}^{s}_{\tau_j,i} - \delta_{\tau_{j-1},i}, 0\right\} \\
=& - (1+\lambda_c)\cdot\frac{|\delta_{\tau_0,i}|}{\eta} + (1-\lambda_d)\cdot\eta\cdot\left(\sum_{j\in[k]}\max\left\{E^{\min} - \widehat{p}^{s}_{\tau_j+1,i} - \delta_{\tau_{j-1},i}, 0\right\}\right) \\
\le& - \frac{|\delta_{\tau_0,i}|}{\eta} - \lambda_c\frac{|\delta_{\tau_0,i}|}{\eta} + (1-\lambda_d)\cdot\eta\cdot|\delta_{\tau_0,i}| \\
=& \eta\cdot|\delta_{\tau_0,i}|\left(1-\lambda_d-(1+\lambda_c)/\eta^2\right) < 0,
\end{aligned}$$

where the last inequality is from Claim 2. Therefore, $c(\tilde{\boldsymbol{p}}) + \boldsymbol{\lambda}^\top g(\tilde{\boldsymbol{p}}) < c(\widehat{\boldsymbol{p}}) + \lambda^\top g(\widehat{\boldsymbol{p}})$, hence a contradiction. □

Lemma 2. *Suppose $E^{\min}_c = E^{\min}_d = 0$. For any $(\lambda_c,\ \lambda_d)$ such that $\lambda_c + \eta^2\cdot\lambda_d > 1-\eta^2$ with a given $\eta \in (0,1]$, let $\boldsymbol{p}$ be an optimal solution to* (Reg-Battery) *problem. Then, $p^d_{t,i}p^{ex}_{t,i} = 0$ for all $t\in\mathcal{T},\ i\in\mathcal{N}$.*

Proof. By contradiction, suppose there exists an optimal solution $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$ of (Reg-Battery) problem such that $\widehat{p}^d_{t,i} > 0$ and $\widehat{p}^{ex}_{t,i} > 0$ for at least one $(t,i)\in\mathcal{T}\times\mathcal{N}$.

Without loss of generality, we show the proof for one such $i\in\mathcal{N}$ as the proof can be extended for any multiple nodes. Let $\tau_0\in\mathcal{T}$ be the first time period such that $\widehat{p}^d_{\tau_0,i} > 0$ and $\widehat{p}^{ex}_{\tau_0,i} > 0$. Let $\tau_1,\dots,\tau_k \in \{\tau_0+1,\dots,T\}$ such that $\widehat{p}^c_{\tau_j,i} > 0$ for $j\in[k]$ and $\widehat{p}^c_{t.i} = 0$ for $t\in\{\tau_0+1,\dots,T\}\setminus\{\tau_1,\dots,\tau_k\}$. We define adjustments to the state-of-charge as the following:

$$\delta_{\tau_j,i} = \begin{cases} 1/\eta\cdot\min\{\widehat{p}^d_{\tau_0,i},\ \widehat{p}^{ex}_{\tau_0,i}\} & j=0, \\ \delta_{\tau_{j-1},i} - \max\left\{\widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - E^{\max}, 0\right\} & \forall j\in[k]. \end{cases}$$

Note that $\delta_{\tau_0,i} < 0$. We proceed to construct a solution $(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{f}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{u}})$ from the current optimal solution $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$, where $\tilde{\boldsymbol{\theta}} = \widehat{\boldsymbol{\theta}},\ \tilde{\boldsymbol{f}} = \widehat{\boldsymbol{f}},\ \tilde{\boldsymbol{u}} = \widehat{\boldsymbol{u}},\ \tilde{\boldsymbol{p}}^{ex} = \widehat{\boldsymbol{p}}^{ex},\ \tilde{\boldsymbol{p}}^{g} = \widehat{\boldsymbol{p}}^{g}$, and changing values only corresponding to node i as follows:

$$\tilde{p}^d_{t,i} = \begin{cases} \widehat{p}^d_{t,i}, & \forall t\in\mathcal{T}\setminus\{\tau_0\}, \\ \widehat{p}^d_{\tau_0,i} - \min\{\widehat{p}^d_{\tau_0,i},\ \widehat{p}^{ex}_{\tau_0,i}\}, & t=\tau_0, \end{cases}$$

$$\tilde{p}^c_{t,i} = \begin{cases} \widehat{p}^c_{t,i}, & \forall t\in\mathcal{T}\setminus\{\tau_1,\dots,\tau_k\}, \\ \widehat{p}^c_{t,i} - \frac{1}{\eta}\cdot\max\left\{\widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - E^{\max}, 0\right\}, & t=\tau_j,\ \forall j\in[k], \end{cases}$$

$$\tilde{p}^{ex}_{t,i} = \begin{cases} \widehat{p}^{ex}_{t,i}, & \forall t\in\mathcal{T}\setminus\{\tau_0,\tau_1,\dots,\tau_k\}, \\ \widehat{p}^{ex}_{t,i} - \min\{\widehat{p}^d_{\tau_0,i},\ \widehat{p}^{ex}_{\tau_0,i}\}, & t=\tau_0, \\ \widehat{p}^{ex}_{t,i} + \frac{1}{\eta}\cdot\max\left\{\widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - E^{\max}, 0\right\}, & t=\tau_j,\ \forall j\in[k], \end{cases}$$

$$\tilde{p}^s_{t,i} = \begin{cases} \widehat{p}^s_{t,i}, & \forall t\in[\tau_0-1], \\ \widehat{p}^s_{t,i} + \delta_{\tau_j,i}, & \forall t\in\{\tau_j,\dots,\tau_{j+1}-1\},\ \forall j\in\{0,\dots,k\}, \end{cases}$$

where $\tau_{k+1} - 1 = T$.

The new solution $\tilde{\boldsymbol{p}}$ above is created in the following fashion: We first reduce both $\widehat{p}^d_{\tau_0,i}$ and $\widehat{p}^{ex}_{\tau_0,i}$, which causes $\widehat{p}^s_{t,i}$ to increase in time periods following τ_0. In particular, it may increase beyond $E^{\max}$. In order to fix this, we need to modify the charging levels (and excess values of corresponding time periods) to ensure that the storage levels meet the maximum requirement $E^{\max}$. We carefully decrease the values of $\widehat{p}^c_{t,i}$ so that the minimum charge level is satisfied and the state-of-charge level is never beyond $E^{\max}$ at the same time.

Claim 3. *$(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{f}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{u}})$ is a feasible solution to* (Reg-Battery) *with given $\boldsymbol{\lambda}$.*

Proof. It suffices to show that $\tilde{\boldsymbol{p}}$ satisfies (1d) – (1f), (1h) and (1j) – (1k).

- (1d): We want to show that $\tilde{p}^c_{t,i} \geq 0 = E^{\min}_c$ for all $t \in \mathcal{T}$. It is sufficient to prove this for $t = \{\tau_1, \ldots, \tau_k\}$. There are two cases:
  - (i) If $\widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - E^{\max} \leq 0$: This case is straightforward as $\tilde{p}^c_{t,i} = \widehat{p}^c_{t,i} \geq 0$.
  - (ii) If $\widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - E^{\max} > 0$: In this case, note that from the proof above of $\tilde{p}^s_{t,i} \leq E^{\max}$, we have that $\tilde{p}^s_{\tau_j,i} = E^{\max}$. Also $\tilde{p}^s_{\tau_j-1,i} \leq E^{\max}$. Therefore, $\tilde{p}^c_{\tau_j,i} = 1/\eta \cdot (\tilde{p}^s_{\tau_j,i} - \tilde{p}^s_{\tau_{j-1},i}) \geq 0$.
- (1e): It is straightforward to verify that $\tilde{p}^d_{t,i} \geq 0 = E^{\min}_d$ for all $t \in \mathcal{T}$.
- (1f): It is straightforward to verify that $\tilde{p}^s_{t,i} = \tilde{p}^s_{t-1,i} + \eta \cdot \tilde{p}^c_{t,i} - 1/\eta \cdot \tilde{p}^d_{t,i}$ for all $t \in \mathcal{T}$.
- (1h): We want to show that $\tilde{p}^s_{t,i} \leq E^{\max}$ for all $t \in \mathcal{T}$. Clearly $\tilde{p}^s_{t,i} = \widehat{p}^s_{t,i} \leq E^{\max}$ for all $t \in [\tau_0 - 1]$. For $t \geq \tau_0$, we show this in three parts:
  - (i) For $t = \tau_0$, from (1f) and the constructions above, we have $\widehat{p}^s_{\tau_0,i} = \widehat{p}^s_{\tau_0-1,i} - 1/\eta \cdot \widehat{p}^d_{\tau_0,i} \leq E^{\max} - 1/\eta \cdot \min\{\widehat{p}^d_{\tau_0,i},\ \widehat{p}^{ex}_{\tau_0,i}\}$. Therefore, $\tilde{p}^s_{\tau_0,i} = \widehat{p}^s_{\tau_0,i} + 1/\eta \cdot \min\{\widehat{p}^d_{\tau_0,i},\ \widehat{p}^{ex}_{\tau_0,i}\} \leq E^{\max}$.
  - (ii) For $t = \tau_j$ for all $j \in [k]$, we have $\tilde{p}^s_{\tau_j,i} = \widehat{p}^s_{\tau_j,i} + \delta_{\tau_j,i} = \widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - \max\{\widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - E^{\max}, 0\} \leq E^{\max}$.
  - (iii) Finally, for any $t \in \{\tau_j + 1, \ldots, \tau_{j+1} - 1\}$ for $j \in \{0, \ldots, k\}$, observe that since $\widehat{p}^c_{\tau_j+1,i} = \cdots = \widehat{p}^c_{\tau_{j+1}-1,i} = 0$, we have that $\tilde{p}^s_{t,i} = \widehat{p}^s_{t,i} + \delta_{\tau_j,i} \leq \widehat{p}^s_{\tau_j,i} + \delta_{\tau_j,i} = \tilde{p}_{\tau_j,i} \leq E^{\max}$, where the last inequality follows from the above.
- (1j): It is straightforward to verify that $\tilde{p}^{ex}_{t,i} \geq 0$ for all $t \in \mathcal{T}$.
- (1k): It is straightforward to verify that $\widehat{p}^c_{t,i} + \widehat{p}^{ex}_{t,i} - \widehat{p}^d_{t,i} - \widehat{p}^{ls}_{t,i} = \tilde{p}^c_{t,i} + \tilde{p}^{ex}_{t,i} - \tilde{p}^d_{t,i} - \tilde{p}^{ls}_{t,i}$ for all $t \in \mathcal{T}$.

□

Observe that $\tilde{p}^d_{\tau_0,i} \cdot \tilde{p}^{ex}_{\tau_0,i} = 0$. Next, we claim that the total additional adjustment made to δ over time is upper bounded.

Claim 4. $\sum_{j=1}^k \max\{\widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - E^{\max}, 0\} \leq |\delta_{\tau_0,i}|$.

Proof. Suppose that $S \subseteq [k]$ such that $\max\{\widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - E^{\max}, 0\} = \widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - E^{\max}$. Let $S = \{j_1, j_2, \ldots, j_m\}$. If $S = \emptyset$, then there is nothing to verify. Otherwise, it is straightforward to verify that:

$$\begin{aligned}
&\sum_{j=1}^{k} \max\{\widehat{p}^s_{\tau_j,i} + \delta_{\tau_j-1,i} - E^{\max}, 0\} \\
&= (\widehat{p}^s_{\tau_{j_1},i} + \delta_{\tau_0,i} - E^{\max}) + (\widehat{p}^s_{\tau_{j_2},i} + \delta_{\tau_{j_2}-1,i} - E^{\max}) + \cdots + (\widehat{p}^s_{\tau_{j_m},i} + \delta_{\tau_{j_m}-1,i} - E^{\max}) \\
&= (\widehat{p}^s_{\tau_{j_1},i} + \delta_{\tau_0,i} - E^{\max}) + (\widehat{p}^s_{\tau_{j_1}}, i - \widehat{p}^s_{\tau_{j_2},i}) + \cdots + (\widehat{p}^s_{\tau_{j_{m-1}},i} - \widehat{p}^s_{\tau_{j_m},i}) \\
&= \widehat{p}^s_{\tau_{j_m},i} + \delta_{\tau_0,i} - E^{\max} \leq |\delta_{\tau_0,i}|.
\end{aligned}$$

The second equality is due to the fact that $\delta_{\tau_{j_l},i} = E^{\max} - \widehat{p}^s_{\tau_{j_l},i}$ and $\delta_{\tau_{j_{l+1}}-1,i} = \delta_{\tau_{j_l},i}$ for all $l = 1, ..., m$. □

Finally, we are ready to compute the difference in objective function value of the two solutions:

$$\begin{aligned}
& c(\tilde{\boldsymbol{p}}) + \boldsymbol{\lambda}^\top g(\tilde{\boldsymbol{p}}) - c(\widehat{\boldsymbol{p}}) - \boldsymbol{\lambda}^\top g(\widehat{\boldsymbol{p}}) \\
=& \sum_{t\in\mathcal{T}} \sum_{i\in N} \left(\tilde{p}^{ls}_{t,i} - \widehat{p}^{ls}_{t,i} + \tilde{p}^{ex}_{t,i} - \widehat{p}^{ex}_{t,i} + \lambda_c(\tilde{p}^c_{t,i} - \widehat{p}^c_{t,i}) + \lambda_d(\tilde{p}^d_{t,i} - \widehat{p}^d_{t,i}) \right) \\
=& - \min\{\widehat{p}^d_{\tau_0,i},\ \widehat{p}^{ex}_{\tau_0,i}\} + \sum_{j\in[k]} \frac{1}{\eta} \cdot \max\left\{\widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - E^{\max}, 0\right\} \\
& - \lambda_c \cdot \sum_{j\in[k]} \frac{1}{\eta} \cdot \max\left\{\widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - E^{\max}, 0\right\} - \lambda_d \cdot \min\{\widehat{p}^d_{\tau_0,i},\ \widehat{p}^{ex}_{\tau_0,i}\} \\
=& - \eta|\delta_{\tau_0,i}| - \lambda_d\eta|\delta_{\tau_0,i}| + \frac{1-\lambda_c}{\eta} \cdot \left(\sum_{j=1}^{k} \max\left\{\widehat{p}^s_{\tau_j,i} + \delta_{\tau_{j-1},i} - E^{\max}, 0\right\} \right) \\
\leq& - \eta|\delta_{\tau_0,i}| - \lambda_d\eta|\delta_{\tau_0,i}| + \frac{1-\lambda_c}{\eta} \cdot |\delta_{\tau_0,i}| \\
=& \eta \cdot |\delta_{\tau_0,i}| \left(-1 - \lambda_d + (1-\lambda_c)/\eta^2\right) < 0.
\end{aligned}$$

When $-1-\lambda_d+(1-\lambda_c)/\eta^2 < 0$ (i.e., under the assumption that $\lambda_c+\eta^2\lambda_d > 1-\eta^2$), $c(\tilde{\boldsymbol{p}})+\boldsymbol{\lambda}^\top g(\tilde{\boldsymbol{p}}) < c(\widehat{\boldsymbol{p}}) + \lambda^\top g(\widehat{\boldsymbol{p}})$, hence a contradiction. □

Appendix B Detailed Formulation for Numerical Studies

B.1 Placement of Batteries under Stochastic Demand in Section 4.4

B.1.1 Problem Formulation

We introduce a two-stage stochastic programming that determines the locations for battery installation and power operations to minimize expected load shedding and excess power under stochastic demand. The first-stage (here-and-now) decision, denoted as $\boldsymbol{x}$, involves placing batteries in the grid subject to a given budget $b \in \mathbb{Z}_+$:

$$\boldsymbol{x} \in \{0,1\}^N, \tag{10a}$$

$$\sum_{i \in \mathcal{N}} x_i \leq b. \tag{10b}$$

The second-stage decision is operating the system with the same objective under the exogenous demand uncertainty. Specifically, let Ξ be the set of demand scenarios. The power balance equation, which previously depended on deterministic demand, now varies with scenario $\boldsymbol{D} \in \Xi$. We use $D_{t,i}$ to represent the stochastic demand for each time t and node i, and the power balance equation is modified as follows:

$$\sum_{j \in \delta_i^+} f_{t,ij} - \sum_{j \in \delta_i^-} f_{t,ji} = p^g_{t,i} - D_{t,i} - p^c_{t,i} + p^d_{t,i} + p^{ls}_{t,i} - p^{ex}_{t,i}, \qquad \forall t \in \mathcal{T},\ i \in \mathcal{N}. \tag{10c}$$

Initial state-of-charge as well as upper and lower bounds of state-of-charge of a battery depends on whether a battery is sited or not:

$$p^s_{0,i} = E_0 x_i, \qquad \forall i \in \mathcal{N}, \tag{10d}$$

$$E^{\min} x_i \leq p^s_{t,i} \leq E^{\max} x_i, \qquad \forall t \in \mathcal{T}, i \in \mathcal{N}. \tag{10e}$$

When there is no battery at node $i \in \mathcal{N}$, $p^s_{t,i} = 0$ for all $t \in \mathcal{T}$, so we can expand the state-of-charge over time to the entire $\mathcal{N}$:

$$p^s_{t,i} = p^s_{t-1,i} + \eta \cdot p^c_{t,i} - 1/\eta \cdot p^d_{t,i}, \qquad \forall t \in \mathcal{T}, i \in \mathcal{N}. \tag{10f}$$

The bounds on charging and discharging, similarly, depend on whether a battery exists at a node and whether the battery is charging or discharging, represented by $u_{t,i}$:

$$\boldsymbol{u} \in \{0,1\}^{T \times N}, \tag{10g}$$

$$E_c^{\min} u_{t,i} \leq p^c_{t,i} \leq E_c^{\max} u_{t,i}, \qquad \forall t \in \mathcal{T}, i \in \mathcal{N}, \tag{10h}$$

$$E_d^{\min}(x_i - u_{t,i}) \leq p^d_{t,i} \leq E_d^{\max}(x_i - u_{t,i}), \qquad \forall t \in \mathcal{T}, i \in \mathcal{N}. \tag{10i}$$

Finally, a battery can only operate when there is a battery installed at the node:

$$u_{t,i} \leq x_i, \qquad \forall t \in \mathcal{T}, i \in \mathcal{N}. \tag{10j}$$

Bounds on generator outputs (1a), limits on transmission lines (1b), power flow approximation (1c), and nonnegativity constraint on load shedding and excess power (1j) do not change. Then the two-stage stochastic formulation is

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \mathcal{E}_{\mathbb{P}}\left[Q(\boldsymbol{x}, \tilde{\boldsymbol{D}})\right], \\ \text{s.t.} \quad & (10a) - (10b), \end{aligned} \tag{11}$$

where for a particular realization $\boldsymbol{D}$, we have

$$Q(\boldsymbol{x}, \boldsymbol{D}) = \min_{\boldsymbol{\theta}, \boldsymbol{f}, \boldsymbol{p}, \boldsymbol{u}} \quad c(\boldsymbol{p}),$$
$$\text{s.t.} \quad (1\text{a}) - (1\text{c}),\ (10\text{c}) - (10\text{j}).$$

Suppose that the distribution $\mathbb{P}$ of random parameters $\tilde{\boldsymbol{D}}$ is an equiprobable empirical one generated from N_S independent and identically distributed (i.i.d.) samples $\boldsymbol{D^j}_{\{j\in[N_S]\}}$, with $\mathbb{P}\{\tilde{\boldsymbol{D}} = \boldsymbol{D}^j\} = 1/N_S$. The first-stage objective function can be written as $\mathcal{E}_{\mathbb{P}}[Q(\boldsymbol{x}, \tilde{\boldsymbol{D}})] = \sum_{j\in[N_S]} 1/N_S Q(\boldsymbol{x}, \boldsymbol{D}^j)$, where $Q(\boldsymbol{x}, \boldsymbol{D}^j)$ is the optimal second-stage value for the scenario $\boldsymbol{D}^j$ given the first-stage decision $\boldsymbol{x}$.

B.1.2 Methods

When modeling the second-stage stage with (Battery), note that any two-stage stochastic optimization with finite scenario support can be reformulated to the exact deterministic formulation. That is to say, in our setting, we solve one large MILP with $N + T \times N \times N_S$ binary variables (N for the first-stage decision $\boldsymbol{x}$ and $T \times N \times N_S$ for N_S number of the second-stage decisions $\boldsymbol{u} \in \{0,\ 1\}^{T\times N}$). Another methodology is to use an integer L-shaped method (see the details in Laporte and Louveaux 1993). We iteratively add valid cuts to the master problem based on the LP-relaxed solution of the second stage and a cut based on the integer solution of the second stage. Let θ^j be an overestimator of the optimal second-stage cost $Q(\tilde{\boldsymbol{x}}, \boldsymbol{D}^j)$ with a feasible first-stage solution $\tilde{\boldsymbol{x}}$. The algorithm first solves an LP relaxation of the second-stage problem given the first-stage solution $\tilde{\boldsymbol{x}}$. If the second-stage cost from LP relaxation for a scenario $j \in [N_S]$, denoted as $Q^{LP}(\tilde{\boldsymbol{x}}, \boldsymbol{D}^j)$, exceeds the overestimator of the optimal second-stage cost θ^j for any $j \in [N_S]$, we add the following optimality cut to the master problem:

$$\theta^j \geq Q^{LP}(\tilde{\boldsymbol{x}}, \boldsymbol{D}^j) + \boldsymbol{\pi}_j(\boldsymbol{x} - \tilde{\boldsymbol{x}}),$$

where $\boldsymbol{\pi}_j$ be the subgradient of $Q^{LP}(\tilde{\boldsymbol{x}}, \boldsymbol{D}^j)$ with respect to $\boldsymbol{x}$. Once an optimality cut is added, the algorithm proceeds to solve the master problem with the added cut. If no optimality cut is added, we solve the integer second-stage problem for a given $\tilde{\boldsymbol{x}}$. If $Q(\tilde{\boldsymbol{x}}, \boldsymbol{D}^j) > \theta^j$, the following integer optimality cut is added:

$$\theta^j \geq Q^{LP}(\tilde{\boldsymbol{x}}, \boldsymbol{D}^j) + \left[Q(\tilde{\boldsymbol{x}}, \boldsymbol{D}^j) - Q^{LP}(\tilde{\boldsymbol{x}}, \boldsymbol{D}^j)\right] \left(1 + \sum_{i\notin S(\tilde{x})} x_i - \sum_{i\in S(\tilde{x})} (1 - x_i)\right),$$

where $S(\tilde{\boldsymbol{x}})$ is the set of indexes where $\tilde{x}_i = 1$, i.e., $S(\tilde{\boldsymbol{x}}) = \{i : \tilde{x}_i = 1, \forall i \in [N]\}$. Notice that feasibility is guaranteed for the second-stage problem as the objective is to reduce both load shedding and excess power, hence we have a complete recourse problem. We take the best solution achieved from either the deterministic formulation or the integer L-shaped method.

We employ two common techniques for the L-shaped method and Bender's decomposition: (i) building a single search tree and generating cuts in a delayed cut generation through callback functionality; and (ii) including one scenario in the master problem so that when scenarios are not too far from each other, this first-stage decision can be a near-optimal solution for other scenarios as well. We acknowledge that there are many acceleration techniques for Bender's decomposition and integer L-shaped method (Magnanti and Wong 1981, Angulo et al. 2016) that may improve computational results.

B.2 $N - k$ Contingency Problem in Section 4.5

B.2.1 Problem Formulation

We formally give a mathematical formulation of this problem. The first decision by a planner, denoted as $\boldsymbol{x}$, is to place batteries in the grid subject to the given budget $b \in \mathbb{Z}_+$.

$$\boldsymbol{x} \in \{0,\ 1\}^{\mathcal{N}}, \tag{12a}$$

$$\sum_{i \in \mathcal{N}} x_i \leq b. \tag{12b}$$

Then a network interdictor decides whether or not to destroy a transmission line $(i, j) \in \mathcal{L}$ by a binary variable y_{ij}. The interruption can happen to at most k transmission lines in the system and impacts the performance of transmission lines throughout the time period considered:

$$\boldsymbol{y} \in \{0,\ 1\}^{\mathcal{L}}, \tag{12c}$$

$$\sum_{e \in \mathcal{L}} y_e \leq k, \tag{12d}$$

$$-F_{ij}(1 - y_{ij}) \leq f_{t,ij} \leq F_{ij}(1 - y_{ij}), \qquad \forall t \in \mathcal{T},\ (i, j) \in \mathcal{L}. \tag{12e}$$

Bounds on generator output (1a), limits on transmission lines (1b), nonnegativity constraint on load shedding and excess power (1j), and power balance equation (1k) do not change from the optimal power flow with the battery problem. Operational constraints for battery (10d) - (10j) are also the same as two-stage stochastic programming studied in the previous section. For purposes of this problem, we omit Ohm's law constraint (see, e.g., Johnson and Dey 2022), and therefore the third-level problem becomes a network flow problem. Throughout the time period considered, the network operator then aims to generate power and send power flows to minimize the load shedding and lost power. We now provide the formulation below:

$$\min_{\boldsymbol{x}} \max_{\boldsymbol{y}} \min_{\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{u}} \left\{ c(\boldsymbol{p}) \colon \text{(1a)},\ \text{(1b)},\ \text{(1j)},\ \text{(1k)},\ \text{(10d)} - \text{(10j)},\ \text{(12a)-(12e)} \right\}. \tag{13}$$

B.2.2 Solution Methodology

Computationally, a min-max-min problem is not an easy problem to solve. In literature, two-stage robust programs, a special case of the min-max-min problem, have been discussed extensively in the literature (see, e.g., Atamtürk and Zhang 2007, Jiang et al. 2014, van Hulst et al. 2017, Mattia et al. 2017). Jeroslow (1985) showed NP-hardness and Ben-Ayed and Blair (1990) also discussed the computational difficulties of bilevel linear problems. Problem (13) is a trilevel problem with binary variables at each level resulting in a particularly challenging optimization problem, which forbids the classical approach to formulating the trilevel problem into a bilevel problem and applying techniques to solve bilevel optimization problems. Using the regularized MIP model, however, enables us to linearize the third-level problem and convert it to a bilevel problem by taking the dual of the third-level problem.

It is often undesirable to have unbounded dual variables. A popular heuristic is to use the big-M method to bound such unbounded variables to a reasonable number. We prove that in our formulation the dual variables are bounded. This allows the exact reformulation for bilinear terms that appear in the objective function.

Once we obtain the bilevel formulation, we apply a generic iterative algorithm to solve the bilevel optimization problem. In particular, we use the algorithm outlined in Bienstock and Özbay

2008. The stopping criterion was either: (i) the iteration reaches maximum iteration of 1000; (ii) the iteration has run more than 6 hours; or (iii) the upper bound and lower bound gap is less than 0.5%.

B.2.3 Detailed Bilevel Formulation

We present here the reformulation to the bilevel min-max problem. Since variables associated with the third level are linear, we dualize the third level. For notational simplicity, let $\boldsymbol{\theta} = (\boldsymbol{\alpha}, \boldsymbol{\beta}^{\pm}, \boldsymbol{\delta}, \boldsymbol{\gamma}^{\pm}, \boldsymbol{\tau}, \boldsymbol{\tau}^0, \boldsymbol{\mu}^{\pm}, \boldsymbol{\nu}^{\pm}, \boldsymbol{\omega}^{\pm}, \boldsymbol{\mu}^{\pm}, \boldsymbol{\phi})$. The primal variable associated with a constraint for the third level is provided on the left.

$$
\begin{aligned}
\min_{\boldsymbol{x}} \max_{\boldsymbol{y},\boldsymbol{\theta},\boldsymbol{z}} \quad & -\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} D_{t,i}\alpha_{t,i} - \sum_{t\in\mathcal{T},(i,j)\in\mathcal{L}} F_{ij}(1-y_{ij})(\beta^+_{t,ij}+\beta^-_{t,ij}) \\
& + \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} G_i^{\min}\gamma^+_{t,i} - \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} G_i^{\max}\gamma^-_{t,i} + \sum_{i\in\mathcal{N}} E^0 x_i \tau_i^0 \\
& + \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} E^{\min} x_i \mu^+_{t,i} - \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} E^{\max} x_i \mu^-_{t,i} \\
& + \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} E_d^{\min} x_i \omega^+_{t,i} - \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} E_d^{\max} x_i \omega^-_{t,i} - \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}} x_i\phi_{t,i}, \qquad (14a)
\end{aligned}
$$

s.t.

$$
\begin{aligned}
p^{ls}_{t,i},\ p^{ex}_{t,i} \quad \cdots\cdots \quad & -1 \le \alpha_{t,i} \le 1, && \forall t\in\mathcal{T}, i\in\mathcal{N}, && (14b)\\
f_{t,ij} \quad \cdots\cdots \quad & \alpha_{t,i} - \alpha_{t,j} + \beta^+_{t,ij} - \beta^-_{t,ij} = 0, && \forall t\in\mathcal{T}, (i,j)\in\mathcal{L}, && (14c)\\
p^g_{t,i} \quad \cdots\cdots \quad & -\alpha_{t,i} + \gamma^+_{t,i} - \gamma^-_{t,i} = 0, && \forall t\in\mathcal{T}, i\in\mathcal{N}, && (14d)\\
p^c_{t,i} \quad \cdots\cdots \quad & \alpha_{t,i} - \eta\cdot\tau_{t,i} + \nu^+_{t,i} - \nu^-_{t,i} = \lambda^c, && \forall t\in\mathcal{T}, i\in\mathcal{N}, && (14e)\\
p^d_{t,i} \quad \cdots\cdots \quad & -\alpha_{t,i} + 1/\eta\cdot\tau_{t,i} + \omega^+_{t,i} - \omega^-_{t,i} = \lambda^d, && \forall t\in\mathcal{T}, i\in\mathcal{N}, && (14f)\\
p^s_{t,i} \quad \cdots\cdots \quad & \tau_{t,i} - \tau_{t+1,i} + \mu^+_{t,i} - \mu^-_{t,i} = 0, && \forall t\in\mathcal{T}\setminus\{T\}, i\in\mathcal{N}, && (14g)\\
p^s_{T,i} \quad \cdots\cdots \quad & \tau_{T,i} + \mu^+_{T,i} - \mu^-_{T,i} = 0, && \forall i\in\mathcal{N}, && (14h)\\
p^s_{0,i} \quad \cdots\cdots \quad & \tau_i^0 - \tau_{1,i} = 0, && \forall i\in\mathcal{N}, && (14i)\\
u_{0,i} \quad \cdots\cdots \quad & E_c^{\min}\nu^+_{t,i} - E_c^{\max}\nu^-_{t,i} - E_d^{\min}\omega^+_{t,i} + E_d^{\max}\omega^-_{t,i} + \phi_{t,i} \ge 0, && \forall t\in\mathcal{T}, i\in\mathcal{N}, && (14j)\\
& \boldsymbol{\beta}^{\pm}, \boldsymbol{\gamma}^{\pm}, \boldsymbol{\nu}^{\pm}, \boldsymbol{\omega}^{\pm}, \boldsymbol{\mu}^{\pm}, \boldsymbol{\phi} \ge \boldsymbol{0}, && && (14k)\\
& \sum_{i\in\mathcal{N}} x_i \le b, && && (14l)\\
& \sum_{l\in\mathcal{L}} y_l \le k, && && (14m)\\
& \boldsymbol{x}\in\{0,1\}^N,\ \boldsymbol{y}\in\{0,1\}^L. && && (14n)
\end{aligned}
$$

Proposition 5. *Independent of the values of λ^c and λ^d, there exists an optimal solution of (14) for which the following inequalities are valid:*

$$0 \le \beta^{\pm}_{t,ij} \le 2, \quad \forall t\in\mathcal{T},\ (i,j)\in\mathcal{L}.$$

Proof. In the objective function (14a), we focus on optimizing $\boldsymbol{\beta}^+$ and $\boldsymbol{\beta}^-$. Notice that $F_{ij} \ge 0$ for all $(i,j)\in\mathcal{L}$ and $t\in T$. Then, for a given $(i,j)\in\mathcal{L}$ and $t\in T$ and the associated $\beta^+_{t,ij}, \beta^-_{t,ij}$, we optimize

$$\max_{\beta^+_{t,ij}\geq 0, \beta^-_{t,ij}\geq 0} -\beta^-_{t,ij} - \beta^+_{t,ij},$$
$$\text{s.t.} \quad \beta^+_{t,ij} = \alpha_{t,j} - \alpha_{t,i} + \beta^-_{t,ij},$$

which is equivalent to optimizing

$$v^{\beta}_{t,ij} = \min_{\beta^+_{t,ij}\geq 0, \beta^-_{t,ij}\geq 0} \beta^-_{t,ij} + \beta^+_{t,ij},$$
$$\text{s.t.} \quad \beta^+_{t,ij} - \beta^-_{t,ij} = \alpha_{t,j} - \alpha_{t,i},$$

where the optimal value is $v^{\beta^*}_{t,ij} = |\alpha_{t,j} - \alpha_{t,i}|$. From constraint (14b), we know $-1 \leq \alpha_{t,i} \leq 1$ and $-1 \leq \alpha_{t,j} \leq 1$, then we have $0 \leq v^{\beta^*}_{t,ij} \leq 2$, which implies that

$$0 \leq \beta^-_{t,ij} + \beta^+_{t,ij} \leq 2.$$

Hence, we have the desired result. □

We can then use McCormick inequalities to exactly reformulate the bilinear terms of the form $\boldsymbol{\beta}^{\pm}\boldsymbol{y}$ that appear in dualizing the third level of this trilevel problem.

In the objective function, the only bilinear terms are $\{y_{ij}\beta^+_{t,ij}\}_{t\in\mathcal{T},(i,j)\in\mathcal{L}}$ and $\{y_{ij}\beta^-_{t,ij}\}_{t\in\mathcal{T},(i,j)\in\mathcal{L}}$. Since we show that all dual variables are bounded, especially $\mathbf{0} \leq \boldsymbol{\beta}^{\pm} \leq \mathbf{2}$, bilinear terms can be reformulated exactly by applying the McCormick Envelopes (see, e.g., McCormick 1976):

$$z^+_{t,ij} \geq 0, \quad z^+_{t,ij} \geq \beta^+_{t,ij} + 2y_{ij} - 2, \quad z^+_{t,ij} \leq \beta^+_{t,ij}, \quad z^+_{t,ij} \leq 2y_{ij}, \qquad \forall t \in \mathcal{T}, (i,j) \in \mathcal{L},$$
$$z^-_{t,ij} \geq 0, \quad z^-_{t,ij} \geq \beta^-_{t,ij} + 2y_{ij} - 2, \quad z^-_{t,ij} \leq \beta^-_{t,ij}, \quad z^-_{t,ij} \leq 2y_{ij}, \qquad \forall t \in \mathcal{T}, (i,j) \in \mathcal{L}.$$

Note that this is an exact reformulation of the bilinear terms, not a relaxation. Hence, the objective function can replaced with the reformulation and additional constraints from (15) are added to the formulation. This completes converting the trilevel formulation to bilevel formulation.

Appendix C Additional Tables for Section 4.5

The algorithm to solve the trilevel min-max-min problem produces upper bound and lower bound at each iteration. The algorithm terminates when the gap between the upper bound and the lower bound is less than the specified limit of 0.5%. For a time limit of 6 hours, the algorithm may not terminate, and we report the final gap produced from the algorithm along with the solution time.

Table 5: Average Solution Time and Optimality Gap of Iteration for $N - k$ Contingency Problem with $b = 2$ and $k = 3$

	(Reg-Battery)		(LP-Battery)	
	Time (sec)	Opt. Gap	Time (sec)	Opt. Gap
IEEE 14	8	0.00%	7	0.00%
IEEE 73	479	0.00%	522	0.00%
PEGASE 89	12	0.00%	70	0.02%
IEEE 118	9970	0.00%	7817	0.19%
IEEE 162	14	0.00%	137	0.00%
IEEE 300	63	0.13%	3238	3.33%
PEGASE 1354	20272	2.21%	19388	13.25%
RTE 1888	8961	0.14%	21600	7.46%

Table 6: Average Solution Time and Optimality Gap of Iteration for $N - k$ Contingency Problem with $b = 2$ and $k = 5$

	(Reg-Battery)		(LP-Battery)	
	Time (sec)	Opt. Gap	Time (sec)	Opt. Gap
IEEE 14	8	0.00%	7	0.00%
IEEE 73	601	0.00%	697	0.05%
PEGASE 89	16	0.00%	113	0.00%
IEEE 118	3752	0.67%	2804	0.25%
IEEE 162	34	0.00%	550	0.00%
IEEE 300	92	0.07%	3618	3.11%
PEGASE 1354	14453	1.83%	21600	13.04%
RTE 1888	12267	0.67%	21600	7.65%

Table 7: Average Solution Time and Optimality Gap of Iteration for $N - k$ Contingency Problem with $b = 3$ and $k = 5$

	(Reg-Battery)		(LP-Battery)	
	Time (sec)	Opt. Gap	Time (sec)	Opt. Gap
IEEE 14	7	0.00%	8	0.00%
IEEE 73	993	0.00%	730	0.05%
PEGASE 89	16	0.00%	245	0.00%
IEEE 118	4441	0.98%	2205	0.22%
IEEE 162	32	0.04%	1325	2.24%
IEEE 300	192	0.12%	3356	3.53%
PEGASE 1354	21600	3.89%	21600	17.70%
RTE 1888	14262	0.24%	21600	14.19%

Table 8: Average Solution Time and Optimality Gap of Iteration for $N-k$ Contingency Problem with $b=5$ and $k=10$

	(Reg-Battery)		(LP-Battery)	
	Time (sec)	Opt. Gap	Time (sec)	Opt. Gap
IEEE 14	9	0.00%	9	0.00%
IEEE 73	1155	0.00%	1010	0.00%
PEGASE 89	21	0.01%	374	0.80%
IEEE 118	16742	0.25%	19982	1.22%
IEEE 162	16	0.21%	523	1.94%
IEEE 300	22	0.09%	1241	6.96%
PEGASE 1354	12653	1.54%	21600	18.14%
RTE 1888	21600	1.29%	21600	24.97%