Aggregation of quadratic inequalities and hidden hyperplane convexity

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Outline

Introduction
- QCQP: Need for convexification
- Two row relaxation

Hidden hyperplane convexity
- Properties of HHC

From HHC to convex hulls

Is HHC condition necessary?

Finiteness of aggregations.

The closed case.

1
Introduction

1.1

QCQP: Need for convexification

Quadratically Constrained Quadratic Program

QCQP

Quadratic objective, quadratic constraints:

$$\begin{aligned} \max \quad & x^\top A_0 x + 2b_0^\top x \\ \text{s.t.} \quad & x^\top A_i x + 2b_i^\top x + c_i \leq 0 \ \forall i \in [m] \end{aligned}$$

Quadratically Constrained Quadratic Program

QCQP

May be equivalently written as:

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & x^\top A_0 x + 2b_0^\top x \geq z \\ & x^\top A_i x + 2b_i^\top x + c_i \leq 0 \ \forall i \in [m] \end{aligned}$$

Quadratically Constrained Quadratic Program

QCQP

So in general, equivalent to:

$$\begin{array}{lll} \max & \tilde{b}_0^\top x & \text{(linear function)} \\ \text{s.t.} & x^\top A_i x + 2b_i^\top x + c_i \leq 0 \ \forall i \in [m] & \text{(quadratic constraints)} \end{array}$$

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1. So, we care about finding:

$$\text{conv}\left\{ x \;\middle|\; x^\top A_i x + 2b_i^\top x + c_i \leq 0 \;\forall i \in [m] \right\}$$

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2. This is challenging to compute! So we can consider convexification of relaxations (similar to integer programming)

1.2
Two row relaxation

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Two row relaxation

- We can select two rows and try and find the convex hull of their intersection:

$$\mathcal{C}2 = \left\{ x \in \mathbb{R}^n \,\middle|\, x^\top A_i x + 2b_i^\top x + c_i \leq 0 \ \forall i \in [2] \right\}$$

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- (For some technical reasons for now), let us consider the “open version" of the above set:

$$\mathcal{O}2 = \left\{ x \in \mathbb{R}^n \,\middle|\, x^\top A_i x + 2b_i^\top x + c_i < 0 \ \forall i \in [2] \right\}$$

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- It turns out convex hull of $\mathcal{O}2$ is well understood!

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Lets first talk about aggregation

- Given $\lambda \in \mathbb{R}^m_+$ and

$$S := \left\{ x \;\middle|\; x^\top A_i x + 2b_i^\top x + c_i \spadesuit 0 \;\forall i \in [m] \right\},$$

where $\spadesuit \in \{\leq, <\}$ (for all constraints).

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Then:

$$S_\lambda := \left\{ x \;\middle|\; x^\top \left(\sum_{i=1}^m \lambda_i A_i\right) x + \left(\sum_{i=1}^m \lambda_i 2b_i\right)^\top x + \left(\sum_{i=1}^m \lambda_i c_i\right) \spadesuit 0 \;\forall i \in [m] \right\}$$

is a relaxation of S.

- Basically, we are multiplying i^{th} constraint by λ_i and then adding them together.

Convex hull of $\mathcal{O}2$

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Theorem ([Yildiran (2009)])

Given a set $\mathcal{O}2 \neq \emptyset$*, such that* $\operatorname{conv}(\mathcal{O}2) \neq \mathbb{R}^n$*, there exists* $\lambda^1, \lambda^2 \in \mathbb{R}^2_+$ *such that:*

$$\operatorname{conv}(\mathcal{O}2) = (\mathcal{O}2)_{\lambda^1} \cap (\mathcal{O}2)_{\lambda^2}.$$

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- The paper [Yildiran (2009)] gives algorithm to compute λ^1 and λ^2.
- The quadratic constraints $(\mathcal{O}2)_{\lambda^i}$ $i \in \{1, 2\}$ has very nice properties:
  - $\sum_{j=1}^2 \lambda_j^i \begin{bmatrix} A_j & b_j \\ b_j^\top & c_j \end{bmatrix}$ has at most one negative eigenvalue for both $i \in \{1, 2\}$.

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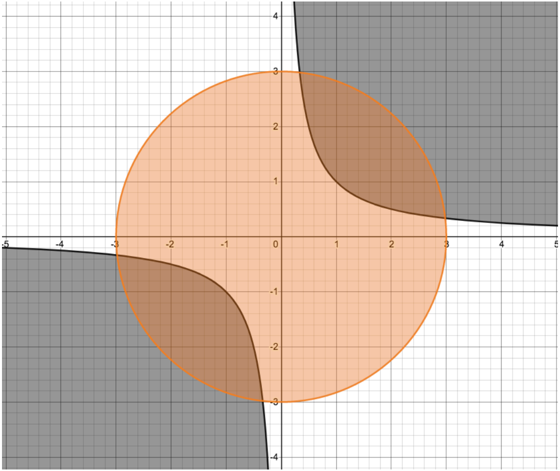
  - Basically, the sets $(\mathcal{O}2)_{\lambda^i}$ $i \in \{1, 2\}$ are either ellipsoid (may be degenarate) or hyperboloid which is union of two convex sets.
  - Henceforth, we call such quadratic constraints (that contain the convex hull) as *good* constraint.

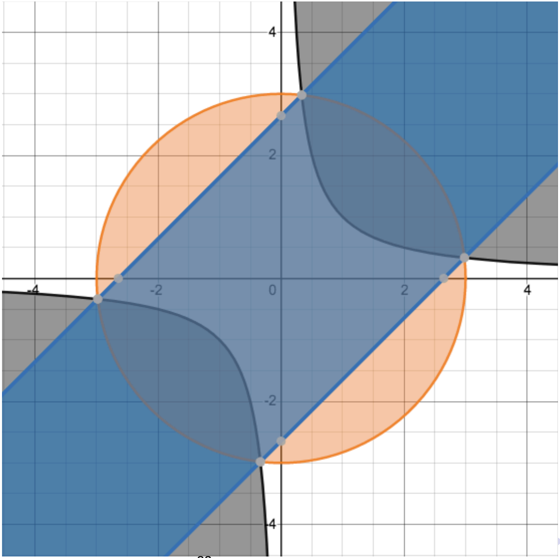
Example

$$S := \left\{ x, y \;\middle|\; \begin{array}{rcl} -xy & < & -1 \\ x^2 + y^2 & < & 9 \end{array} \right\}$$



Example - contd 1

$$\operatorname{conv}(S) := \left\{ x, y \;\middle|\; \begin{array}{lcl} (x-y)^2 & < & 7 \\ x^2 + y^2 & < & 9 \end{array} \right\}$$



Example - contd 2

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- Understanding the blue quadratic: $\lambda^1 = (2, 1)$

$$\begin{array}{lrcrl} & (-xy & < & -1) & \times 2 \\ + & (x^2 + y^2 & < & 9) & \times 1 \\ \hline \end{array}$$

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- $\lambda^2 = (0, 1)$, so the second aggregated constraints is $x^2 + y^2 < 9$.

Literature survey (incomplete!)

Related results:

- [Yildiran (2009)]
- [Burer, Kılınc-Karzan (2017)] (second order cone intersection with a nonconvex quadratic)
- [Modaresi, Vielma (2017)] (closed version of results)

Literature survey (incomplete!)

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Other related papers:

- [Tawarmalani, Richard, Chung (2010)] (Covering bilinear knapsack)
- [Santana, D. (2020)] (polytope and one quadratic constraint)
- [Ye, Zhang (2003)], [Burer, Anstreicher (2013)], [Beinstock (2014)] [Burer (2015)], [Burer, Yang (2015)], [Anstreicher (2017)] (extended trust-region problem)
- [Burer, Ye (2019)], [Wang, Kılınc-Karzan (2020, 2021)], [Argue, Kılınc-Karzan, Wang (2020)] (general conditions for the SDP relaxation being tight)
- [Gu, D., Richard (2023)] [Bienstock, Chen, Muñoz (2020)], [Muñoz and Serrano (2020)] (Cut for QCQPs)
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Questions we consider...

The main goal of this study: understand when aggregation produces convex hull.

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Hidden hyperplane convexity

Hidden convexity

- We call a map $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ a quadratic map, if there exist m symmetric matrices $Q_1, \ldots, Q_m$ such that:

$$\varphi(x) = \left(x^\top Q_1 x, \ldots, x^\top Q_m x\right) \text{ for all } x \in \mathbb{R}^n.$$

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Definition (Hidden Convexity)

A quadratic map $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ satisfies *hidden convexity* if $\text{image}\,(\varphi) = \{\varphi(x) : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$ is convex.

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Theorem (Dines [1941])

Let $Q_i \in \mathbb{S}^n$ for $i \in [2]$, then the image of $\varphi : \mathbb{R}^n \to \mathbb{R}^2$ defined as $\varphi(x) = \left(x^\top Q_1 x, x^\top Q_2 x\right)$ is convex.

Hidden hyperplane convexity (HHC)

Definition (Hidden hyperplane convexity (HHC))

A quadratic map $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ satisfies hidden hyperplane convexity (HHC) if for all linear hyperplanes $H \subseteq \mathbb{R}^n$,

$$\text{image}\,(\varphi|_H) = \{\varphi(x) : x \in H\} \subseteq \mathbb{R}^m$$

is a convex set.

2.1
Properties of HHC

Some properties of HHC: Comparison with hidden convexity

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Example

Let $\varphi(x) = \left(x^\top D_1 x, \ldots, x^\top D_m x\right)$, where $D_1, \ldots, D_m$ are diagonal matrices.

- Any diagonal quadratic map φ is known to satisfy hidden convexity. [Polyak (1998)]
- Let $\varphi : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by the three matrices:

$$Q_1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, Q_2 := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$Q_3 := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $H := \{x \in \mathbb{R}^4 : x_1 + x_2 - x_3 + x_4 = 0\}$. Note that $(1, 0, 1, 0)$ and $(0, 1, 1, 0) \in H$, $\varphi(1, 0, 1, 0) = (0, -2, 0)$ and $\varphi(0, 1, 1, 0) = (-2, 0, 0)$. Thus $(-1, -1, 0) \in \text{conv}(\text{image}(\varphi|_H))$. However, we observe that $(-1, -1, 0) \notin \text{image}(\varphi|_H)$.

Operations preserving HHC

Lemma

Suppose that $Q_1, \ldots, Q_m$ satisfy HHC. Then the following matrices also satisfy HHC:

1. $P^\top Q_1 P, \ldots, P^\top Q_m P$ *where P is any invertible matrix.*
2. $Q'_1, \ldots, Q'_k$ *where* $\text{span}(Q'_1, \ldots, Q'_k) \subseteq \text{span}(Q_1, \ldots, Q_m)$.
   (Equivalently, there exists a $k \times m$ matrix Λ such that $Q'_i = \sum_{j=1}^m \Lambda_{ij} Q_j$ for all $i \in [k]$.)

Example of maps that satisfy HHC

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- Let Q_1, Q_2, Q_3 be symmetric matrices of dimension $n \geq 4$.
  - We say Q_1, Q_2, Q_3 *positive definite linear combination* (PDLC) if Q_1, Q_2, Q_3 satisfy the following condition:

$$\exists \theta \in \mathbb{R}^3, \sum_{i=1}^{3} \theta Q_i \succ 0.$$

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  - In the case of 3 quadratics, PDLC implies hidden convexity [Calabi (1982)].
  - It follows that for 3 quadratics, PDLC implies HHC.

A more non-trivial example of HHC

Theorem (Non-trivial example of HHC with more constraints)

Fix integers $n > m + 1$, $m \geq 2$. Let $\varphi = (f_0, \ldots, f_m)$ where $f_0, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ are quadratic forms on $\mathbb{R}^n$ such that:

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- *f_0 is positive definite,*
- *There exists linear form $\ell : \mathbb{R}^n \to \mathbb{R}$ and $\ell_i : \mathbb{R}^n \to \mathbb{R}$ for all $1 \leq i \leq m$, such that $f_i(x) = \ell(x)\ell_i(x)$ for some linear form $\ell_i : \mathbb{R}^n \to \mathbb{R}$.*

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Then $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ satisfies HHC.

3

From HHC to convex hulls

Main result

- It 'makes sense' to consider good aggregations λ (which have at most one negative eigenvalue for $\sum_i \lambda_i \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix}$), so that the set defined by the aggregated constraint has at most two connected components that are both convex.

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- Furthermore S_λ clearly should contain the convex hull in one of its connected components.

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$$\Omega = \{\lambda \in \mathbb{R}^m_+ \backslash \{0\} : \text{conv}(S) \subseteq S_\lambda \text{ and } Q_\lambda \text{ has at most one negative eigenvalue.}\}$$

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$$\Omega = \{\lambda \in \mathbb{R}^m_+ \backslash \{0\} : \mathrm{conv}(S) \subseteq S_\lambda \text{ and } Q_\lambda \text{ has at most one negative eigenvalue.}\}$$

Theorem

Let $n \geq 3$ and $f_i : \mathbb{R}^n \to \mathbb{R}$ be the functions $f_i(x) = x^\top A_i x + 2b_i^\top x + c_i, i \in [m]$. Let $S = \{x \in \mathbb{R}^n : f_i(x) < 0, i \in [m]\}$. Suppose that the associated homogeneous quadratic map satisfies the hidden hyperplane convexity. If $S \neq \emptyset$ and $\mathrm{conv}(S) \neq \mathbb{R}^n$, then

Main result

- It 'makes sense' to consider good aggregations λ (which have at most one negative eigenvalue for $\sum_i \lambda_i \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix}$), so that the set defined by the aggregated constraint has at most two connected components that are both convex.
- Furthermore S_λ clearly should contain the convex hull in one of its connected components.

$$\Omega = \{\lambda \in \mathbb{R}^m_+ \backslash \{0\} : \mathrm{conv}(S) \subseteq S_\lambda \text{ and } Q_\lambda \text{ has at most one negative eigenvalue.}\}$$

Theorem

Let $n \geq 3$ and $f_i : \mathbb{R}^n \to \mathbb{R}$ be the functions $f_i(x) = x^\top A_i x + 2b_i^\top x + c_i, i \in [m]$. Let $S = \{x \in \mathbb{R}^n : f_i(x) < 0, i \in [m]\}$. Suppose that the associated homogeneous quadratic map satisfies the hidden hyperplane convexity. If $S \neq \emptyset$ and $\mathrm{conv}(S) \neq \mathbb{R}^n$, then

$$\mathrm{conv}(S) = \bigcap_{\lambda \in \Omega} S_\lambda.$$

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Previous results

Theorem

Suppose that $Q_1, \ldots, Q_m$ satisfy the following:

- *There exists two indices $i_1, i_2 \in [m]$ such that $Q_1, \ldots, Q_m$ belong to the span of Q_{i_1}, Q_{i_2}, (generalizes [Yildiran (2009)]) or,*
- *There exists three indices $i_1, i_2, i_3 \in [m]$ such that $Q_1, \ldots, Q_m$ belong to the span of $Q_{i_1}, Q_{i_2}, Q_{i_3}$ and $Q_{i_1}, Q_{i_2}, Q_{i_3}$ satisfy PDLC (generalizes [D., Muñoz, Serrano (2022)]).*

Let $S = \{x \in \mathbb{R}^n : f_i(x) < 0, i \in [m]\}$ where $f_i(x) = [x^\top 1]Q_i \begin{bmatrix} x \\ 1 \end{bmatrix}$. If $\emptyset \subsetneq \mathrm{conv}(S) \subsetneq \mathbb{R}^n$, then $\mathrm{conv}(S)$ is given by aggregations, i.e.,

$$\mathrm{conv}(S) = \bigcap_{\lambda \in \Omega_1} S_\lambda.$$

Example

- $$S := \left\{ (x, y, z) \;\middle|\; \begin{array}{rcll} x^2 + y^2 & < & 2 & \heartsuit \\ -x^2 - y^2 & < & -1 & \spadesuit \\ -x^2 + y^2 + z^2 + 6x & < & 0 & \clubsuit \end{array} \right\}$$
- PDLC condition holds, $\mathrm{conv}(S) \neq \mathbb{R}^3$

Example

- $$S := \left\{ (x,y,z) \;\middle|\; \begin{array}{rcll} x^2+y^2 & < & 2 & \heartsuit \\ -x^2-y^2 & < & -1 & \spadesuit \\ -x^2+y^2+z^2+6x & < & 0 & \clubsuit \end{array} \right\}$$
- PDLC condition holds, $\text{conv}(S) \neq \mathbb{R}^3$
- $$\text{conv}(S) := \left\{ (x,y,z) \;\middle|\; \begin{array}{rcll} x^2+y^2 & < & 2 & \heartsuit \\ -2x^2+z^2+6x & < & -1 & \spadesuit + \clubsuit \\ -x^2+y^2+z^2+6x & < & 0 & \clubsuit \end{array} \right\}$$

Example -contd 1

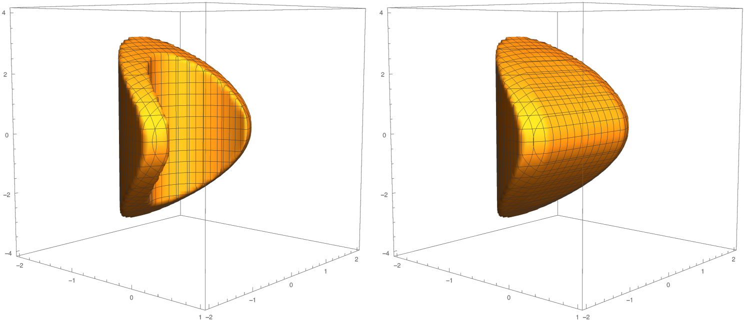

Figure: Plots of sets S (left) and $\mathrm{conv}(S)$ (right).

A new result: linear and sphere constraints

Theorem (Linear and sphere constraints)

Let $f_i(x) = x^\top A_i x + 2b_i^\top x + c_i, 1 \leq i \leq m$ be quadratic functions on $\mathbb{R}^n$, where A_i is either I_n (inside sphere), $-I_n$ (outside sphere) or 0 (linear).

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Example

- $$S := \left\{ (x,y,z) \;\middle|\; \begin{array}{rcll} x^2+y^2+z^2 & < & 3 & \spadesuit \\ x^2+y^2+z^2-2x+y & < & 3 & \clubsuit \\ -x^2-y^2-z^2-3x-2y & < & 1 & \heartsuit \end{array} \right\}$$
- PDLC condition holds, $\text{conv}(S) \neq \mathbb{R}^3$
- $$\text{conv}(S) := \left\{ (x,y,z) \;\middle|\; \begin{array}{rcll} x^2+y^2+z^2 & < & 3 & \spadesuit \\ x^2+y^2+z^2-2x+y & < & 3 & \clubsuit \\ -3x-2y-4 & < & 0 & \spadesuit+\heartsuit \\ -5x-y-4 & < & 0 & \clubsuit+\heartsuit \end{array} \right\}$$

Example -contd 1

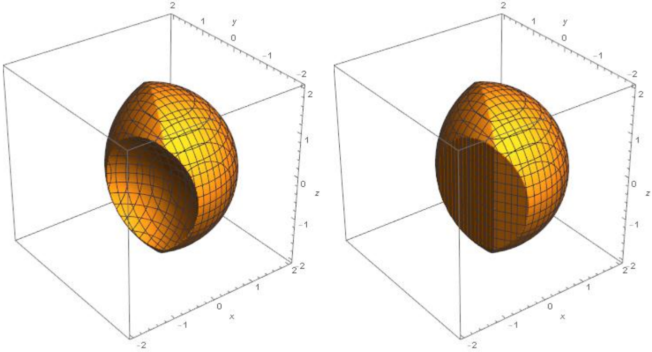

Figure: Plots of sets S (left) and $\mathrm{conv}(S)$ (right).

4

Is HHC condition necessary?

Is HHC condition necessary: No

Theorem (Separable quadratic maps)

Let $n \geq 2$ and $f_i : \mathbb{R}^n \to \mathbb{R}$ be the functions $f_i(x) = x^\top D_i x + c_i, i \in [m]$. Let $S = \{x \in \mathbb{R}^n : f_i(x) < 0, i \in [m]\}$. Assume $D_1, \ldots, D_m$ are diagonal.

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Is HHC condition necessary: No

Theorem (Separable quadratic maps)

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5

Finiteness of aggregations.

PDLC implies finiteness.

Theorem

Let $n \geq 3$ and $f_i : \mathbb{R}^n \to \mathbb{R}$ be the functions $f_i(x) = [x^\top 1] Q_i \begin{bmatrix} x \\ 1 \end{bmatrix}$. Let $S = \{x \in \mathbb{R}^n : f_i(x) < 0, i \in [m]\}$. Assume:

- *(Standard, HHC) $S \neq \emptyset$ and $\mathrm{conv}(S) \neq \mathbb{R}^n$ and HHC holds for the associated homogeneous quadratic map f^h.*
- *(PDLC - for every subset of cardinality 3) Assume for all distinct $i, j, k \in [m]$ there exist scalars $p_{ijk}, q_{ijk}, r_{ijk} \in \mathbb{R}$ such that $p_{ijk} Q_i + q_{ijk} Q_j + r_{ijk} Q_k \succ 0$.*

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Let $n \geq 3$ *and* $f_i : \mathbb{R}^n \to \mathbb{R}$ *be the functions* $f_i(x) = [x^\top 1] Q_i \begin{bmatrix} x \\ 1 \end{bmatrix}$. *Let* $S = \{x \in \mathbb{R}^n : f_i(x) < 0, i \in [m]\}$. *Assume:*

- *(Standard, HHC)* $S \neq \emptyset$ *and* $\mathrm{conv}(S) \neq \mathbb{R}^n$ *and HHC holds for the associated homogeneous quadratic map* f^h.
- *(PDLC - for every subset of cardinality* 3*)* *Assume for all distinct* $i, j, k \in [m]$ *there exist scalars* $p_{ijk}, q_{ijk}, r_{ijk} \in \mathbb{R}$ *such that* $p_{ijk} Q_i + q_{ijk} Q_j + r_{ijk} Q_k \succ 0$.

Then there exist $\lambda^{(1)}, \ldots, \lambda^{(r)} \in \Omega_2$ *such that*

$$\mathrm{conv}(S) = \bigcap_{i=1}^{r} S_{\lambda^{(i)}},$$

where $\Omega_2 = \{\lambda \in \Omega_1 : |\{i : \lambda_i > 0\}| \leq 2\}$ *and* $r \leq m^2 - m$.

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Let $n \geq 3$ *and* $f_i : \mathbb{R}^n \to \mathbb{R}$ *be the functions* $f_i(x) = [x^\top 1]Q_i \begin{bmatrix} x \\ 1 \end{bmatrix}$. *Let* $S = \{x \in \mathbb{R}^n : f_i(x) < 0, i \in [m]\}$. *Assume:*

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Then there exist $\lambda^{(1)}, \ldots, \lambda^{(r)} \in \Omega_2$ *such that*

$$\mathrm{conv}(S) = \bigcap_{i=1}^{r} S_{\lambda^{(i)}},$$

where $\Omega_2 = \{\lambda \in \Omega_1 : |\{i : \lambda_i > 0\}| \leq 2\}$ *and* $r \leq m^2 - m$. *Moreover:*

- *Given any* $u, v \in [m]$, $u \neq v$, *there are at most two* $\lambda^{(i)}$*s with support* u, v.
- *These* $\lambda^{(i)}$*s can be written as* $\alpha' e_u + (1 - \alpha')e_v, \alpha'' e_u + (1 - \alpha'')e_v$, *where* α', α'' *are roots of* $\det(\alpha Q_u + (1 - \alpha)Q_v) = 0$.

Second order cone representable

Corollary

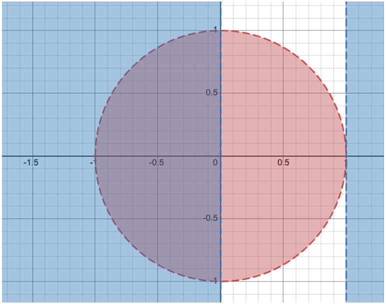
Let $n \geq 3$ *and* $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ *be the functions* $f_i(x) = [x^\top 1]Q_i \begin{bmatrix} x \\ 1 \end{bmatrix}$. *Let* $S = \{x \in \mathbb{R}^n : f_i(x) < 0, i \in [m]\}$. *Assume:*

- *(Standard, HHC)* $S \neq \emptyset$ *and* $\text{conv}(S) \neq \mathbb{R}^n$ *and HHC holds for the associated homogeneous quadratic map* f^h.
- *(PDLC - for every subset of cardinality* 3*)* *Assume for all distinct* $i, j, k \in [m]$ *there exist scalars* $p_{ijk}, q_{ijk}, r_{ijk} \in \mathbb{R}$ *such that* $p_{ijk}Q_i + q_{ijk}Q_j + r_{ijk}Q_k \succ 0$.

Then $\text{conv}(S)$ *is interior of an SOCP-representable set.*

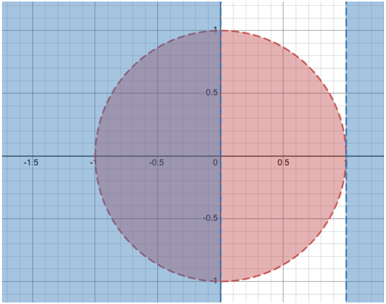
5
The closed case.

Example of “differences" with open case.



$$S := \left\{ (x,y,z) \;\middle|\; \begin{array}{cccc} -x^2 + x & < & 0 & \spadesuit \\ x^2 + y^2 & < & 1 & \clubsuit \end{array} \right\} = \mathrm{conv}(S)$$

Example of “differences" with open case.



$$S := \left\{ (x, y, z) \;\middle|\; \begin{array}{llll} -x^2 + x & < & 0 & \spadesuit \\ x^2 + y^2 & < & 1 & \clubsuit \end{array} \right\} = \text{conv}(S)$$

$$T := \left\{ (x, y, z) \;\middle|\; \begin{array}{llll} -x^2 + x & \leq & 0 & \spadesuit \\ x^2 + y^2 & \leq & 1 & \clubsuit \end{array} \right\} \neq \text{conv}(T)$$

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Closed sets

Theorem

Let $n \geq 3$ and $f_i : \mathbb{R}^n \to \mathbb{R}$ be the functions $f_i(x) = [x^\top 1] Q_i \begin{bmatrix} x \\ 1 \end{bmatrix}$ for $i \in [m]$. Let $S = \{x \in \mathbb{R}^n : f_i(x) < 0, i \in [m]\}$ and let $T = \{x : f_i(x) \leq 0, i \in [m]\}$ and $G = \text{int}(\overline{\text{conv}(T)})$.

Closed sets

Theorem

Let $n \geq 3$ and $f_i : \mathbb{R}^n \to \mathbb{R}$ be the functions $f_i(x) = [x^\top 1]Q_i \begin{bmatrix} x \\ 1 \end{bmatrix}$ for $i \in [m]$. Let $S = \{x \in \mathbb{R}^n : f_i(x) < 0, i \in [m]\}$ and let $T = \{x : f_i(x) \leq 0, i \in [m]\}$ and $G = \text{int}(\overline{\text{conv}(T)})$. Assume $Q_1, \ldots, Q_m$ satisfy:

- *hidden hyperplane convexity*
- *$\emptyset \subsetneq G \subsetneq \mathbb{R}^n$, and furthermore,*
- *$Q_\lambda \neq 0$ for all nonzero $\lambda \geq 0$.*

Closed sets

Theorem

Let $n \geq 3$ *and* $f_i : \mathbb{R}^n \to \mathbb{R}$ *be the functions* $f_i(x) = [x^\top 1] Q_i \begin{bmatrix} x \\ 1 \end{bmatrix}$ *for* $i \in [m]$*. Let* $S = \{x \in \mathbb{R}^n : f_i(x) < 0, i \in [m]\}$ *and let* $T = \{x : f_i(x) \leq 0, i \in [m]\}$ *and* $G = \text{int}(\text{conv}(T))$*. Assume* $Q_1, \ldots, Q_m$ *satisfy:*

- *hidden hyperplane convexity*
- $\emptyset \subsetneq G \subsetneq \mathbb{R}^n$*, and furthermore,*
- $Q_\lambda \neq 0$ *for all nonzero* $\lambda \geq 0$*.*

Then

$$G = \bigcap_{\lambda \in \Omega_T} S_\lambda,$$

where $S_\lambda = \{x : \sum_{i=1}^m \lambda_i f_i(x) < 0\}$ *and* $\Omega_T \subseteq \mathbb{R}^m_+ \setminus \{0\}$ *is the set of* λ *where* $Q_\lambda = \sum_{i=1}^m \lambda_i Q_i$ *has at most one negative eigenvalue and* $G \subseteq S_\lambda$*.*

Example continued.

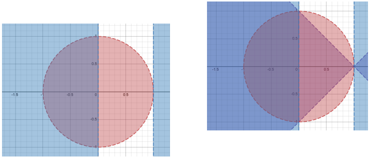

Figure: Plots of sets S (left) and $\text{conv}(G)$ (right).

- $$S := \left\{ (x,y,z) \;\middle|\; \begin{array}{ccc} -x^2 + x & < & 0 & \spadesuit \\ x^2 + y^2 & < & 1 & \clubsuit \end{array} \right\} = \text{conv}(S)$$
- HHC holds, $\emptyset \subsetneq G \subsetneq \mathbb{R}^n$, $Q_\lambda \neq 0$ for all nonzero $\lambda \geq 0$
- $$\text{conv}(G) := \left\{ (x,y,z) \;\middle|\; \begin{array}{cccc} x^2 + y^2 & < & 1 & \clubsuit \\ -x^2 + y^2 + 2x & < & 1 & 2 \cdot \spadesuit + \clubsuit \end{array} \right\}$$

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Discussion

Classify: conv.hull of QCQP substructure is SOCr?

Is SOCP representable:

1. One quadratic constraint $\cap$ polytope [Santana, D. (2020)]
2. Two quadratic inequalities (Bienstock, Michalka[2014], Burer, Klinc-Karzan [2017], Modaresi, Vielma [2017])
3. HHC satisfying quadratic inequalities under PDLC condition (including the special case of three quadratic constraints satisfying PDLC)

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3. HHC satisfying quadratic inequalities under PDLC condition (including the special case of three quadratic constraints satisfying PDLC)

Is not SOCP representable:

1. Already in 10 variables, 5 quadratic equalities, 4 quadratic inequalities, 3 linear inequalities (Fawzi [2018])

Thank You