

Branching with a pre-specified finite list of k -sparse split sets for binary MILPs

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Abstract

When branching for binary mixed integer linear programs with disjunctions of sparsity level 2, we observe that there exists a finite list of 2-sparse disjunctions, such that any other 2-sparse disjunction is dominated by one disjunction in this finite list. For sparsity level greater than 2, we show that a finite list of disjunctions with this property cannot exist. This leads to the definition of covering number for a list of splits disjunctions. Given a finite list of split sets \mathcal{F} of k -sparsity, and a given k -sparse split set S , let $\mathcal{F}(S)$ be the minimum number of split sets from the list \mathcal{F} , whose union contains $S \cap [0, 1]^n$. Let the covering number of \mathcal{F} be the maximum value of $\mathcal{F}(S)$ over all k -sparse split sets S . We show that the covering number for any finite list of k -sparse split sets is at least $\lfloor k/2 \rfloor$ for $k \geq 4$. We also show that the covering number of the family of k -sparse split sets with coefficients in $\{-1, 0, 1\}$ is upper bounded by $k - 1$ for $k \leq 4$.

1 Introduction

Land and Doig [19] invented the branch-and-bound procedure to solve mixed integer linear programs (MILP). Today, all state-of-the-art MILP solvers use the branch-and-bound procedure at its core. An important decision in formalizing a branch-and-bound algorithm is to decide the method to partition the feasible region of the linear program corresponding to a node in the branch-and-bound tree. Given $\pi \in \mathbb{Z}^n$ and $\eta \in \mathbb{Z}$, a general way to partition a feasible region where all variables are binary is to use the following disjunction for $x \in \{0, 1\}^n$: $(\pi^\top x \leq \eta) \vee (\pi^\top x \geq \eta + 1)$, in order to create two child nodes. The open set

$$S(\pi, \eta) := \{x \in \mathbb{R}^n \mid \eta < \pi^\top x < \eta + 1\},$$

is called *split set* and the associated disjunction is called *split disjunction*. We say a split set $S(\pi, \eta)$ is k -sparse if the number of non-zero entries of π , denoted by $\|\pi\|_0$, is at most k , that is, $\|\pi\|_0 \leq k$.

Most state-of-the-art MILP solvers are based on branch-and-bound trees built using 1-sparse split disjunctions; such branch-and-bound trees are called simple branch-and-bound trees [11]. One rationale for using 1-sparse split disjunctions is to maintain the sparsity of linear programs solved at child nodes; see discussion in [12, 13]. Recently, [10, 5] showed that on random

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instances, using 1-sparse split disjunctions is sufficient to obtain a polynomial size branch-and-bound tree when the number of constraints are fixed. However, several papers have shown the power of constructing branch-and-bound trees with dense disjunctions. See, for example, the papers [24, 1, 20, 22, 18, 7, 21, 26, 23] which present several evidences of dramatic reduction in the number of nodes in a branch-and-bound tree when using dense disjunctions in comparison to branch-and-bound trees based on 1-sparse disjunctions. Moreover, the papers [17, 6] present examples of MILPs where every 1-sparse branching scheme leads to exponential size branch-and-bound trees, although these instances can be solved using polynomial-size branch-and-bound trees when using denser inequalities see [26, 4]. While the worst-case size of a branch-and-tree may be exponential even when using dense disjunctions [8, 11, 14, 16], the papers [25, 4] present other compelling theoretical evidence on the importance of branching using dense disjunctions.

The papers [24, 20, 22, 26] show significant improvement in the size of the branch-and-bound tree by using split disjunctions of a specified sparsity level together which having the coefficients of the associated split sets being in $\{-1, 0, 1\}$. One perspective to view this line of work, is that they explore the paradigm of expanding the list of disjunctions used to build the branch-and-bound tree, from the typically used 1-sparse disjunctions, to a finite list of pre-specified denser disjunctions. In this paper, we explore a geometric problem motivated by the use of such pre-specified finite lists of dense disjunctions to solve binary MILPs.

2 Main results

2.1 Dominance result for 2-sparse disjunctions

Consider two split sets $S(\pi^1, \eta^1)$ and $S(\pi^0, \eta^0)$ in \mathbb{R}^n . We say that $S(\pi^1, \eta^1)$ dominates $S(\pi^0, \eta^0)$ if

$$S(\pi^1, \eta^1) \cap [0, 1]^n \supseteq S(\pi^0, \eta^0) \cap [0, 1]^n. \quad (1)$$

If (1) holds, then in any branch-and-bound tree that solves a binary MILP using the disjunction corresponding to $S(\pi^0, \eta^0)$, we may replace this disjunction by the disjunction corresponding to $S(\pi^1, \eta^1)$, resulting in a branch-and-bound tree that cannot increase in size in comparison to the original branch-and-bound tree.

Let \mathcal{F}_k be the finite list of k -sparse split sets, such that $S(\pi, \eta) \in \mathcal{F}_k$ if $\pi \in \{-1, 0, 1\}^n$, $\|\pi\|_0 \leq k$, and $\eta \in \{-k, \dots, -1, 0, 1, \dots, k\}$. If we only use 1-sparse disjunctions, then clearly there are only n possible split sets in \mathcal{F}_1 , none of which dominate each other. Next let us consider the case of 2-sparse disjunctions.

Proposition 1. *Consider any arbitrary 2-sparse split set $S(\pi, \eta) \subseteq \mathbb{R}^n$, that is, $\pi \in \mathbb{Z}^n$ and $\|\pi\|_0 \leq 2$. Then there exists a split set in \mathcal{F}_2 that dominates $S(\pi, \eta)$.*

Proposition 1 shows that if one decides to branch using 2-sparse disjunctions only for solving binary MILPs, there is no reason to use general 2-sparse split disjunctions – in particular, one may restrict the use of disjunctions to the finite list described in Proposition 1. Indeed, the paper [26] shows the importance of branching using 2-sparse disjunctions by employing exactly the split sets described in Proposition 1 and shows significant improvement over sizes of tree constructed using the 1-sparse disjunctions. See Section 4 for a proof of Proposition 1.

Generalizing the result of Proposition 1, we would like to fix the level of sparsity k of the split disjunctions used to build branch-and-bound tree and ask the question: Does there exist a finite list of k -sparse disjunctions, such that it is sufficient to restrict attention to this finite list in order to get the full power of branching with k -sparse disjunctions. Unfortunately, as shown in the next result, such finite lists do not exist for k -sparse disjunctions with $k \geq 3$.

Theorem 2. *Let $k \geq 3$. There does not exist any finite list \mathcal{F} of k -sparse split sets such that any arbitrary k -sparse split set is dominated by exactly one of the split sets from \mathcal{F} .*

This negative result in the context of the use of split disjunctions in branch-and-bound is in striking contrast to the case of cutting planes computed using split sets: for any rational

polyhedral there exists a finite list of split sets such that cutting planes derived from an arbitrary split set are dominated by cutting planes derived using one split set from this finite list [2, 3, 9]. See Section 5 for a proof of Theorem 2.

2.2 Lower bound on covering number for general finite list of dense disjunctions with $k \geq 4$

Given the negative result of Theorem 2, the next natural question to ask is if there exists finite list of k -sparse split sets, such that any other arbitrary k -sparse split set is a subset of an union of a small number of split sets from the list. Formally, given split sets $\{S(\pi^i, \eta^i)\}_{i=0}^p \subseteq \mathbb{R}^n$, we say that $\{S(\pi^i, \eta^i)\}_{i=1}^p$ dominates $S(\pi^0, \eta^0)$ if:

$$\left(\bigcup_{i=1}^p S(\pi^i, \eta^i) \right) \cap [0, 1]^n \supseteq S(\pi, \eta) \cap [0, 1]^n. \quad (2)$$

If (2) holds, then in any branch-and-bound tree that solves a binary MILP using the disjunction corresponding to $S(\pi^0, \eta^0)$, we may replace this disjunction by the disjunctions corresponding to $\{S(\pi^i, \eta^i)\}_{i=1}^p$ resulting in a branch-and-bound tree whose size is no more than 2^{p-1} times the original branch-and-bound tree.

Definition 1 (Covering number for a finite list of k -sparse split sets). *Let \mathcal{F} be a finite list of k -sparse split sets. Given an arbitrary k -sparse split set S , let $\mathcal{F}(S)$ be the smallest number of split sets from \mathcal{F} that dominates S . We define the covering number of \mathcal{F} , denoted as $C(\mathcal{F})$, as:*

$$C(\mathcal{F}) := \max\{\mathcal{F}(S) \mid S \text{ is a } k\text{-sparse split set}\}.$$

If one can show that a finite list of k -sparse disjunctions has a small covering number, then it could be considered a theoretical justification for using just this finite list of pre-specified k -sparse disjunctions instead of general k -sparse disjunctions.

The covering number of \mathcal{F}_1 is k , since, for example, in order to dominate the split set $\{x \in \mathbb{R}^k \mid k-1 < \sum_{i=1}^k x_i < k\}$ we require all the k disjunctions $0 < x_i < 1$ for $i \in \{1, \dots, k\}$. Unfortunately, the next result indicates that it is not possible to find a finite list of disjunctions with significantly smaller covering number.

Theorem 3. *Let \mathcal{F} be any finite list of k -sparse split sets. Then $C(\mathcal{F}) \geq \lfloor \frac{k}{2} \rfloor$.*

See Section 6 for a proof of Theorem 3.

2.3 Covering number of $\{-1, 0, 1\}$ -disjunctions

Finally, since a number of papers have successfully employed the very natural list of disjunctions with coefficients only in $\{-1, 0, 1\}$, we explore the covering number of such finite list of disjunctions for sparsity level less or equal than 4.

Proposition 4. *For $k = 2, 3, 4$ we have that $C(\mathcal{F}_k) \leq k - 1$.*

See Section 7 for a proof of Proposition 4.

3 Conclusions

The results of this paper justify the use of pre-specified list of disjunctions with coefficients in $\{-1, 0, 1\}$ for low levels of sparsity. For $k = 2$, Proposition 1 provides this justification. For a branch-and-bound tree using 3-sparse disjunctions, Theorem 3 and Proposition 4 imply that any finite list has a covering number of at least 2 and \mathcal{F}_3 also has a covering number of 2. Thus with respect to covering number, it is optimal to limit the use of disjunctions from \mathcal{F}_3 . It is an open question if \mathcal{F}_k is optimal for higher values of k with respect to covering number. In order

to answer this question, results of both Theorem 3 and Proposition 4 may need to be tightened and generalized.

More generally, Theorem 3 may also be an indication that the use of pre-specified list of disjunctions may not be the best way to generate small branch-and-bound trees. While using disjunctions in \mathcal{F}_k already produces smaller branch-and-bound trees than those produced using 1-sparse disjunctions [24, 22, 15, 26], in order to truly obtain significantly smaller branch-and-bound trees, one may need to further develop and expand on methods to select problem-specific dense disjunctions that are not pre-specified [1, 20, 18, 7, 21, 26, 23].

4 Proof of Proposition 1

In order to prove Proposition 1 ($k = 2$) and Proposition 4 ($k = 3, 4$) in Section 7, we have to show that for any given arbitrary split set $S = \{x \in \mathbb{R}^k \mid \eta < \pi^\top x < \eta + 1\}$ at most $k - 1$ split sets from \mathcal{F}_k are needed to dominate it. Without loss of generality, we may assume that $0 \leq \pi_1 \leq \pi_2 \leq \dots \leq \pi_k$. This is because, if $\pi_i < 0$ we can change x_i to $1 - x_i$, and then permute the order of the variables. Note that this is fine because \mathcal{F}_k is closed under taking the same operations.

Proof for $k = 2$. We assume $\|\pi\|_0 = 2$, since otherwise the result is trivial. Let $x \in S \cap [0, 1]^2$. We consider the following cases.

- $0 \leq \eta < \eta + 1 \leq \pi_1$: For $x \in S$, we have $x_1 + x_2 \leq x_1 + \frac{\pi_2}{\pi_1}x_2 = \frac{\pi_1 x_1 + \pi_2 x_2}{\pi_1} < \frac{\eta + 1}{\pi_1} \leq 1$. Since $(0, 0) \notin S$, we obtain $S \cap [0, 1]^2 \subseteq \{x \in [0, 1]^2 \mid 0 < x_1 + x_2 < 1\}$.
- $0 \leq \pi_1 \leq \eta$; $\eta + 1 \leq \pi_2$: We have $0 \leq \frac{\eta - \pi_1}{\pi_2} \leq \frac{\eta - \pi_1 x_1}{\pi_2} < x_2$ for $x \in S$. On the other hand, for $x \in S$ we have $x_2 \leq \frac{\pi_1}{\pi_2}x_1 + x_2 < \frac{\eta + 1}{\pi_2} \leq 1$. Thus, $S \cap [0, 1]^2 \subseteq \{x \in [0, 1]^2 \mid 0 < x_2 < 1\}$.
- $0 \leq \pi_1 \leq \eta$; $0 \leq \pi_2 \leq \eta$: We have $1 < \frac{\pi_1}{\eta}x_1 + \frac{\pi_2}{\eta}x_2 \leq x_1 + x_2$ for $x \in S$. Since $(1, 1) \notin S$, we obtain $S \cap [0, 1]^2 \subseteq \{x \in [0, 1]^2 \mid 1 < x_1 + x_2 < 2\}$.

□

5 Proof of Theorem 2

We will prove Theorem 2 for $k = 3$. A similar proof can be given for $k \geq 4$, but the result in this case is implied by Theorem 3 so we do not consider it in this section.

Proof of Theorem 2. In order to prove Theorem 2, we show that for the infinite family of split sets

$$S_\gamma = \{x \in \mathbb{R}^3 \mid \gamma < x_1 + \gamma x_2 + (\gamma + 1)x_3 < \gamma + 1\},$$

where $\gamma \in \mathbb{Z}_+$, $\gamma \geq 1$, there is no split set in \mathbb{R}^3 that contains $S_\gamma \cap [0, 1]^3$ for infinitely many values of γ . Assume for a contradiction that there exists a split set $S = \{x \in \mathbb{R}^3 \mid \eta < \pi^\top x < \eta + 1\}$, where $\pi \in (\pi_1, \pi_2, \pi_3) \in \mathbb{Z}^3$, $\eta \in \mathbb{Z}$, such that S dominates S_γ for infinitely many $\gamma \in \mathbb{Z}_+$, that is,

$$S_\gamma \cap [0, 1]^3 \subseteq S \cap [0, 1]^3 \quad \forall \gamma \in \Gamma \quad (\Rightarrow \overline{S_\gamma} \cap [0, 1]^3 \subseteq \overline{S} \cap [0, 1]^3, \quad \forall \gamma \in \Gamma), \quad (3)$$

where $\overline{S_\gamma}$ and \overline{S} are closure of S_γ and S respectively, and $\Gamma \subseteq \mathbb{Z}_+$ is an infinite set.

We first show that:

$$H^0 := \{x \in [0, 1]^3 \mid \pi^\top x \leq \eta\} \subseteq \{x \in [0, 1]^3 \mid x_1 + \gamma x_2 + (\gamma + 1)x_3 \leq \gamma\} =: H_\gamma^0, \quad (4)$$

$$H^1 := \{x \in [0, 1]^3 \mid \pi^\top x \geq \eta + 1\} \subseteq \{x \in [0, 1]^3 \mid x_1 + \gamma x_2 + (\gamma + 1)x_3 \geq \gamma + 1\} =: H_\gamma^1. \quad (5)$$

Notice that we must have $H_\gamma^j \cap \{0, 1\}^3 \neq \emptyset$ for all $j \in \{0, 1\}$; otherwise if, for instance $H_\gamma^0 \cap \{0, 1\}^3 = \emptyset$, then we would have $\{0, 1\}^3 \subseteq H_\gamma^1$ which implies $[0, 1]^3 \subseteq H_\gamma^1$, a contradiction with the fact $S_\gamma \cap [0, 1]^3 \neq \emptyset$. On the other hand, observe that S_γ being dominated by S is

equivalent to: for all $i \in \{0, 1\}$ there exists $j \in \{0, 1\}$ such that $H^i \subseteq H_\gamma^j$. Since $(H^0 \cup H^1) \cap \{0, 1\}^3 = \{0, 1\}^3$, we conclude that it cannot happen that $H^0 \subseteq H_\gamma^j$ and $H^1 \subseteq H_\gamma^j$ for the same j since $H_\gamma^j \cap \{0, 1\}^3 \neq \emptyset$ for $i \neq j$. Therefore, (4) and (5) hold (we may assume that we have $H^i \subseteq H_\gamma^i$ for $i \in \{0, 1\}$ by considering S to be defined by $\hat{\pi} = -\pi$ and $\hat{\eta} = -\eta - 1$ if necessary).

Since $(0, 1, 0)$ satisfies the equation $x_1 + \gamma x_2 + (\gamma + 1)x_3 = \gamma$ and by (4) we have $H^0 \cap \{0, 1\}^3 = H_\gamma^0 \cap \{0, 1\}^3$, we must have that $(0, 1, 0)$ satisfies the inequality $\pi^T x \leq \eta$. We now show that $(0, 1, 0)$ must satisfy $\pi^T x = \eta$. Assume for a contradiction that it satisfies $\pi^T x < \eta$. Let $x_0 \in S_\gamma \cap [0, 1]^3$ be an arbitrary point. For $\lambda > 0$ small enough we have that the point $x_\lambda = (0, 1, 0) + \lambda(x_0 - (0, 1, 0))$ satisfies $\pi^T x_\lambda < \eta$ and, by convexity of $S_\gamma \cap [0, 1]^3$, that $x_\lambda \in S_\gamma \cap [0, 1]^3$. Since $S_\gamma \cap [0, 1]^3 \subseteq S \cap [0, 1]^3$, it follows that $x_\lambda \in S \cap [0, 1]^3$, a contradiction with the fact that $\pi^T x_\lambda < \eta$. Thus, we must have that $\pi_2 = \eta$. By a similar argument, since $(1, 1, 0)$ and $(0, 0, 1)$ satisfy $x_1 + \gamma x_2 + (\gamma + 1)x_3 = \gamma + 1$, it follows from (5) that we must have that these points satisfy $\pi^T x = \eta + 1$, and therefore $\pi_1 + \pi_2 = \eta + 1$ and $\pi_3 = \eta + 1$. Therefore, we obtain that $\pi_1 = 1$, $\pi_2 = \eta$ and $\pi_3 = \eta + 1$.

Since $(1, 0, \gamma/(\gamma + 1)) \in S_\gamma \cap [0, 1]^3$, by (3) we obtain $1 + (\eta + 1)\frac{\gamma}{\gamma + 1} \leq \eta + 1 \Leftrightarrow 1 \leq \frac{\eta + 1}{\gamma + 1}$. Since this inequality holds for any $\gamma \in \Gamma$, we obtain $1 \leq 0$, a contradiction. \square

6 Proof of Theorem 3

We first prove the result when the sparsity level is an even positive integer $2k$.

Consider the following family of split sets parameterized by a positive integer θ :

$$S^\theta = \left\{ (x, y) \in \mathbb{R}^k \times \mathbb{R}^k \mid \sum_{i=1}^k \theta^i < \sum_{i=1}^k \theta^i (x_i + y_i) < 1 + \sum_{i=1}^k \theta^i \right\}.$$

In order to prove Theorem 3 it is sufficient to prove the following result:

Lemma 1. *For every finite collection of split sets \mathcal{F} , there exists $\theta \in \mathbb{Z}_+$ with $\theta \geq 1$, such that one needs at least an union of k split sets from \mathcal{F} to dominate $S^\theta \cap [0, 1]^{2k}$.*

Before presenting the proof of Lemma 1, we introduce some notation. Consider a list of split sets $A^{(i)} := \{(x, y) \mid c^{(i)} < a^{(i)}x + b^{(i)}y < c^{(i)} + 1\}$, for $i = 1, \dots, p$. For each i , we denote the two connected components of the complement set to $A^{(i)}$ by

$$A_0^{(i)} := \{(x, y) \mid a^{(i)}x + b^{(i)}y \leq c^{(i)}\} \quad \text{and} \quad A_1^{(i)} := \{(x, y) \mid a^{(i)}x + b^{(i)}y \geq c^{(i)} + 1\}.$$

Given a binary vector $u \in \{0, 1\}^p$, we further define $A_u = \bigcap_{i=1}^p A_{u_i}^{(i)}$. Note that the fact that $\bigcup_{i=1}^p A^{(i)}$ dominates S^θ can be written as:

$$S^\theta \cap [0, 1]^{2k} \subseteq \left(\bigcup_{i \in [p]} A^{(i)} \right) \cap [0, 1]^{2k} \Leftrightarrow [0, 1]^{2k} \setminus S^\theta \supseteq [0, 1]^{2k} \setminus \left(\bigcup_{i \in [p]} A^{(i)} \right).$$

So dominance of the given list of split sets is equivalent to:

$$\forall u \in \{0, 1\}^p, \text{ either } A_u \cap [0, 1]^{2k} \subseteq S_0^\theta \cap [0, 1]^{2k} \text{ or } A_u \cap [0, 1]^{2k} \subseteq S_1^\theta \cap [0, 1]^{2k}. \quad (6)$$

Now we present a proof of Lemma 1.

Proof. We argue by contradiction. Suppose one needs at most $k-1$ split sets from \mathcal{F} to dominate S^θ for all θ . Since \mathcal{F} is finite, but there are infinitely many choices of S^θ , there must exist p split sets from \mathcal{F} , where $p \leq k-1$, and an infinite set $\Theta \subseteq \mathbb{Z}_+$ such that those p split sets dominate $S^\theta \cap [0, 1]^{2k}$ for all $\theta \in \Theta$. We denote those split sets by

$$A^{(i)} := \{(x, y) \mid c^{(i)} < a^{(i)}x + b^{(i)}y < c^{(i)} + 1\}, \text{ for } i = 1, \dots, p.$$

We will show that (6) fails for sufficiently large $\theta \in \Theta$. Our main idea is to construct a certain point $z \in [0, 1]^{2k}$ such that $z \in A_u \cap [0, 1]^{2k}$ for some u but z violates (6).

Consider the following linear system:

$$a^{(i)}x + b^{(i)}y = 0 \quad \text{for } i = 1, \dots, p \quad (7)$$

$$y_i = 0 \quad \text{for } i = 1, \dots, k. \quad (8)$$

This linear system has $2k$ variables and $k + p$ constraints. Since $k + p < 2k$ it has at least one non-zero solution (x^*, y^*) . Without loss of generality, we may assume that $\|(x^*, y^*)\|_2 = 1$ and $x_j^* > 0$ where j is the largest index $i = 1, \dots, k$ such that $x_i^* \neq 0$.

By (8) and for sufficiently large $\theta \in \Theta$ we have that

$$\sum_{i=1}^k \theta^i (x_i^* + y_i^*) = \sum_{i=1}^j \theta^i x_i^* > 0. \quad (9)$$

We now construct a binary vector $(s, t) \in \{0, 1\}^{2k}$ in the following way:

$$s_i = 0, t_i = 1 \text{ if } x_i^* \geq 0 \quad \text{and} \quad s_i = 1, t_i = 0 \text{ if } x_i^* < 0.$$

Notice that since (s, t) is a integer vector, it must belong to either $A_0^{(i)}$ or $A_1^{(i)}$ for all $i = 1, \dots, p$ and therefore $(s, t) \in A_{u^*} \cap [0, 1]^{2k}$ for some u^* .

We now verify that $(s, t) + \lambda(x^*, y^*) \in A_{u^*} \cap [0, 1]^{2k}$ for some sufficiently small $\lambda > 0$. Indeed, $(s, t) + \lambda(x^*, y^*)$ stays in A_{u^*} for any $\lambda > 0$ because of (7). On the other hand, $(s, t) + \lambda(x^*, y^*)$ stays in $[0, 1]^{2k}$ for sufficiently small $\lambda > 0$ because $t_i + \lambda y_i^*$ does not change due to (8), components associated to $s_i = 1$ decrease a little and components associated to $s_i = 0$ increase a little.

Now observe that $\sum_{i \in [k]} \theta^i (s_i + t_i) = \sum_{i \in [k]} \theta^i$ and $\sum_{i \in [n]} \theta^i (x_i^* + y_i^*) > 0$ by (9), hence we obtain that

$$\sum_{i=1}^k \theta^i < \sum_{i=1}^k \theta^i (s_i + \lambda x_i^* + t_i + \lambda y_i^*) < 1 + \sum_{i=1}^k \theta^i,$$

for sufficiently small $\lambda > 0$. In other words, for sufficiently small $\lambda > 0$ and large enough $\theta \in \Theta$ we have that $(s, t) + \lambda(x^*, y^*) \in S^\theta \cap (A_{u^*} \cap [0, 1]^{2k})$. We conclude that (6) is not satisfied for the point $(s, t) + \lambda(x^*, y^*)$, a contradiction. \square

In order to prove Theorem 3 for odd sparsity levels of split disjunctions, a similar proof can be presented using the family of split sets:

$$S^\theta := \left\{ (x, y) \in \mathbb{R}^k \times \mathbb{R}^{k+1} : \sum_{i=1}^k \theta^i < \sum_{i=1}^k \theta^i (x_i + y_i) + y_{k+1} < 1 + \sum_{i=1}^k \theta^i \right\}.$$

7 Proof of Proposition 4

The case $k = 2$ is proven in Proposition 1. We now consider the cases $k = 3, 4$.

Proof for $k = 3$. Let $S = \{x \in \mathbb{R}^3 \mid \eta < \pi^\top x < \eta + 1\}$, recall that we may assume that $0 \leq \pi_1 \leq \pi_2 \leq \pi_3$ (see Section 4). We have to show that at most 2 split sets from \mathcal{F}_3 are needed to dominate it. There are three cases:

- $\pi_3 \geq \eta + 1$: In this case, observe that $x \in S$, implies that $x_3 < 1$. By Proposition 1, we know there exist 1 split set of sparsity 2 (or lesser) from \mathcal{F}_3 whose union contains the set $\{x \in [0, 1]^3 \mid x \in S, x_3 = 0\}$. The set of points in $\{x \in [0, 1]^3 \mid x \in S, 0 < x_3 < 1\}$ is contained in the split set $0 < x_3 < 1$.

- $\pi_1 + \pi_2 \leq \eta$: In this case, observe that $x \in S$, implies that $x_3 > 0$. By Proposition 1, we know there exist 1 split set of sparsity 2 (or lesser) from \mathcal{F}_3 whose union contains the set $\{x \in [0, 1]^3 \mid x \in S, x_3 = 1\}$. The set of points in $\{x \in [0, 1]^3 \mid x \in S, 0 < x_3 < 1\}$ is contained in the split set $0 < x_3 < 1$.
- $\pi_3 \leq \eta$ and $\pi_1 + \pi_2 \geq \eta + 1$: Since $\pi_1 \leq \pi_2 \leq \pi_3 \leq \eta$, if $\sum_{j=1}^3 x_j \leq 1$, then $\sum_{j=1}^3 \pi_j x_j \leq \eta$. Thus, $x \in S$ implies that $\sum_{j=1}^3 x_j > 1$. Moreover, if $\sum_{j=1}^3 x_j \geq 2$, then $\sum_{j=1}^3 \pi_j x_j \geq \pi_1 + \pi_2 \geq \eta + 1$. Thus $x \in S$ implies that $\sum_{j=1}^3 x_j < 2$. Therefore, S is dominated by the split set $\{x \in \mathbb{R}^3 \mid 1 < \sum_{j=1}^3 x_j < 2\}$.

□

Proof for $k = 4$. Let $S = \{x \in \mathbb{R}^4 \mid \eta < \pi^\top x < \eta + 1\}$ with $0 \leq \pi_1 \leq \pi_2 \leq \pi_3 \leq \pi_4$. There are ten cases:

- $\pi_4 \geq \eta + 1$: In this case, observe that $x \in S$, implies that $x_4 < 1$. By Proposition 4 for $k = 3$ case, we know there at most 2 split set of sparsity 3 (or lesser) from \mathcal{F}_4 whose union contains the set $\{x \in [0, 1]^4 \mid x \in S, x_4 = 0\}$. The set of points in $\{x \in [0, 1]^4 \mid x \in S, 0 < x_4 < 1\}$ is contained in the split set $0 < x_4 < 1$.
- $\pi_1 + \pi_2 + \pi_3 \leq \eta$: In this case, observe that $x \in S$, implies that $x_4 > 0$. By Proposition 4 for $k = 3$, we know there at most 2 split set of sparsity 3 (or lesser) from \mathcal{F}_4 whose union contains the set $\{x \in [0, 1]^4 \mid x \in S, x_4 = 1\}$. The set of points in $\{x \in [0, 1]^4 \mid x \in S, 0 < x_4 < 1\}$ is contained in the split set $0 < x_4 < 1$.
- $\pi_1 + \pi_2 \geq \eta + 1$: We may assume that $\pi_4 \leq \eta$. Thus, we have $x \in S$ implies $1 < x_1 + x_2 + x_3 + x_4$. On the other hand, we also must have $x_1 + x_2 + x_3 + x_4 < 2$, since otherwise, $\sum_{j=1}^4 \pi_j x_j \geq \pi_1 + \pi_2 \geq \eta + 1$. Thus $S \cap [0, 1]^4$ is contained in $1 < x_1 + x_2 + x_3 + x_4 < 2$.
- $\pi_1 + \pi_3 \geq \eta + 1$: We may assume $\pi_4 \leq \eta$ and $\pi_1 + \pi_2 \leq \eta$. We claim that S is contained in the union of $1 < x_1 + x_2 + x_3 + x_4 < 2$ and $0 < x_3 + x_4 < 1$. Consider the following cases for $x \in S$:
 - If $x_1 + x_2 \leq 1$: We claim that that $x_1 + x_2 + x_3 + x_4 < 2$. Assume by contradiction $x_1 + x_2 + x_3 + x_4 \geq 2$. Then we have $x_3 + x_4 \geq 1$ and thus $\sum_{j=1}^4 \pi_j x_j \geq \pi_1 \cdot \min\{1, 2 - x_3 - x_4\} + \pi_3 \cdot \max\{1, x_3 + x_4\} \geq \pi_1 + \pi_3 \geq \eta + 1$. On the other hand, since $\pi_4 \leq \eta$, we have $1 < x_1 + x_2 + x_3 + x_4$. Thus, in this case x belongs to $1 < x_1 + x_2 + x_3 + x_4 < 2$.
 - If $x_1 + x_2 > 1$: Then note that $x_3 + x_4 < 1$, since otherwise $\sum_{j=1}^4 \pi_j x_j \geq \pi_1 + \pi_3 \geq \eta + 1$. Also note that if $x_3 + x_4 = 0$, then $\sum_{j=1}^4 \pi_j x_j \leq \pi_1 + \pi_2 \leq \eta$. Thus, in this case we have that x belongs $0 < x_3 + x_4 < 1$.
- $\pi_3 + \pi_4 \leq \eta$: We assume that $\pi_1 + \pi_2 + \pi_3 \geq \eta + 1$. First, note that S is contained in $x_1 + x_2 + x_3 + x_4 > 2$. Also, since $\pi_1 + \pi_2 + \pi_3 \geq \eta + 1$, we have that S is contained in $x_1 + x_2 + x_3 + x_4 < 3$. Thus, S is contained in $2 < x_1 + x_2 + x_3 + x_4 < 3$.
- $\pi_2 + \pi_4 \leq \eta$: We may assume $\pi_1 + \pi_2 + \pi_3 \geq \eta + 1$ and $\pi_3 + \pi_4 \geq \eta + 1$. We claim that S is contained in the union of $2 < x_1 + x_2 + x_3 + x_4 < 3$ and $1 < x_3 + x_4 < 2$. Consider the following cases for $x \in S$:
 - If $x_3 + x_4 \leq 1$: We claim that that $x_1 + x_2 + x_3 + x_4 > 2$. Assume by contradiction $x_1 + x_2 + x_3 + x_4 \leq 2$. Thus, $\sum_{j=1}^4 \pi_j x_j \leq \pi_2 \cdot \max\{1, 2 - x_3 - x_4\} + \pi_4 \cdot \min\{1, x_3 + x_4\} \leq \pi_2 + \pi_4 \leq \eta$. On the other hand, since $\pi_1 + \pi_2 + \pi_3 \geq \eta + 1$, we have $3 > x_1 + x_2 + x_3 + x_4$. Thus, in this case x belongs to $2 < x_1 + x_2 + x_3 + x_4 < 3$.
 - Also note that $x_3 + x_4 = 2$ is not possible, since then $\sum_j \pi_j x_j \geq \pi_3 + \pi_4 \geq \eta + 1$.

Thus, $S \cap [0, 1]^4$ is contained in the union of $2 < x_1 + x_2 + x_3 + x_4 < 3$ and $1 < x_3 + x_4 < 2$.

- $\pi_2 + \pi_3 \geq \eta + 1$ and $\pi_1 + \pi_4 \geq \eta + 1$: We may assume $\pi_4 \leq \eta$ and $\pi_1 + \pi_3 \leq \eta$. We claim that S is contained in the union of $1 < x_2 + x_3 + x_4 < 2$, $0 < x_1 < 1$ and $0 < x_4 < 1$. Consider the following cases:

- $x_1 = 0$: In this case, note that because of $\pi_2 + \pi_3 \geq \eta + 1$ and $\pi_4 \leq \eta$, we have that x belongs to $1 < x_2 + x_3 + x_4 < 2$.
- $0 < x_1 < 1$: In this case, note that x belongs to $0 < x_1 < 1$.
- $x_1 = 1$: Clearly, due to $\pi_2 + \pi_3 \geq \eta + 1$ we have that $x_2 + x_3 + x_4 < 2$. If $1 < x_2 + x_3 + x_4$, then x belongs to $1 < x_2 + x_3 + x_4 < 2$.
Otherwise suppose, $x_2 + x_3 + x_4 \leq 1$. We claim that $0 < x_4 < 1$. By contradiction, if $x_4 = 0$, then note that $\sum_j \pi_j x_j \leq \pi_1 + \pi_3 \leq \eta$. If $x_4 = 1$, then note that $\sum_j \pi_j x_j \geq \pi_1 + \pi_4 \geq \eta + 1$.
- $\pi_2 + \pi_3 \leq \eta$ and $\pi_1 + \pi_4 \leq \eta$: We may assume $\pi_1 + \pi_2 + \pi_3 \geq \eta + 1$ and $\pi_2 + \pi_4 \geq \eta + 1$. We claim that S is contained in the union of $0 < x_2 + x_3 < 1$, $2 < x_1 + x_2 + x_3 < 3$ and $0 < x_4 < 1$. Consider the following cases:
 - $x_4 = 0$: In case, note that due to $\pi_1 + \pi_2 + \pi_3 \geq \eta + 1$, we have that $x_1 + x_2 + x_3 < 3$. Also, since $\pi_2 + \pi_3 \leq \eta$, we have that $x_1 + x_2 + x_3 > 2$. Thus, x is contained in $2 < x_1 + x_2 + x_3 < 3$.
 - $0 < x_4 < 1$: In this case, note that x belongs to $0 < x_4 < 1$.
 - $x_4 = 1$: In this case note that $x_2 + x_3 \geq 1$ is not possible, since $\sum_j \pi_j x_j \geq \pi_2 + \pi_4 \geq \eta + 1$. Also note that $x_2 + x_3 = 0$ is not possible, since that $\sum_j \pi_j x_j \leq \pi_1 + \pi_4 \leq \eta$. Thus, in this case, x belongs to $0 < x_2 + x_3 < 1$.
- $\pi_2 + \pi_3 \geq \eta + 1$ and $\pi_1 + \pi_4 \leq \eta$: We may assume $\pi_1 + \pi_2 + \pi_3 \geq \eta + 1$, $\pi_1 + \pi_3 \leq \eta$ and $\pi_4 \leq \eta$. We claim that S is contained in the union of $1 < x_2 + x_3 + x_4 < 2$, $0 < x_1 < 1$, and $0 < x_2 + x_3 < 1$. Consider the following cases:
 - $x_1 = 0$: In this case, note that because of $\pi_2 + \pi_3 \geq \eta + 1$ and $\pi_4 \leq \eta$, we have that x belongs to $1 < x_2 + x_3 + x_4 < 2$.
 - $0 < x_1 < 1$: In this case, note that x belongs to $0 < x_1 < 1$.
 - $x_1 = 1$: Clearly, due to $\pi_2 + \pi_3 \geq \eta + 1$ we have that $x_2 + x_3 + x_4 < 2$. If $1 < x_2 + x_3 + x_4$, then x belongs to $1 < x_2 + x_3 + x_4 < 2$.
Otherwise suppose, $x_2 + x_3 + x_4 \leq 1$. In this case, note that $x_2 + x_3 = 1$ is not possible, since that $x_4 = 0$ and we have $\sum_j \pi_j x_j \leq \pi_1 + \pi_3 \leq \eta$. Also note that $x_2 + x_3 = 0$ is not possible, since that $\sum_j \pi_j x_j \leq \pi_1 + \pi_4 \leq \eta$. Thus, in this case, x belongs to $0 < x_2 + x_3 < 1$.
- $\pi_2 + \pi_3 \leq \eta$ and $\pi_1 + \pi_4 \geq \eta + 1$: We may assume $\pi_1 + \pi_2 + \pi_3 \geq \eta + 1$ and $\pi_4 \leq \eta$. We claim that S is contained in the union of $1 < x_1 + x_2 + x_3 + x_4 < 2$, $2 < x_1 + x_2 + x_3 < 3$ and $0 < x_4 < 1$. Consider the following cases:
 - $x_4 = 0$: In case, note that due to $\pi_1 + \pi_2 + \pi_3 \geq \eta + 1$, we have that $x_1 + x_2 + x_3 < 3$. Also, since $\pi_2 + \pi_3 \leq \eta$, we have that $x_1 + x_2 + x_3 > 2$. Thus, x is contained in $2 < x_1 + x_2 + x_3 < 3$.
 - $0 < x_4 < 1$: In this case, note that x belongs to $0 < x_4 < 1$.
 - $x_4 = 1$: In this case note that since $\pi_1 + \pi_4 \geq \eta + 1$, we have that $x_1 + x_2 + x_3 + x_4 < 2$. Also note that since $\pi_4 \leq \eta$, we have that $x_1 + x_2 + x_3 + x_4 > 1$. Thus, in this case, x belongs to $1 < x_1 + x_2 + x_3 + x_4 < 2$.

□

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