Mixing Inequalities and Maximal Lattice-Free Triangles

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Outline

1. Mixing Inequalities
2. Connection With Lattice-free Triangles
3. Result: Strengthening Mixing Inequalities For Use In Simplex Tableau

Mixing Inequalities.

The mixing set

[Günlük and Pochet (2001)]

$$
\begin{array}{ccccccccccc}
y_0 & + & x_1 & & & & & & & \geq & b_1 \\
y_0 & & & + & x_2 & & & & & \geq & b_2 \\
y_0 & & & & & + & x_3 & & & \geq & b_3 \\
\vdots & & & & & & & \ddots & & & \vdots \\
y_0 & & & & & & & + & x_n & \geq & b_n
\end{array}
$$

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\begin{aligned}
y_0 &\in \mathbb{R}_+ \\
x_i &\in \mathbb{Z} \qquad \forall 1 \leq i \leq n
\end{aligned}
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Mixing Inequality is facet-defining for the Mixing Set:

$$
y_0 \geq \sum_{i=1}^{n} (\tilde{b}_i - \tilde{b}_{i-1})(\lceil b_i \rceil - x_i) \tag{1}
$$

where $\tilde{b}_i = b_i - \lceil b_i \rceil + 1$, $\tilde{b}_i \geq \tilde{b}_{i-1}$ and $\tilde{b}_0 = 0$.

The mixing set appears as a 'substructure' in many problems

The mixing inequality can be used to derive facets for:

1. Production Planning (Constant capacity lot-sizing)
2. Capacitated Facility Location
3. Capacitated Network Design

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Can we use mixing inequalities for general problems?

Rearranging the mixing set for simplicity

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} y_0 \geq \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$x_1, x_2 \in \mathbb{Z}, y_0 \in \mathbb{R}_+$$

Let $r_i = b_i (mod\, 1)$. We assume $0 < r_1 < r_2 < 1$.

Mixing Inequality: $y_0 \geq (r_2 - r_1)(\lceil b_2 \rceil - x_2) + r_1(\lceil b_1 \rceil - x_1)$

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Introduce non-negative slack variables:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} y_0 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} y_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$x_1, x_2 \in \mathbb{Z}, y_0, y_1, y_2 \in \mathbb{R}_+$$

$$\text{Mixing Inequality: } \frac{1 - r_2}{D} y_0 + \frac{r_1}{D} y_1 + \frac{r_2 - r_1}{D} y_2 \geq 1$$

where $D = (r_2 - r_1)(1 - r_2) + r_1(1 - r_1)$.

Using mixing inequalities for general simplex tableau

$$\left.\begin{array}{l} x_1 + y_0 - y_1 = 1.4 \\ x_2 + y_0 - y_2 = 0.6 \\ x_1, x_2 \in \mathbb{Z}, y_0, y_1, y_2 \in \mathbb{R}_+ \end{array}\right\} \Rightarrow \frac{10}{8} y_0 + \frac{10}{8} y_1 + \frac{5}{8} y_2 \geq 1$$

Idea: Rewrite/Relax rows of simplex tableau to 'look' like the Mixing Set

$$\begin{aligned} u_1 + 0u_2 + 0.5u_3 + 0.9u_4 &= 1.4 \\ 0u_1 + 1u_2 + 0.1u_3 + 0.5u_4 &= 0.6 \\ u &\in \mathbb{Z}^4_+ \end{aligned}$$

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$$\tfrac{10}{8}(0.5u_3 + 0.9u_4) + \tfrac{10}{8}(0) + \tfrac{5}{8}(0.4u_3 + 0.3u_4) \geq 1$$

They are many possible ways to relax simplex tableau

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The 'best' cut

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For the simplex tableau:

$$\sum_{i=1}^{n} \begin{pmatrix} a_1^i \\ a_2^i \end{pmatrix} u_i + \sum_{j=1}^{m} \begin{pmatrix} c_1^j \\ c_2^j \end{pmatrix} v_j = \begin{pmatrix} 1.4 \\ 0.6 \end{pmatrix}. \qquad u_i \in \mathbb{Z}_+, v_j \in \mathbb{R}_+ \tag{2}$$

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The 'best' cut by relaxation of the simplex tableau is

$$\sum_{i=1}^{n} \phi^0(a_1^i, a_2^i) u_i + \sum_{j=1}^{m} \pi^0(c_1^j, c_2^j) v_j \geq 1 \tag{3}$$

where,

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$$\phi^0(a_1, a_2) = \begin{array}{ll} \min & \frac{10}{8} y_0 + \frac{10}{8} y_1 + \frac{5}{8} y_2 \\ \text{s.t.} & x_1 + y_0 - y_1 = a_1 \\ & x_2 + y_0 - y_2 = a_2 \\ & x_1, x_2 \in \mathbb{Z}, y_0, y_1, y_2 \in \mathbb{R}_+ \end{array}$$

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$$\begin{array}{lll} \pi^0(c_1, c_2) = & \min & \frac{10}{8} y_0 + \frac{10}{8} y_1 + \frac{5}{8} y_2 \\ & \text{s.t.} & y_0 - y_1 = c_1 \\ & & y_0 - y_2 = c_2 \\ & & y_0, y_1, y_2 \in \mathbb{R}_+ \end{array}$$

Closed form for ϕ^0 and π^0

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Proposition

Consider two rows of a simplex tableau: $\sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{m} c_i y_i = b$, $x_i \in \mathbb{Z}_+$, $y_i \in \mathbb{R}_+$. *Then the inequality* $\sum_{i=1}^{n} \phi^0(\mathcal{F}(a_i))x_i + \sum_{i=1}^{m} \pi^0(c_i)y_i \geq 1$ *is valid inequality where,*

$$\phi^0(w_1, w_2) = \begin{cases} \sigma_1(1 - w_1) + \sigma_2(1 - w_2) & (w_1, w_2) \in R^1 \\ \sigma_3(w_1) + \sigma_2(1 - w_2) & (w_1, w_2) \in R^2 \\ \sigma_1(1 - w_1) + \sigma_2(1 - w_2) & (w_1, w_2) \in R^3 \\ \sigma_1(1 - w_1) + \sigma_4(w_2) & (w_1, w_2) \in R^4 \\ \sigma_3(w_1) + \sigma_2(-w_2) & (w_1, w_2) \in R^5 \\ \sigma_1(-w_1) + \sigma_4(w_2) & (w_1, w_2) \in R^6 \end{cases} \tag{4}$$

where $(r_1, r_2) = \mathcal{F}(c)$ *and* $\sigma_1 = \frac{r_1}{r_2 - (r_1)^2 - r_2(r_2 - r_1)}$, $\sigma_2 = \frac{r_2 - r_1}{r_2 - (r_1)^2 - r_2(r_2 - r_1)}$, $\sigma_3 = \frac{1 - r_1}{r_2 - (r_1)^2 - r_2(r_2 - r_1)}$, $\sigma_4 = \frac{1 - r_2 + r_1}{r_2 - (r_1)^2 - r_2(r_2 - r_1)}$ *and*

$$\pi^0(c) = lim_{h \downarrow 0} \frac{\phi(ch)}{h} \tag{5}$$

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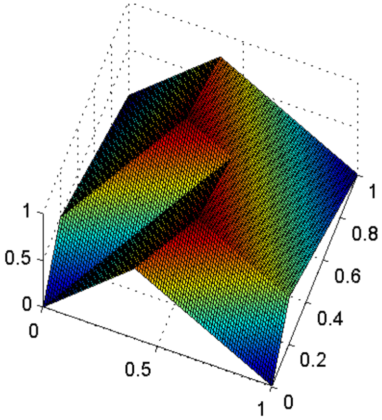
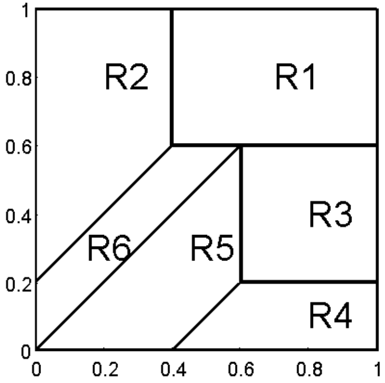
Observation: The closed form of ϕ^0 depends only on the fractional part of columns of integer variables.

Closed form for ϕ^0 contd.





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- We show that there exist functions (ϕ^M, π^0) that strictly dominate (ϕ^0, π^0) and are extreme for the infinite group relaxation of mixed integer programs.

Is it possible to get a stronger general-purpose 'mixing-type' cut?

Yes!

More precisely:

- (ϕ^0, π^0) represents a valid inequality for the *infinite group relaxation* of mixed integer programs.
- We show that there exist functions (ϕ^M, π^0) that strictly dominate (ϕ^0, π^0) and are extreme for the infinite group relaxation of mixed integer programs.
- Upshot: Better cut coefficients can be obtained using the function (ϕ^M, π^0) that improve upon the coefficients obtained by (ϕ^0, π^0).

We proceed step-by-step in the following slides.

The Framework: Infinite Group Relaxation.

Infinite relaxation of two-rows of simplex tableau

The mixed integer relaxation: $MG(I^2, \mathbb{R}^2, f)$:

$$\sum_{a \in I^2} ax(a) + \sum_{w \in \mathbb{R}^2} wy(w) + f = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_{B_1} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_{B_2}$$
$$x_{B_1}, x_{B_2} \in \mathbb{Z}, x(a) \in \mathbb{Z}_+, y(w) \in \mathbb{R}_+$$
$$x, y \text{ have finite support }, \quad (6)$$

where $I^2 = \{(a_1, a_2) \in \mathbb{R}^2 \,|\, 0 \leq a_1, a_2 \leq 1\}$, i.e. set of all columns of two fractions.

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Assuming all the nonbasic variables are continuous: $MG(\emptyset, \mathbb{R}^2, f)$:

$$\sum_{w \in \mathbb{R}^2} wy(w) + f = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_{B_1} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_{B_2}$$
$$x_{B_1}, x_{B_2} \in \mathbb{Z}, y(w) \in \mathbb{R}_+$$
$$y \text{ has a finite support }. \tag{7}$$

Problem statement precisely

Now that we have defined the infinite relaxation, we proceed to ask the following precise questions...

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Now that we have defined the infinite relaxation, we proceed to ask the following precise questions...

The function (ϕ^0, π^0) represents a valid inequality for $MG(I^2, \mathbb{R}^2, f)$ as:
If $(\bar{x}, \bar{y})$ satisfies:

$$\sum_{a \in I^2} a x(a) + \sum_{w \in \mathbb{R}^2} w y(w) + f = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_{B_1} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_{B_2}$$

$$x_{B_1}, x_{B_2} \in \mathbb{Z}, x(a) \in \mathbb{Z}_+, y(w) \in \mathbb{R}_+,$$

then $(\bar{x}, \bar{y})$ satisfies:

$$\sum_{a \in I^2} \phi^0(a) x(a) + \sum_{w \in \mathbb{R}^2} \pi^0(w) y(w) \geq 1 \tag{8}$$

Question: Do there exist functions (ϕ', π'), $\phi' : I^2 \rightarrow \mathbb{R}_+$ and $\pi' : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ such that: $\phi' \leq \phi^0$ and $\pi' \leq \pi^0$ and

$$\sum_{a \in I^2} \phi'(a) x(a) + \sum_{w \in \mathbb{R}^2} \pi'(w) \geq 1 \tag{9}$$

is a valid inequality?

Analysis of π^0: The Strength of Continuous Coefficients.

Maximal lattice-free convex sets

[Lovász(1989)]

Definition

A set S is called a maximal lattice-free convex set in $\mathbb{R}^2$ if it is convex and

1. $\text{interior}(S) \cap \mathbb{Z}^2 = \emptyset$,
2. There exists no convex set S' satisfying (1), such that $S \subsetneq S'$. □

[Borozan and Cornuéjols (2007), Andersen, Louveaux, Weismantel, and Wolsey (2007)]

Theorem

For the system $MG(\emptyset, \mathbb{R}^2, f)$, an inequality of the form $\sum_{w \in \mathbb{R}^2} \pi(w)y(w) \geq 1$ is un-dominated, if the set

$$P(\pi) = \{w \in \mathbb{R}^2 | \pi(w - f) \leq 1\} \quad (10)$$

is a maximal lattice-free convex set. □

Strength of continuous coefficients, i.e., π^0

$$\begin{aligned}\pi^0(w_1, w_2) = \quad \min \quad & \frac{10}{8} y_0 + \frac{10}{8} y_1 + \frac{5}{8} y_2 \\ \text{s.t.} \quad & y_0 - y_1 = w_1 \\ & y_0 - y_2 = w_2 \\ & y_0, y_1, y_2 \in \mathbb{R}_+\end{aligned}$$

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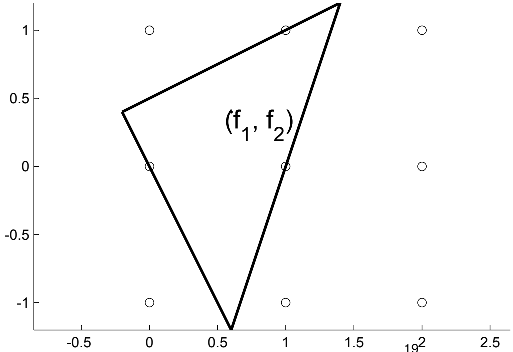
We construct $P(\pi^0) : \{w \in \mathbb{R}^2 | \pi(w - f) \leq 1\}$:



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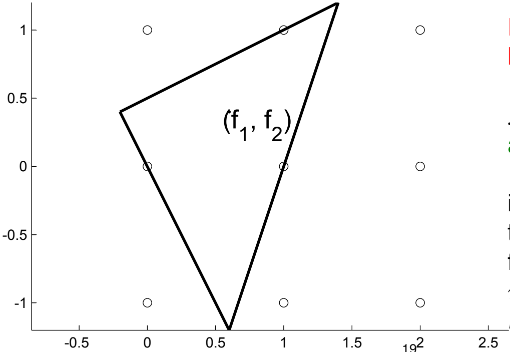

For all r, $P(\pi^0)$ is a maximal lattice-free triangle.

$\therefore$ π^0 is undominated by any inequality:

i.e. $\nexists\ \pi' : \mathbb{R}^2 \to \mathbb{R}_+$ such that $\pi' \leq \pi^0$, $\pi'(w) < \pi^0(w)$ for some $w \in \mathbb{R}^2$ where π' is valid inequality for $MG(\emptyset, \mathbb{R}^2, r)$.

π^0 is an extreme inequality for $MG(\emptyset, \mathbb{R}^2, r)$

[Cornuéjols and Margot(2008)]

Theorem

If $P(\pi)$ is a maximal lattice-free triangle, then π represents an extreme inequality for $MG(\emptyset, \mathbb{R}^2, f)$, i.e, $\nexists\ \pi_1, \pi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ such that π_1 and π_2 represent valid inequalities, $\pi_1 \neq \pi_2$, and $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$.

Corollary

The function representing coefficient of continuous variable obtained using mixing inequalities, i.e., π^0, is extreme for $MG(\emptyset, \mathbb{R}^2, r)$.

We cannot improve the coefficient of continuous variables in the mixing cut.

Analysis of ϕ^0: The Strength of Integer Coefficients.

It is possible to strengthen ϕ^0

$$\phi^0(a_1, a_2) = \begin{array}{ll} \min & \frac{10}{8}y_0 + \frac{10}{8}y_1 + \frac{5}{8}y_2 \\ \text{s.t.} & x_1 + y_0 - y_1 = a_1 \\ & x_2 + y_0 - y_2 = a_2 \\ & x_1, x_2 \in \mathbb{Z}, y_0, y_1, y_2 \in \mathbb{R}_+ \end{array}$$

$$\pi^0(c_1, c_2) = \begin{array}{ll} \min & \frac{10}{8}y_0 + \frac{10}{8}y_1 + \frac{5}{8}y_2 \\ \text{s.t.} & y_0 - y_1 = c_1 \\ & y_0 - y_2 = c_2 \\ & y_0, y_1, y_2 \in \mathbb{R}_+ \end{array}$$

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We can rewrite:

$$\phi^0(a_1, a_2) = \min_{x_1, x_2 \in \mathbb{Z}}(\pi^0(a_1 - x_1, a_2 - x_2)). \tag{11}$$

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(11) is the fill-in function [Gomory and Johnson (1972)].

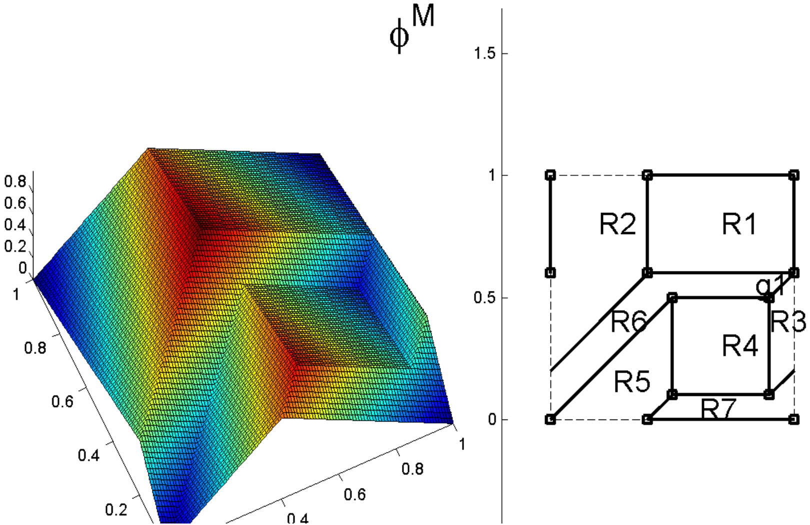
[D. and Wolsey (2008)]

Theorem

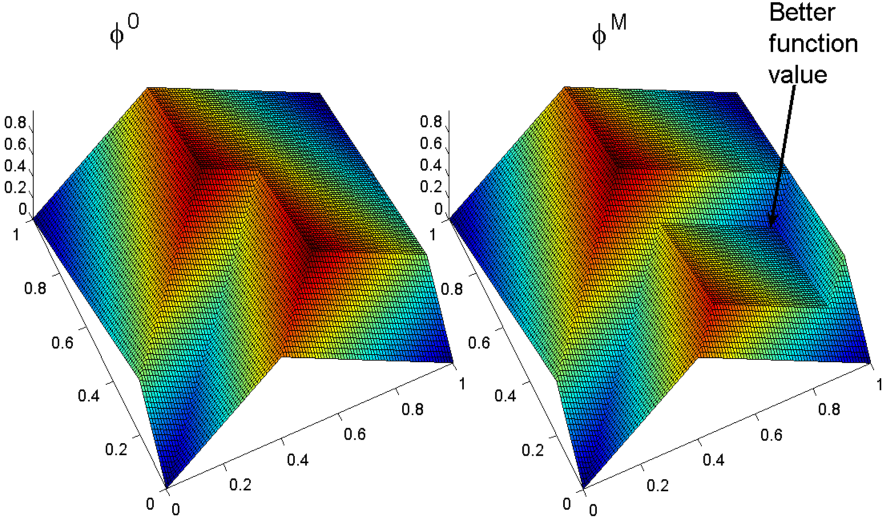
If $P(\pi)$ is a lattice-free triangle with non-integral vertices and exactly one integer point in the interior on each side, then $\phi^0 : I^2 \to \mathbb{R}_+$ defined as in (11) is not an undominated inequality. □

$\exists$ a function $\phi' : I^2 \to \mathbb{R}_+$ such that ϕ' represents a valid inequality and $\phi' < \phi^0$.

A stronger inequality ϕ^M



Comparing ϕ^M with ϕ^0



Main result: (ϕ^M, π^0) is extreme inequality for $MG(I^2, \mathbb{R}^2, r)$

Theorem

Given $0 < r_1 < r_2 < 1$ *such that* $r_1 + r_2 \leq 1$*, the functions* (ϕ^M, π^0) *represent an extreme inequality for* $MG(I^2, \mathbb{R}^2, r)$

Note: The condition $r_1 + r_2 \leq 1$ is not very 'serious', since given any two rows of tableau such that $r_1 + r_2 > 1$, multiple both the rows with -1. Then for the resulting rows, $r_1 + r_2 \leq 1$.

Steps in Proof:

1. Prove function results in a valid inequality:

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1. Prove function results in a valid inequality: Show that $\phi^M(u) + \phi^M(v) \geq \phi^M(u + v) \forall u, v \in I^2$, i.e., prove $\phi^M : I^2 \rightarrow \mathbb{R}_+$ is subadditive.

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2. Prove function is un-dominated:

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2. Prove function is un-dominated: Easily verified that, $\phi^M(u) + \phi^M(r - u) = 1 \ \forall u \in I^2$. Therefore, by [Johnson's (1974)] ϕ^M is an un-dominated inequality.

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3. Prove function is extreme.

Proving ϕ^M is a subadditive function

[D. and Richard (2008)]

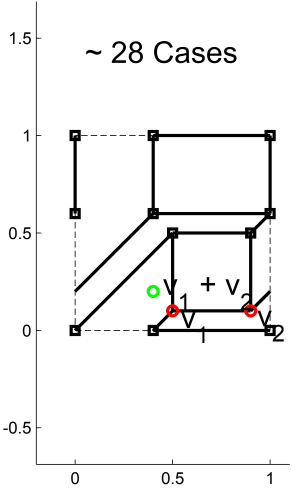
Proposition

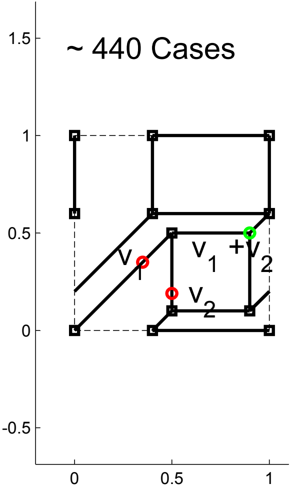
Let ϕ be a continuous, piecewise linear and nonnegative function on I^2 such that $\phi(u) + \phi(r - u) = 1$. Let $\mathbb{V}$ and $\mathbb{E}$ be the set of 'vertices' and 'edges' of ϕ. Then ϕ is subadditive iff

$$\phi(v_1) + \phi(v_2) \geq \phi(v_1 + v_2) \qquad \forall v_1, v_2 \in \mathbb{V}(\phi)$$
$$\phi(e_1) + \phi(e_2) \geq \phi(v_3) \textit{ where}$$
$$e_1 \in q_1, e_2 \in q_2, e_1 + e_2 = v_3, \forall v_3 \in \mathbb{V}(\phi) \cup \mathbb{V}'(\phi) \quad , \forall q_1, q_2 \in \mathbb{E}(\phi).$$

To check subadditivity of a piecewise linear function we need to check only the function at the 'vertices' and 'edges'.

Proving ϕ^M is a subadditive function: Too many cases!





After checking these case we prove that the function ϕ^M is a valid inequality.

A value-function interpretation of ϕ^M

In fact, by the subadditivity of ϕ^M we obtain the following result:

Proposition

If $r_1 + r_2 \leq 1$, then for the problem:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 1 + \frac{r_1 - r_2}{2} \\ \frac{r_1 + r_2}{2} \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} y_0 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} y_2 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

$$x_1, x_2 \in \mathbb{Z}, x_3 \in \mathbb{Z}_+, y_0, y_1, y_2 \in \mathbb{R}_+$$

The following inequality is valid:

$$\frac{(r_2 - r_1)(1 - r_2)}{2D} x_3 + \frac{1 - r_2}{D} y_0 + \frac{r_1}{D} y_1 + \frac{r_2 - r_1}{D} y_2 \geq 1$$

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Then ϕ^M can be obtained as follows:

$$\begin{aligned} \phi^M(a_1, a_2) = \quad \min \quad & \frac{(r_2 - r_1)(1 - r_2)}{2D} x_3 + \frac{1 - r_2}{D} y_0 + \frac{r_1}{D} y_1 + \frac{r_2 - r_1}{D} y_2 \\ \text{s.t.} \quad & x_1 + (1 + \frac{r_1 - r_2}{2}) x_3 + y_0 - y_1 = a_1 \\ & x_2 + \frac{r_1 + r_2}{2} x_3 + y_0 - y_2 = a_2 \\ & x_1, x_2 \in \mathbb{Z}, x_3 \in \mathbb{Z}_+, y_0, y_1, y_2 \in \mathbb{R}_+ \end{aligned}$$

Without the terms corresponding to x_3 the above reduces to the problem corresponding to ϕ^0.

Proving (ϕ^M, π^0) is extreme inequality for $MG(I^2, \mathbb{R}^2, r)$

[D. and Wolsey (2008)]

Theorem

1. *Let* $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ *be a extreme inequality for* $MG(\emptyset, \mathbb{R}^2, r)$.
2. *Let* $u^0 \in I^2$ *and define* $V = max_{n \in \mathbb{Z}_+, n \geq 1}\{\frac{1-\pi(w)}{n} | \mathcal{F}(u^0 n + w) = r\}$. . . . *Lifting.*
3. *Define* $\phi : I^2 \rightarrow \mathbb{R}_+$ *as* $\phi(v) = min_{n \in \mathbb{Z}_+}\{nV + \pi(w) | \mathcal{F}(u^0 n + w) = v\}$. . . . *Fill-in.*

If (ϕ, π) *is an un-dominated inequality for* $MG(I^2, \mathbb{R}^2, r)$, *then* (ϕ, π) *is an extreme inequality for* $MG(I^2, \mathbb{R}^2, r)$.

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In our case, $u^0 = (1 + \frac{r_1 - r_2}{2}, \frac{r_1 + r_2}{2})$ and

$$\frac{(r_2 - r_1)(1 - r_2)}{2D} = \max_{n \in \mathbb{Z}_+, n \geq 1}\{\frac{1 - \pi^0(w)}{n} | \mathcal{F}(u^0 n + w) = r\}$$

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2. We showed that when applying the mixing inequalities for general two-rows of a simplex tableau, the inequality can be strengthened.
3. A new class of extreme inequality for two-row mixed integer infinite group problem.

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Challenges:

1. The proof of validity of the stronger inequality is *not elegant*: More importantly the proof does not extend to more rows of a mixing inequalities.

Thank You.