Analysis of MILP techniques for the Pooling Problem

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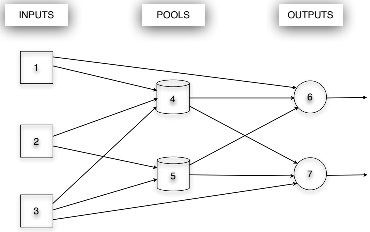
[1] Support from EXXONMOBIL Research and Engineering is gratefully acknowledged.
[2] Many thanks to Akshay Gupte for helping prepare the Figures.

1

The Pooling Problem: Model, Relaxations and Restrictions

The Pooling Problem: Network Flow on Tripartite Graph



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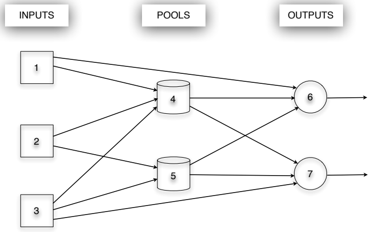

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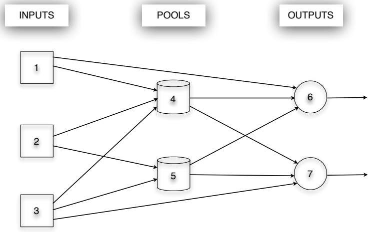

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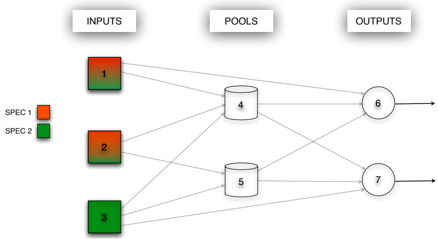
- Send flow from input nodes via pool nodes to output nodes.
- Each of the arcs and nodes have capacities of flow.

The Pooling Problem: Other Constraints

- Raw material has specifications (like sulphur, carbon, etc.).



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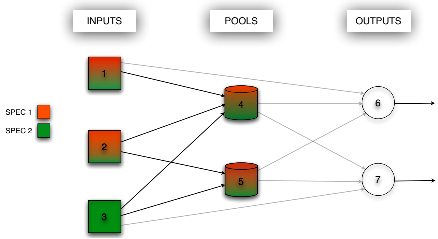

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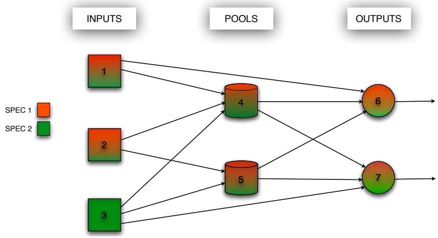

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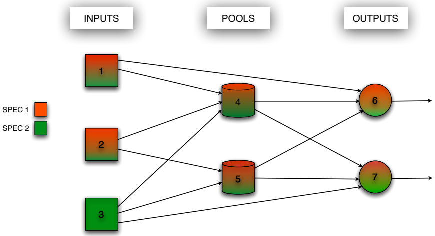

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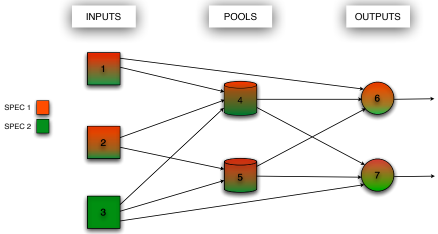
- The material gets further mixed at the output nodes.
- The output node has required levels for each specification.

Tracking Specification



Data:

- λ_i^k: The value of specification k at input node i.

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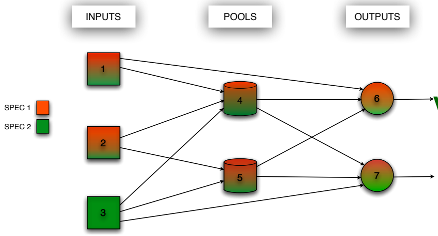

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Variable:

- p_l^k: The value of specification k at node l
- y_{ab}: Flow along the arc (ab).

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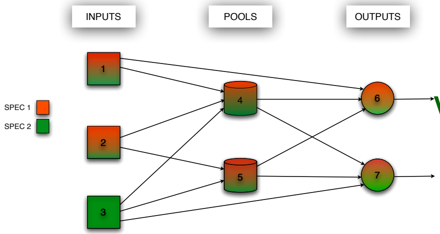

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Specification Tracking: $$\underbrace{\sum_{i \in I} \lambda_i^k y_{il}}_{\text{Inflow of Spec k}} = \underbrace{p_l^k \left(\sum_{j \in J} y_{lj} \right)}_{\text{Out flow of Spec k}}$$

The pooling problem: ‘P’ formulation

$$\max \quad \sum_{ij \in \mathcal{A}} w_{ij} y_{ij} \quad \text{(Maximize profit due to flow)}$$

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Subject To:

1. Node and arc capacities.
2. Total flow balance at each node.
3. Specification balance at each pool.

$$\sum_{i \in I} \lambda_i^k y_{il} = p_l^k \left(\sum_{j \in J} y_{lj} \right)$$

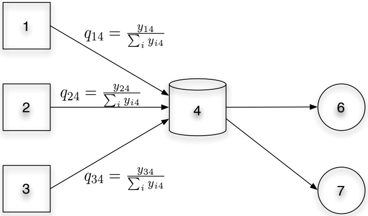
4. Bounds on p_j^k for all out put nodes j and specification k.

PQ Model

New Variable:

- q_{il} : fraction of flow to l from $i \in I$

$$\sum_{i \in I} q_{il} = 1, q_{il} \geq 0, i \in I.$$



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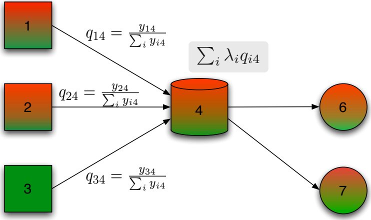

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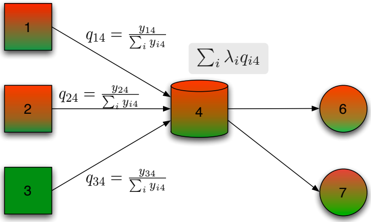

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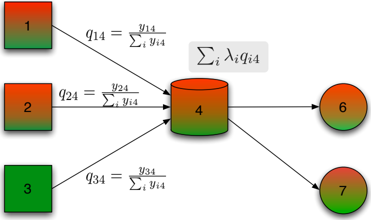

PQ Model Complete

$$\max \quad \sum_{i\in I, j\in J} w_{ij} y_{ij} + \sum_{i\in I, l\in L, j\in J} (w_{il} + w_{lj}) v_{ilj}$$

$$\text{s.t.} \quad v_{ilj} = q_{il} y_{lj} \; \forall i \in I, l \in L, j \in J$$

$$\sum_{i\in I} q_{il} = 1 \; \forall l \in L$$

$$a_j^k \left(\sum_{i\in I} y_{ij} + \sum_{l\in L} y_{lj} \right) \le \sum_{i\in I} \lambda_i^k y_{ij} + \sum_{i\in I, l\in L} \lambda_i^k v_{ilj} \le b_j^k \left(\sum_{i\in I} y_{ij} + \sum_{l\in L} y_{lj} \right)$$

Capacity constraints

All variables are non-negative

$$\sum_{i\in I} v_{ilj} = y_{lj} \; \forall l \in L, j \in J$$

$$\sum_{j\in J} v_{ilj} \le c_l q_{il} \; \forall i \in I, l \in L.$$

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I am an integer programmer, so I am going to talk about IP methods....

Using Integer Linear Programming to Construct Relaxation

1. $S := \{(x, y, w) \in \mathbb{R}^3 \mid w = xy,\ x_l \le x \le x_u,\ y_l \le y \le y_u\}$.

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2. Let $g(x, y) : [x_l, x_u] \times [y_l, y_u] \to \mathbb{R}$ be a piece-wise linear continuous function (pwl) such that $g(x, y) \geq xy$.
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4. Then $S \subseteq \{(x, y, w) \in \mathbb{R}^3 \mid h(x, y) \leq w \leq g(x, y),\ x_l \leq x \leq x_u,\ y_l \leq y \leq y_u\}$.

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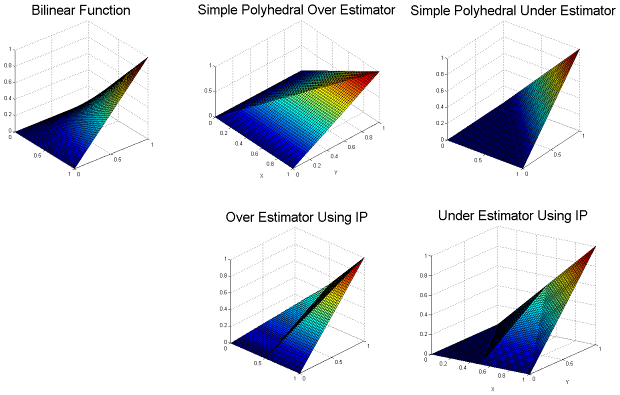
6. **Gounaris et al. (2009), Misener and Floudas (2009)** used this for Pooling Problem very succesfully.

Using Integer Linear Programming to Construct Relaxation: Example



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1. Restrictions typically obtained by restricting a subset of variables, say y, to take values in a pre-determined finite set.
2. This restriction can be modeled using extra 0/1 variables and unary or binary expansion models.
3. **Pham et al. (2009), Alfaki et al. (2011), Gupte et al.(2012)** used for pooling problem with some success.

Using Mixed Integer Linear Programming to Construction Restriction: Example

- $w = xy, x \in [0, 1], y \in [0, 1]$

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$$\begin{aligned} x\left(\frac{1}{M}\sum_{i=1}^{N} 2^{i-1}u_i\right) &= w \\ \sum_{i=1}^{N} 2^{i-1}u_i &\le M \\ u_i \in \{0,1\}\ \forall i \in N \end{aligned}$$

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$$\left.\begin{array}{rcl} \frac{1}{M}\sum_{i=1}^{N} 2^{i-1} v_i & = & w \\ \sum_{i=1}^{N} 2^{i-1} u_i & \leq & M \\ v_i \leq u_i, v_i \leq x, v_i & \geq & x + u_i - 1 \\ u_i \in \{0,1\}\ \forall i \in N & & \end{array}\right\}\text{MILP restriction}$$

2
Main Results

Main Results - Relaxations

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2. *Quality of analysis: Suppose we choose a pwl MILP relaxation $\mathcal{P}$. Then for any $\epsilon > 0$, there exists an instance of the pooling problem with*

$$z^{\mathcal{P}} \geq (n - \epsilon)z^*.$$

Main Results - Computational Complexity

Corollary

There exists a polynomial-time algorithm that produces a feasible solution with objective function value $z^{\mathcal{A}}$ that satisfies $z^{\mathcal{A}} \geq \frac{z^*}{n}$.

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Proposition

If there exists a polynomial-time approximation algorithm with guarantee $z^{\mathcal{A}} \geq \frac{z^*}{n-\epsilon}$ for any $\epsilon > 0$ for the pooling problem, then any problem in NP has randomized polynomial time algorithm.

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   2.1 MILP's feasible region is a restriction of the pooling problem.
   2.2 All solution produced by the approximation algorithnm belong to the MILP $\Rightarrow$ **MILP produces solutions within a factor of n.**
3. The MILP produces good results in short time. In particular, for a number of problems in the literature, we have **found the best known solutions.**

3

Quality of MILP relaxation

Quality of dual bound

Proposition

Let n denote the number of output nodes. Let z^* denote the optimal solution of pooling problem. For any pwl MILP relaxation $\mathcal{P}$, let $z^{\mathcal{P}}$ be the optimal value of the MILP. Then

$$z^* \leq z^{\mathcal{P}} \leq nz^*.$$

High level proof technique

1. We construct a general relaxation $\mathcal{R}$ of the pooling problem, such that $\mathcal{R} \supseteq \mathcal{P}$, i.e. $\mathcal{R}$ is a relaxation of $\mathcal{P}$, for all $\mathcal{P}$.

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Inconsistent Pool Outflow Problem (IPOP)

$$
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\text{s.t.} \quad & v_{ilj} = q_{il} y_{lj} \ \forall i \in I, l \in L, j \in J \\
& \sum_{i \in I} q_{il} = 1 \ \forall l \in L \\
& a_j^k \left(\sum_{i \in I} y_{ij} + \sum_{l \in L} y_{lj} \right) \leq \sum_{i \in I} \lambda_i^k y_{ij} + \sum_{i \in I, l \in L} \lambda_i^k v_{ilj} \leq b_j^k \left(\sum_{i \in I} y_{ij} + \sum_{l \in L} y_{lj} \right) \\
& \text{Capacity constriants} \\
& \text{All variables are non-negative} \\
& \sum_{i \in I} v_{ilj} = y_{lj} \ \forall l \in L, j \in J \\
& \sum_{j \in J} v_{ilj} \leq c_l q_{il} \ \forall i \in I, l \in L.
\end{aligned}
$$

Inconsistent Pool Outflow Problem (IPOP)

IPOP:

$$\max \sum_{i \in I, j \in J} w_{ij} y_{ij} + \sum_{i \in I, l \in L, j \in J} (w_{il} + w_{lj}) v_{ilj}$$

s.t.

$$a_j^k \left(\sum_{i \in I} y_{ij} + \sum_{l \in L} y_{lj} \right) \leq \sum_{i \in I} \lambda_i^k y_{ij} + \sum_{i \in I, l \in L} \lambda_i^k v_{ilj} \leq b_j^k \left(\sum_{i \in I} y_{ij} + \sum_{l \in L} y_{lj} \right)$$

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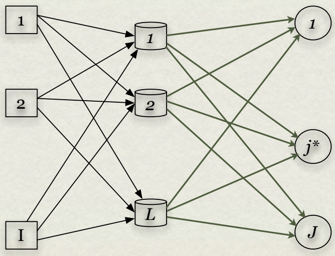
1. This is a relaxation of $\mathcal{P}$ (any piecewise linear MILP relaxation).
2. $\frac{v_{ilj}}{y_{lj}} = q_{il}$: Implies the spec value in the different outer arc from a pool to be consistent.

Rounding IPOP Solution

Let (y, v) be optimal to IPOP and $j^* \in J$ be most 'profitable output':

$$j^* \in \text{argmax}_{j \in J} \left\{ \sum_{i \in I} w_{ij} y_{ij} + \sum_{i \in I, l \in L} (w_{il} + w_{lj}) v_{ilj} \right\}$$



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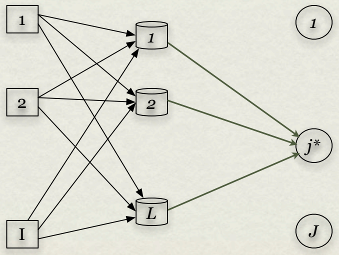

$$\bar{v}_{ilj} = \begin{cases} 0 & j \neq j^* \\ v^*_{ilj} & j = j^* \end{cases}$$

$$\bar{y}_{lj} = \begin{cases} 0 & j \neq j^* \\ y^*_{lj} & j = j^* \end{cases}$$

$$\bar{q}_{il} = \begin{cases} \dfrac{\bar{v}_{ilj^*}}{\bar{y}_{lj^*}} & \bar{y}_{lj^*} > 0 \\ \dfrac{1}{\text{indeg}(l)} & \bar{y}_{lj^*} = 0 \end{cases}$$

Final details

Lemma

$(\bar{y}, \bar{v}, \bar{q})$ *is a valid solution to the pooling problem.*

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1. (y, v) is optimal solution of *IPOP*
2. $(\bar{y}, \bar{v}, \bar{q})$ is a valid solution for pooling problem.
3. Because j^* was picked greedily, we have that $n(\text{Obj. value}(\bar{y}, \bar{v}, \bar{q})) \geq (\text{Obj. value}(y, v))$.

Analysis is tight

Proposition (Quality of analysis)

Suppose we choose a pwl MILP relaxation $\mathcal{P}$. Then for any $\epsilon > 0$, there exists an instance of the pooling problem with

$$z^{\mathcal{P}} \geq (n - \epsilon) z^*.$$

Analysis is tight : High level idea

Construct a pooling problem with:

1. 2 input nodes, 1 pool node, *n* out put nodes, 2 specs, no direct arcs between input and output nodes.

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4. Maximize the total flow through the pool.

1. Pooling Problem: In the actual problem flow can be sent to at most one output node. Therefore max flow $= 1$.
2. IPOP: On the other hand in the pwl relaxation since $v_{ilj} \approx q_{il} y_{lj}$ and since $a_j \approx a_k,\ b_j \approx b_k$, we can set $y_{lj} \approx 1$ for all $j \in J$.

4 Approximation Algorithm

Algorithm

'Algorithm':

1. Solve IPOP
2. Round it to make it feasible for pooling problem.

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Is it possible to obtain a better approximation ratio in polynomial time?

Approximation preserving reduction from Stable Set

Theorem (Hardness to Approximate Max Stable Set (Håstad))

In a graph with n nodes, the max stable set problem cannot be approximated in polynomial time within a factor $n^{1-\epsilon}$, for any constant $\epsilon > 0$, unless any problem in NP can be solved in probabilistic polynomial time.

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Given a simple graph with n vertices, there exists an instance of the pooling problem with n output nodes and of size polynomial in the size of the input graph such that

1. *The size of the maximum stable set of the input graph is less than or equal to the optimal objective function value of the instance of the pooling problem.*
2. *Given any feasible solution for the instance of the pooling problem with objective function value t, it is possible to construct a stable set in the input graph of cardinality greater than or equal to t in polynomial time.*

5
A new MILP restriction and computational result

‘Improving’ the solutions from the approximation algorithm - I

1. Construct an IP whose feasible region contains all the feasible solutions generated by the approximation algorithm.

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We constructed a feasible solution of the pooling problem such that only one output node $j^* \in J$ receives positive flow.

$$\Updownarrow$$

Solve a multicommodity flow problem with the additional condition that all flow must go to one output node

'Improving' the solutions from the approximation algorithm - II

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$\subseteq$ Solve a multicommodity flow problem with the additional condition that flow from every pool goes to atmost one output node.

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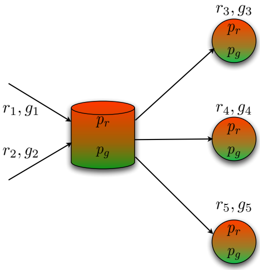
$\boxed{\text{Can we try and do better?}}$

Using IP technology: A new method to discretize



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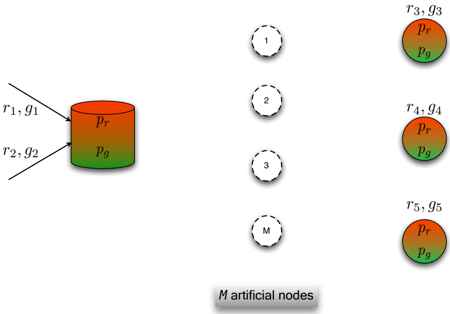

1. Introduce *M* artificial nodes.

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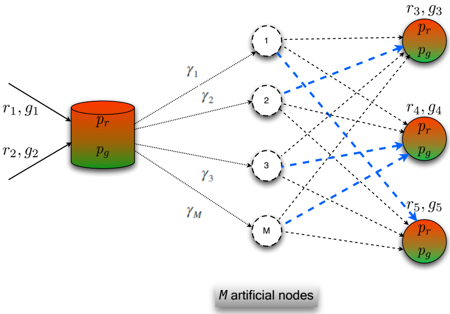

1. Introduce *M* artificial nodes.
2. Each artificial node *t* receives γ_t fraction of total flow.

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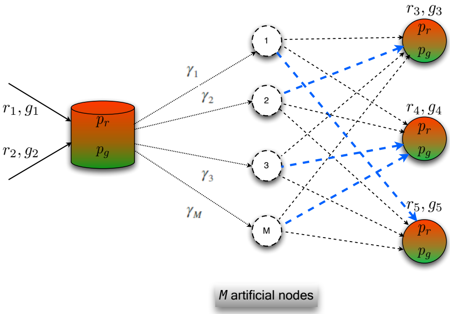

1. Introduce *M* artificial nodes.
2. Each artificial node *t* receives γ_t fraction of total flow.
3. Each artificial node *t* sends flow to one output only.

MILP formulation

$$\max_{y,v,w,\zeta} \quad \sum_{i\in I, j\in J} w_{ij} y_{ij} \; + \sum_{i\in I, l\in L, j\in J} (w_{il} + w_{lj})\, v_{ilj} \tag{1a}$$

$$\text{s.t.} \quad v_{ilj} = \sum_{t=1}^{\tau} f_{il_tj} \qquad \forall i\in I, l\in L, j\in J \tag{1b}$$

$$\sum_{j\in J} f_{il_tj} = \gamma_{lt} \sum_{j\in J} v_{ilj} \qquad \forall i\in I, l\in L, t\in\{1,\dots,\tau\} \tag{1c}$$

$$y_{lj} = \sum_{i\in I} v_{ilj} \qquad \forall l\in L, j\in J \tag{1d}$$

$$0 \le f_{il_tj} \le c_{lj}\zeta_{l_tj} \qquad \forall i\in I, l\in L, t\in\{1,\dots,\tau\}, j\in J \tag{1e}$$

$$\sum_{j\in J} \zeta_{l_tj} = 1 \qquad \forall l\in L, t\in\{1,\dots,\tau\} \tag{1f}$$

$$\zeta_{l_tj} \in \{0,1\} \qquad \forall l\in L, t\in\{1,\dots,\tau\}, j\in J \tag{1g}$$

Spec bounds for $j\in J, k\in K$, y is feasible flow.

More details

Some reasonable choices for γ:

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$$\tilde{z} \geq \frac{z^*}{n}.$$
2. *Let γ be a rational vector. Then there exists a pooling instance such that*
$$\tilde{z} = \frac{z^*}{n}.$$

Computational Experiment

Instances:

Source	Label	Inputs	Pools	Outputs	Specs
		$\|I\|$	$\|L\|$	$\|J\|$	$\|K\|$
Alfaki et al. (2012)	stdA0-9	20	10	15	12
Alfaki et al. (2012)	stdB0-5	35	17	21	17
Alfaki et al. (2012)	stdC0-3	60	30	40	20
Random	randstd11-20	25	18	25	8
Random	randstd21-30	25	22	30	10
Random	randstd31-40	30	22	35	10
Random	randstd41-50	40	30	45	10
Random	randstd51-60	40	30	50	14

Experiment Details

Solvers

1. BARON 9.0.7: (24 hours)
2. SNOPT (1 hr)
3. Alternating LP technique
4. MILP (1hr) (CPLEX 12.2)

Metric

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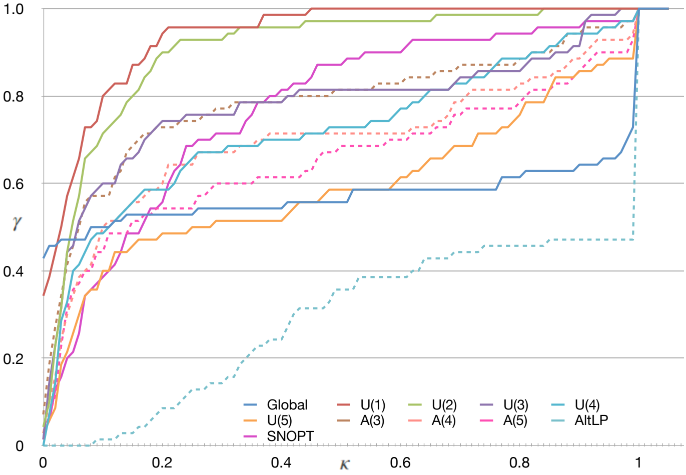
1. $\%\text{gap} = 100 \times \left(\frac{BestRelax}{BestFeas} - 1\right)$
2. $\tau_M(I) :=$
   the best solution value for instance I until termination of method M.
3. $\eta_M(I) := \frac{\tau^{max}(I) - \tau_M(I)}{\tau^{max}(I) - \tau^{min}(I)}$

Performance profile



Geometric Average of % gap versus number of outputs $|J| = n$.

n	Global	$U(1)$	$U(2)$	$U(3)$	$A(3)$	SNOPT	AltLP
15	**1.13**%	2.95%	2.09%	1.70%	1.63%	2.81%	25%
21	8%	**1.00**%	1.57%	1.72%	1.89%	5.18%	12%
25	**0.60**%	3.75%	2.15%	2.18%	1.82%	10%	63%
30	3.50%	2.78%	**1.75**%	1.82%	1.87%	4.09%	39%
35	6%	**2.73**%	3.54%	5.63%	4.71%	5.79%	29%
40	370%	**12**%	15%	15%	18%	115%	38%
45	485%	**5**%	26%	203%	135%	22%	71%
50	550%	**1.87**%	10%	30%	59%	11%	35%
G. Ave.	15%	**3.00**%	4.36%	7.16%	7.18%	8.46%	36%

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3. If we fix the number of specifications, what is the complexity of the problem?
4. Is it possible to obtain a better guarantee than n (n is the number of output nodes), by using some other MILP restrictions of the pooling problem?

Thank you

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