Lagrangian dual with zero duality gap that admits decomposition

Diego Cifuentes Santanu S. Dey Jingye Xu

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Section 1

Introduction

Block-structure MILP with coupling constraint

We consider a mixed-integer linear optimization:

$$\begin{aligned}
\mathbf{OPT} := \min_{(\mathbf{x},\mathbf{y})} \ & \sum_{i\in\{1,2\}} \left\langle \mathbf{c}^{(i)}, \mathbf{x}^{(i)} \right\rangle + \left\langle \mathbf{d}^{(i)}, \mathbf{y}^{(i)} \right\rangle \\
\text{s.t. } & (\mathbf{x}^{(1)}, \mathbf{y}^{(1)}) \in \mathcal{X}^{(1)} \\
& (\mathbf{x}^{(2)}, \mathbf{y}^{(2)}) \in \mathcal{X}^{(2)} \\
& \mathbf{x}^{(1)} = \mathbf{x}^{(2)} \in \{0,1\}^n.
\end{aligned}$$

- $\mathcal{X}^{(i)}$ is a mixed integer linear set (linear constraints + some integrality requirement).

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- We assume $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \{0,1\}^n$.
- We focus on the two-block case for simplicity. More general case later.
- Related to N-fold integer programming. [De Loera, Hemmecke, Onn, Weismantel (2008)], [Hemmecke, Onn, Weismantel (2013)] [Cslovjecsek et al. (2021)], [Cslovjecsek et al. (2024)]

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$$\mathbf{DUAL\text{-}OPT} := \max_{\lambda} L(\lambda).$$

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- Sub-gradient algorithm, Bundle methods, etc.

Duality gap

Theorem ([Geoffrion (1974)])

$$\textbf{DUAL-OPT} = \min_{(\mathbf{x},\mathbf{y})} \sum_{i\in\{1,2\}} \left\langle \mathbf{c}^{(i)}, \mathbf{x}^{(i)} \right\rangle + \left\langle \mathbf{d}^{(i)}, \mathbf{y}^{(i)} \right\rangle$$

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- Because of non-convexity of $\mathcal{X}^{(i)}$, it is possible that **DUAL-OPT** $<$ **OPT**

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- **We want: decomposable + zero duality gap**

Main idea: add redundant constraints

- Suppose

$$\mathbf{x}^{(1)} = \mathbf{x}^{(2)} \in \{0,1\}^n \implies \alpha^{(1)}(\mathbf{x}^{(1)}) + \alpha^{(2)}(\mathbf{x}^{(2)}) \geq 0.$$

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$$\mathbf{x}^{(1)} = \mathbf{x}^{(2)} \in \{0,1\}^n \implies (x_1^{(1)} + x_2^{(1)}) - (x_1^{(2)} \cdot x_2^{(2)}) \geq 0.$$

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- Trivially, we have $\mathbf{OPT}' = \mathbf{OPT}$.

Close gap with more constraints

- $(x_1^{(1)} + x_2^{(1)}) - (x_1^{(2)} \cdot x_2^{(2)}) \geq 0$ is a coupling constraint.

$$L(\lambda, \mu)^{ex} := \min_{(\mathbf{x},\mathbf{y})} \left(\sum_{i \in \{1,2\}} \left\langle \mathbf{c}^{(i)}, \mathbf{x}^{(i)} \right\rangle + \left\langle \mathbf{d}^{(i)}, \mathbf{y}^{(i)} \right\rangle \right)$$
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- Lagrangian relaxation still decomposes into two blocks:

$$\min \left\langle \mathbf{c}^{(1)} + \lambda, \mathbf{x}^{(1)} \right\rangle + \left\langle \mathbf{d}^{(1)}, \mathbf{y}^{(1)} \right\rangle + \mu \cdot (x_1^{(1)} + x_2^{(1)})$$
$$\text{s.t. } (\mathbf{x}^{(i)}, \mathbf{y}^{(i)}) \in \mathcal{X}^{(i)}$$

$$\min \left\langle \mathbf{c}^{(2)} - \lambda, \mathbf{x}^{(2)} \right\rangle + \left\langle \mathbf{d}^{(2)}, \mathbf{y}^{(2)} \right\rangle - \mu \cdot x_1^{(2)} x_2^{(2)}$$
$$\text{s.t. } (\mathbf{x}^{(i)}, \mathbf{y}^{(i)}) \in \mathcal{X}^{(i)}$$

Section 2

Strong Duality + Decomposability

On the choice of redundant constraints

We know that

- $\textbf{OPT}' = \textbf{OPT}$
- How about $\textbf{DUAL-OPT}'$? When we can expected $\textbf{DUAL-OPT}' > \textbf{DUAL-OPT}$?

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If either $\alpha^{(1)}(\cdot)$ *or* $\alpha^{(2)}(\cdot)$ *is an affine function, then* **DUAL-OPT**$'$ $=$ **DUAL-OPT**.

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If either $\alpha^{(1)}(\cdot)$ *or* $\alpha^{(2)}(\cdot)$ *is an affine function, then* **DUAL-OPT**$'$ $=$ **DUAL-OPT**.

- In example, $\alpha^{(1)}(\mathbf{x}^{(1)}) = x_1^{(1)} + x_2^{(1)}$ is an affine function. So no dual bound improvement.
- Both $\alpha^{(1)}(\cdot)$ and $\alpha^{(2)}(\cdot)$ need to be non-linear to improve **Duality-gap**.
- We can add multiply redundant constraints.

One choice of redundant constraints: M-constraints

- Let $\mathcal{S} \subseteq 2^{[n]}$ be a collection of subsets of $[n]$, the primal-redundant constraints take form of

$$\prod_{s \in S} x_s^{(1)} = \prod_{s \in S} x_s^{(2)}, \forall S \in \mathcal{S}.$$

  For example, when $S = \{1, 2\}$, the constraint looks like

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- Not the same as RLT. Traditional RLT constraint:

$$(x_1^{(1)} - x_1^{(2)}) \cdot (x_2^{(1)} - x_2^{(2)}) = 0,$$

which is not decomposable after dualizing.

Another (slightly different) choice of redundant constraints: V-constraints

- Let $V \subseteq \{0,1\}^n$, the primal-redundant constraints take form of

$$\prod_{j \in [n]} \sigma_{v_j}(x_j^{(1)}) = \prod_{j \in [n]} \sigma_{v_j}(x_j^{(2)}), \text{ for each vertex } \mathbf{v} \text{ in } V,$$

where $\sigma_{v_j}(u) := \begin{cases} u & \text{if } v_j = 1 \\ 1-u & \text{if } v_j = 0 \end{cases}$.
For example, when $n = 3$ and $v = (1,0,1)$ the constraint looks like

$$x_1^{(1)}(1 - x_2^{(1)})x_3^{(1)} = x_1^{(2)}(1 - x_2^{(2)})x_3^{(2)}.$$

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- Eventually leads to a "Column generation-type algorithm"

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M-Lagrangian Dual

Theorem

If $\mathcal{S} = 2^{[n]}$, *then* $\textbf{OPT} = \textbf{DUAL-OPT}'$

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V-Lagrangian Dual

Theorem

If $V = \{0,1\}^n$, then $\mathbf{OPT} = \mathbf{DUAL\text{-}OPT}'$

Proof of strong duality (M-Lagrangian dual)

- Let

$$f^{(i)}(\mathbf{x}) := \min \left\langle \mathbf{c}^{(i)}, \mathbf{x}^{(i)} \right\rangle + \left\langle \mathbf{d}^{(i)}, \mathbf{y}^{(i)} \right\rangle$$
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- For simplicity, assume the above problem is feasible for all $\mathbf{x} \in \{0,1\}^n$. (Otherwise there is a simple fix)
- Then $f^{(i)} : \{0,1\}^n \to \mathbb{R}$ for $i \in \{1,2\}$ are pseudo-Boolean functions, so there exists $\mu^*_S \in \mathbb{R}$ for all $S \subseteq 2^{[n]}$, such that

$$f^{(2)}(\mathbf{x}) = \sum_{S \subseteq 2^{[n]}} \mu^*_S \prod_{j \in S} \mathbf{x}_j$$

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& \mathbf{OPT} \\
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= \ & \max_{\mu} \left(\min_{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}} \left((f^{(1)}(\mathbf{x}^{(1)}) + f^{(2)}(\mathbf{x}^{(2)})) + \sum_{S \subseteq 2^{[n]}} \mu_S (\prod_{j \in S} \mathbf{x}_j^{(1)} - \prod_{j \in S} \mathbf{x}_j^{(2)}) \right) \right) \\
\geq \ & \min_{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}} \left((f^{(1)}(\mathbf{x}^{(1)}) + f^{(2)}(\mathbf{x}^{(2)})) + \sum_{S \subseteq 2^{[n]}} \mu_S^* (\prod_{j \in S} \mathbf{x}_j^{(1)} - \prod_{j \in S} \mathbf{x}_j^{(2)}) \right) \\
= \ & \min_{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}} \left((f^{(1)}(\mathbf{x}^{(1)}) + \cancel{f^{(2)}(\mathbf{x}^{(2)})}) + \left(f^{(2)}(\mathbf{x}^{(1)}) - \cancel{f^{(2)}(\mathbf{x}^{(2)})} \right) \right)
\end{aligned}$$

Proof of strong duality (M-Lagrangian dual)

$$f^{(2)}(\mathbf{x}) = \sum_{S \subseteq 2^{[n]}} \mu_S^* \prod_{j \in S} \mathbf{x}_j$$

$$\begin{aligned}
& \textbf{OPT} \\
\geq\ & \textbf{DUAL-OPT}' \\
=\ & \max_\mu \left(\min_{\mathbf{x}^{(1)},\mathbf{x}^{(2)}} \left((f^{(1)}(\mathbf{x}^{(1)}) + f^{(2)}(\mathbf{x}^{(2)})) + \sum_{S \subseteq 2^{[n]}} \mu_S (\prod_{j \in S} \mathbf{x}_j^{(1)} - \prod_{j \in S} \mathbf{x}_j^{(2)}) \right) \right) \\
\geq\ & \min_{\mathbf{x}^{(1)},\mathbf{x}^{(2)}} \left((f^{(1)}(\mathbf{x}^{(1)}) + f^{(2)}(\mathbf{x}^{(2)})) + \sum_{S \subseteq 2^{[n]}} \mu_S^* (\prod_{j \in S} \mathbf{x}_j^{(1)} - \prod_{j \in S} \mathbf{x}_j^{(2)}) \right) \\
=\ & \min_{\mathbf{x}^{(1)},\mathbf{x}^{(2)}} \left((f^{(1)}(\mathbf{x}^{(1)}) + \cancel{f^{(2)}(\mathbf{x}^{(2)})}) + \left(f^{(2)}(\mathbf{x}^{(1)}) - \cancel{f^{(2)}(\mathbf{x}^{(2)})} \right) \right) \\
=\ & \min_{\mathbf{x}^{(1)},\mathbf{x}^{(2)}} \left(f^{(1)}(\mathbf{x}^{(1)}) + f^{(2)}(\mathbf{x}^{(1)}) \right) \\
=\ & \textbf{OPT}.
\end{aligned}$$

Section 3

Bounds as a function of degree of monomial

Bounds for packing and covering sets.

- Packing:
  - We use "min" as our objective, so objective is non-positive.
  - Assumption: $\text{proj}_{\mathbf{x}^{(1)}}(\mathcal{X}^{(1)}) = \text{proj}_{\mathbf{x}^{(2)}}(\mathcal{X}^{(2)})$. (Easy to achieve)

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Theorem

Fix some number $k \geq 1$, let $t = \frac{k}{n}$ and $\mathcal{S} = \binom{[n]}{\leq k}$. (i.e, all monomials of degree up to k)

- *For packing instance,* $(2 + \frac{1}{t-2}) \cdot \mathbf{OPT} \leq \mathbf{DUAL\text{-}OPT}' \leq \mathbf{OPT}$
- *For covering instance,* $\frac{1}{2-t} \cdot \mathbf{OPT} \leq \mathbf{DUAL\text{-}OPT}' \leq \mathbf{OPT}$

$k = 1$ case for packing

$$\textbf{DUAL-OPT} = \min_{(\mathbf{x},\mathbf{y})} \sum_{i \in \{1,2\}} \left\langle \mathbf{c}^{(i)}, \mathbf{x}^{(i)} \right\rangle + \left\langle \mathbf{d}^{(i)}, \mathbf{y}^{(i)} \right\rangle$$

$$\text{s.t. } (\mathbf{x}^{(i)}, \mathbf{y}^{(i)}) \in \text{conv}\{\mathcal{X}^{(i)}\}, \forall i \in \{1,2\},$$

$$\mathbf{x}^{(1)} = \mathbf{x}^{(2)}. \quad (\text{“outside conv”})$$

- Suppose $(\bar{x}, \bar{y}_1, \bar{x}, \bar{y}_2)$ is opt. solution of above.

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- Suppose $(\bar{x}, \bar{y}_1, \bar{x}, \bar{y}_2)$ is opt. solution of above.
- So $(\bar{x}, \bar{y}_1, \bar{x}, \bar{y}_2) = \sum_{j=1}^{r} \lambda_j \left(\underbrace{(x^{(1)})^j, (y^{(1)})^j}_{\in \mathcal{X}^{(1)}}, \underbrace{(x^{(2)})^j, (y^{(2)})^j}_{\in \mathcal{X}^{(2)}} \right), \lambda \in \Delta^r.$

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- By packing and assumption, we have

$$(\bar{x}, \bar{y}_1, \bar{x}, 0) = \sum_{j=1}^{r} \lambda_j \left(\underbrace{(x^{(1)})^j, (y^{(1)})^j}_{\in \mathcal{X}^{(1)}}, \underbrace{(x^{(1)})^j, 0}_{\in \mathcal{X}^{(2)}} \right) \in \text{convex hull of IP.}$$

$k = 1$ case for packing -contd.

1. Similarly, $(\bar{x}, \bar{y}_1, \bar{x}, 0), (\bar{x}, 0, \bar{x}, \bar{y}_2), (0, \bar{y}_1, 0, \bar{y}_2) \in$ convex hull of IP.

$k = 1$ case for packing -contd.

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2. So obj-val$(\bar{x}, \bar{y}_1, \bar{x}, 0) \geq$ **OPT**, obj-val$(\bar{x}, 0, \bar{x}, \bar{y}_2) \geq$ **OPT**, obj-val$(0, \bar{y}_1, 0, \bar{y}_2) \geq$ **OPT**.

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3. On the other hand obj-val$((\bar{x}, \bar{y}_1, \bar{x}, 0) + (\bar{x}, 0, \bar{x}, \bar{y}_2) + (0, \bar{y}_1, 0, \bar{y}_2)) =$ obj-val$(2 \cdot (\bar{x}, \bar{y}_1, \bar{x}, \bar{y}_2)) = 2 \cdot$ **DUAL-OPT**.

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4. So by (2.) and (3.), we have $\frac{3}{2} \cdot$ **OPT** $\leq$ **DUAL-OPT**.

Main proof ingredient for $k \geq 1$

Let $\mathcal{S}$ be a down-closed collection of subsets of $2^{[n]}$. Then:

$$\begin{aligned}
\mathbf{DUAL\text{-}OPT}' = \min_{(\mathbf{x},\mathbf{y},\mathbf{w})} & \sum_{i \in \{1,2\}} \left\langle \mathbf{c}^{(i)}, \mathbf{x}^{(i)} \right\rangle + \left\langle \mathbf{d}^{(i)}, \mathbf{y}^{(i)} \right\rangle \\
\text{s.t. } & (\mathbf{x}^{(i)}, \mathbf{y}^{(i)}, \mathbf{w}^{(i)}) \in \text{conv}\{\mathcal{X}_M^{(i)}(\mathcal{S})\}, \forall i \in \{1,2\}, \\
& \mathbf{w}_S^{(1)} = \mathbf{w}_S^{(2)}, \forall S \in \mathcal{S}.
\end{aligned}$$

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- (proj) of feasible region of the above problem:

$$\mathcal{A}(\mathcal{S}) := \left\{ (\mathbf{x},\mathbf{y}) \;\middle|\; \begin{array}{l} \exists \mathbf{w}, (\mathbf{x}^{(i)}, \mathbf{y}^{(i)}, \mathbf{w}^{(i)}) \in \operatorname{conv}\{\mathcal{X}_M^{(i)}(\mathcal{S})\}, \forall i \in \{1,2\}, \\ \mathbf{w}_S^{(1)} = \mathbf{w}_S^{(2)}, \forall S \in \mathcal{S} \end{array} \right\}.$$

-

$$\mathcal{B}(\mathcal{S}) := \bigcap_{\mathcal{U}\in\mathcal{S}} \operatorname{conv} \left\{ (\mathbf{x},\mathbf{y}) \;\middle|\; \begin{array}{l} (\mathbf{x}^{(i)}, \mathbf{y}^{(i)}) \in \mathcal{X}^{(i)}, \forall i \in \{1,2\}, \\ x_j^{(1)} = x_j^{(2)}, \forall j \in \mathcal{U} \quad \text{“inside conv” for some vars.} \end{array} \right\}.$$

Theorem

If $\mathcal{S}$ is down-closed, then we have $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{B}(\mathcal{S})$.

Section 4

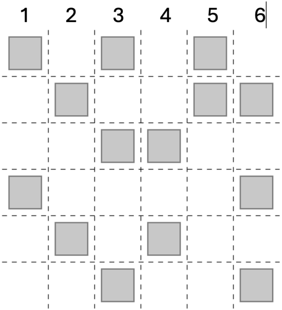
General Setting

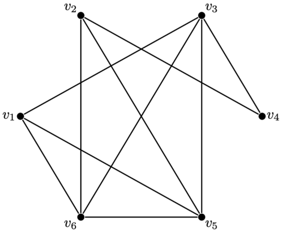
General Setting

$$\min\{\mathbf{c}^\top \mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \text{ is mixed-binary}\}. \quad (1)$$

Definition

The intersection graph of (2) is a simple undirected graph that has a vertex for each variable in (2) and two vertices are adjacent if and only if their associated variables appear in any common constraint of $Ax \leq b$.





General Setting

$$\min\{\mathbf{c}^\top \mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \text{ is mixed-binary}\}. \tag{2}$$

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Definition

Let G be a simple undirected graph. A tree decomposition of G is a pair of $(\mathcal{T}, \mathcal{Q})$ where T is a tree and $\mathcal{Q} = \{\mathcal{Q}_t : t \in V(\mathcal{T})\}$ is a collection of vertices of $V(G)$ such that the following holds:

1. For each $v \in V(G)$, the set $\{t \in V(\mathcal{T}) : v \in \mathcal{Q}_t\}$ forms a subtree of $\mathcal{T}$,
2. If $(u, v) \in E(G)$, then there exists $t \in V(\mathcal{T})$ such that $u, v \in \mathcal{Q}_t$,
3. $\bigcup_{t \in V(\mathcal{T})} \mathcal{Q}_t = V(G)$.

General case -contd.

$(\mathcal{T}, \mathcal{Q})$ is a tree decomposition:

$$
\begin{aligned}
\min \quad & \sum_{i \in V(\mathcal{T})} \left\langle \mathbf{c}^{(i)}, \mathbf{x}^{(i)} \right\rangle \\
\text{s.t.} \quad & \mathbf{x}^{(i)} \in \mathcal{X}^{(i)}, \forall i \in V(\mathcal{T}), \quad \text{(Local copy of each variable for every bag)} \\
& x_v^{(i)} = x_v^{(j)}, \forall (i,j) \in E(\mathcal{T}) \text{ and } v \in \mathcal{Q}_i \cap \mathcal{Q}_j, \\
& \text{(Matching local copy of variable on edge of tree)}
\end{aligned} \tag{3}
$$

General case -contd.

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Adding redundant constraints corresponding to a collection of monomial per edge $\mathcal{S}_{ij}$:

$$\begin{aligned} \min \quad & \sum_{i \in V(\mathcal{T})} \left\langle \mathbf{c}^{(i)}, \mathbf{x}^{(i)} \right\rangle \\ \text{s.t.} \quad & (\mathbf{x}^{(i)}, \mathbf{w}^{(i)}) \in \mathcal{X}_M^{(i)}, \forall i \in V(\mathcal{T}), \\ & \mathbf{w}_S^{(i)} = \mathbf{w}_S^{(j)}, \forall (i,j) \in E(\mathcal{T}), S \in \mathcal{S}_{ij}. \end{aligned} \tag{4}$$

General case -contd.

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Theorem

If $\mathcal{S}_{ij} = 2^{\mathcal{Q}_i \cap \mathcal{Q}_j}$, then zero-duality gap.

Packing and covering bounds

Definition (Good and k-good)

Given a subset $\mathcal{W}$ of variables $\mathbf{x}$ in (2), let $\mathcal{V} := \{i \in V(\mathcal{T}) : \mathcal{Q}_i \cap \mathcal{W} \neq \emptyset\}$. Consider the sub-graph $\mathcal{T}(\mathcal{W})$ of $\mathcal{T}$ induced by $\mathcal{V}$. We say $\mathcal{W}$ is *good* if every connected component $\mathcal{C}$ of $\mathcal{T}(\mathcal{W})$ satisfies either

1. For any $(i,j) \in E(\mathcal{C})$, $|\mathcal{Q}_i \cap \mathcal{Q}_j \cap \mathcal{W}| \leq k$.
2. There exists $i \in V(\mathcal{C})$ such that $(\mathcal{Q}_j \cap \mathcal{W}) \subseteq (\mathcal{Q}_i \cap \mathcal{W}), \forall j \in V(\mathcal{C})$.

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$$\eta_k := \left\{ \min \sum_{\mathcal{W} \text{ is good}} \alpha_{\mathcal{W}} : \sum_{\mathcal{W} \text{ is good}} \alpha_{\mathcal{W}} \chi_{\mathcal{W}} \geq \mathbf{1} \text{ and } \alpha_{\mathcal{W}} \geq 0. \right\} \tag{5}$$

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Theorem

For any packing instance, we have that
$\eta_k \cdot \mathbf{OPT} \leq \mathbf{DUAL\text{-}OPT}' \leq \mathbf{OPT}$.

Theorem

For any covering instance, let $\tau := \max_{v \in V(\mathcal{G})} |\{i \in V(\mathcal{T}) : v \in \mathcal{Q}_i\}|$.

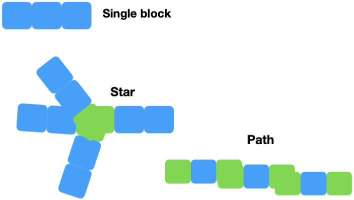
Then we have that $\dfrac{\theta_k}{1 - \tau + \tau \cdot \theta_k} \cdot \mathbf{OPT} \leq \mathbf{DUAL\text{-}OPT}' \leq \mathbf{OPT}$.

Section 5

Preliminary computational study

Preliminary computational study

- 10 blocks, each block is a some stable set problem on a random graph of 100 nodes. Number of variables shared between blocks is 33.
- We consider two block-structures.



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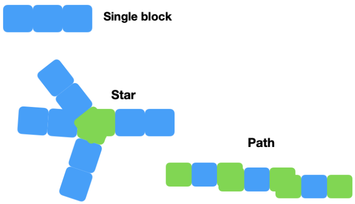

- We consider four methods:
  - (L): classical Lagrangian ; (QL) Lagrangian with all quadratic terms; (VL) Vertex Lagrangian; Gurobi
- Lagrangian dual solved using bundle method. Sub-problems solved sequentially!

Results

Table: STAR-STAB

Methods	Primal-dual gap	Time(s)	Iterations
Gurobi	10.0%	1200	-
L	6.0%	1200	127
QL	4.4%	1200	96
VL	3.9%	1200	103

Table: PATH-STAB

Methods	Primal-dual gap	Time(s)	Iterations
Gurobi	10.8%	1200	-
L	3.5%	1200	369
QL	1.2%	1200	254

Comments

- Lagrangian duals which achieve the twin goal of zero duality gap and maintaining decomposability of the sub-problems:

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  - Can we achieve Lagrangian duals with decomposability and zero duality gap while solving an "easier" subproblem in each iteration?
  - Better bounds (we do not know if our bounds are tight)
- Preliminary computational results are encouraging.
  - Significant more engineering in the implementation of our methods can be done.

Thank you!