Sequential pairing of mixed integer inequalities

Yongpei Guan, Shabbir Ahmed, George L. Nemhauser
School of Industrial & Systems Engineering,
Georgia Institute of Technology, 765 Ferst Drive, Atlanta, GA 30332.

December 22, 2004

Abstract

We present a scheme for generating new valid inequalities for mixed integer programs by taking pair-wise combinations of existing valid inequalities. Our scheme is related to mixed integer rounding and mixing. The scheme is in general sequence-dependent and therefore leads to an exponential number of inequalities. For some important cases, we identify combination sequences that lead to a manageable set of non-dominated inequalities. We illustrate the framework for some deterministic and stochastic integer programs and we present computational results which show the efficiency of adding the new generated inequalities as cuts.

1 Introduction

We develop a scheme for generating new valid inequalities for mixed integer programs by taking pair-wise combinations of existing valid inequalities. Our scheme is related to the mixed integer rounding (MIR) procedure of Nemhauser and Wolsey [7, 8] and the mixing procedure of Günluk and Pochet [5]. We derive new inequalities iteratively by a very simple combination of two inequalities at a time, which we call *pairing*. As will be seen, the order in which the inequalities are paired is important since the resulting new inequalities depend on the order.

We describe the pairing procedure for pure integer programs and present a simple extension to MIPs in the next section. We study two structures in Sections 3 and 4 for which our pairing procedure gives nice results. We say that a set of inequalities is *nested* if component by component the coefficients in each successive inequality are no smaller than the coefficients in the previous inequalities. In the nested case, we show that there is a unique order for combining the inequalities that gives all of the nondominated inequalities that can be generated by the procedure. In this case, we obtain only a small number of inequalities and separation is fast. Moreover, we provide sufficient conditions for which the resulting inequalities are facet-defining. We say that a set of inequalities is *disjoint* if each integer variable appears in only one of the inequalities. Such disjoint sets arise in two-stage stochastic integer programming. Here we are again able to characterize the nondominated inequalities generated by the procedure, and we give a polynomial time separation algorithm. We also provide sufficient facet-defining conditions.

Section 5 focuses on some applications of our procedure. In Section 6, we present computational results for nested and disjoint sets to demonstrate the strength of the inequalities in improving linear programming relaxation bounds. Final remarks are presented in Section 7.

2 The pairing scheme

Given a set of non-negative integer vectors $X \subset \mathbb{Z}^n_+$, a vector $a \in \mathbb{R}^{n+1}$ *defines* a valid inequality for X if

$$\sum_{j=1}^{n} a_j x_j - a_{n+1} \geq 0 \quad \text{for all } x \in X.$$

Given two such valid inequalities defined by vectors a and b, the one defined by a *dominates* the one defined by b if $a_j \leq b_j$ for all $j = 1, \ldots, n$ and $a_{n+1} \geq b_{n+1}$. We write $a \succeq b$.

The inequality $a \le b$ for two vectors a and b of the same dimension is meant to hold component-wise. Similarly, $\min(a,b)$ and $\max(a,b)$ is understood to be carried out component-wise. For brevity, given a vector a and a scalar γ, we define $a+\gamma = a+\gamma\mathbb{1}$ and $\min\{a,\gamma\} = \min\{a,\gamma\mathbb{1}\}$, where $\mathbb{1}$ is a vector of ones of the same dimension as a.

Definition 1 *Given $a, b \in \mathbb{R}^{n+1}$ with $b_{n+1} \ge a_{n+1}$, we define the pairing of a and b as*

$$a \circ b = \min\{a + b_{n+1} - a_{n+1}, \max(a,b)\},$$

i.e., $(a \circ b)_{n+1} = b_{n+1}$ and

$$(a \circ b)_j \quad = \quad \begin{cases} a_j & \text{if } a_j \ge b_j \\ b_j & \text{if } a_j \le b_j,\ b_j \le a_j + b_{n+1} - a_{n+1} \\ a_j + b_{n+1} - a_{n+1} & \text{if } a_j \le b_j,\ b_j \ge a_j + b_{n+1} - a_{n+1}, \end{cases}$$

for all $j = 1, \dots, n$.

Theorem 1 *If $a, b \in \mathbb{R}^{n+1}$ define two valid inequalities for X, then $a \circ b$ defines a valid inequality for X.*

Proof: Without loss of generality, assume that $b_{n+1} \ge a_{n+1}$. Thus $(a \circ b)_{n+1} = b_{n+1}$. Then, given $x \in X$, we need to show that

$$\sum_{j=1}^{n} (a \circ b)_j x_j \ge b_{n+1}. \tag{1}$$

Let $J = \{j \in \{1, \dots, n\} : a_j + b_{n+1} - a_{n+1} < \max(a_j, b_j)\}$ and $\overline{J} = \{1, \dots, n\} \setminus J$. Then the left-hand side of (1) can be written as

$$\sum_{j \in J} a_j x_j + \sum_{j \in \overline{J}} \max(a_j, b_j) x_j + (b_{n+1} - a_{n+1}) \sum_{j \in J} x_j. \tag{2}$$

If there exists $j^* \in J$ such that $x_{j^*} \ge 1$, then (2) is

$$\ge \sum_{j=1}^{n} a_j x_j + (b_{n+1} - a_{n+1}) \ge b_{n+1},$$

where the last inequality follows from the validity of the inequality defined by a. On the other hand, if $x_j = 0$ for all $j \in J$, then (2) is

$$\ge \sum_{j \in J} b_j x_j + \sum_{j \in \overline{J}} b_j x_j \ge b_{n+1},$$

where the last inequality follows from the validity of the inequality defined by b. Thus $a \circ b$ defines a valid inequality for X. □

In addition to the above simple and direct proof of Theorem 1, there is an alternate proof that uses the MIR procedure, and a third proof, for the case of nonnegative coefficients, that follows from Günluk and Pochet mixing.

Example. Consider the set

$$X = \left\{ x \in \mathbb{Z}^3_+ : 3x_1 + 5x_2 \ge 3,\ 5x_2 + 4x_3 \ge 5 \right\}.$$

The two original inequalities for X are defined by $a = (3, 5, 0, 3)$ and $b = (0, 5, 4, 5)$. The valid inequality defined by $a \circ b$ is

$$3x_1 + 5x_2 + 2x_3 \ge 5. \tag{3}$$

To see that (3) can be useful, note that it cuts off the fractional point $(0, 3/5, 1/2)$ which is feasible to the LP relaxation of X.

The pairing scheme can be easily applied to mixed-integer sets. The pair $(a, g) \in \mathbb{R}^{n+1} \times \mathbb{R}^p$, defines a valid inequality for a mixed-integer set $Y \subset \mathbb{Z}^n_+ \times \mathbb{R}^p_+$ if

$$\sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{p} g_j y_j \geq a_{n+1} \quad \text{for all } (x, y) \in Y.$$

Corollary 1.1 *If (a^1, g^1) and (a^2, g^2) define two valid inequalities for Y, then $(a^1 \circ a^2, \max\{g^1, g^2\})$ defines a valid inequality for Y.*

Note that the standard disjunctive inequality (see, e.g. [7]), obtained from the inequalities (a^1, g^1) and (a^2, g^2) for Y,

$$\sum_{i=1}^{n} \max\{a_i^1, a_i^2\} x_i + \sum_{j=1}^{p} \max\{g_j^1, g_j^2\} y_j \geq \min\{a_{n+1}^1, a_{n+1}^2\},$$

is dominated by the pairing inequality in Corollary 1.

We now consider the pairing inequalities obtained from a set of inequalities. Suppose we have K valid inequalities for X defined by the vectors $\{a^1, \dots, a^K\} \subset \mathbb{R}^{n+1}$. Given a subset of these K vectors, we can obtain new valid inequalities by carrying out a sequence of pairing operations. For example, the valid inequality defined by the vector $((a^{k_1} \circ a^{k_2}) \circ (a^{k_2} \circ a^{k_3})) \circ a^{k_4}$ is obtained from $\{a^{k_1}, a^{k_2}, a^{k_3}, a^{k_4}\}$ with the parentheses distinguishing the sequence in which the pairings are carried out. Since the $\circ$ operation is not associative, the valid inequalities obtained from a given set of vectors *depends on the sequence* in which the pairings are done. Thus from the set of K valid inequalities defined by $\{a^1, \dots, a^K\}$ we can generate an exponential number of inequalities depending on the subset of valid inequalities chosen and the sequence in which they are mixed. A key problem is to identify pairing sequences that lead to good sets of valid inequalities, i.e., strong inequalities over which separation can be done efficiently.

In the following two sections, we investigate a pairing sequence that leads to two such families of inequalities. This pairing sequence is defined by

Definition 2 *Given a finite set of vectors, i.e., $A = \{a^1, \dots, a^K\}$, where $a_{n+1}^1 \leq a_{n+1}^2 \leq \dots \leq a_{n+1}^K$, we define sequential pairing of the vectors in A by*

$$\Delta(A) = ((\dots((a^1 \circ a^2) \circ a^3) \circ \dots) \circ a^K).$$

3 The nested case

Consider a set $A = \{a^1, \dots, a^K\} \subset \mathbb{R}^{n+1}$ such that $a^1 \leq \dots \leq a^K$. We say that the valid inequalities defined by the vectors in A are (or the set A itself is) *nested*. Here we consider mixed integer systems where the coefficients of the integer variables are nested. Nested sets arise, for example, in the dynamic knapsack problem considered by Loparic, Marchand and Wolsey [6] where the feasible region is given by

$$X = \Big\{(x, y) \in \{0,1\}^n \times \mathbb{R}_+ : \quad \sum_{j=1}^{i} a_j x_j + y \geq \sum_{j=1}^{i} d_j, \quad i = 1, \dots, n\Big\}, \tag{4}$$

with $a \in \mathbb{R}^n_+$ and $d \in \mathbb{R}^n_+$. Here, y is a continuous inventory variable, $x_j \in \{0, 1\}$ represents whether the amount a_j is produced in period j, and d_j is the demand in period j.

Let $A_k = \{a^1, \dots, a^k\}$ for $k = 1, \dots, K$, and let $\Phi(A) \in \mathbb{R}^{n+1}$ be a vector obtained by an arbitrary sequence of pairings of the vectors in A. Next, we show that $\Delta(A) \succeq \Phi(A)$.

Theorem 2 *If $A = \{a^1, \dots, a^K\}$ is nested, then*

$$\Delta(A) = \min\{a^1 + a_{n+1}^K - a_{n+1}^1, a^2 + a_{n+1}^K - a_{n+1}^2, \dots, a^{K-1} + a_{n+1}^K - a_{n+1}^{K-1}, a^K\}.$$

Proof: The proof is by induction. For $K = 2$, we have $a^1 \leq a^2$, then

$$\begin{aligned} \Delta(A_2) &= \min\{a^1 + a_{n+1}^2 - a_{n+1}^1, \max\{a^1, a^2\}\} \\ &= \min\{a^1 + a_{n+1}^2 - a_{n+1}^1, a^2\}. \end{aligned}$$

Assume that the claim holds for $K = k$, i.e., $\Delta(A_k) = \min\{a^1 + a^k_{n+1} - a^1_{n+1}, a^2 + a^k_{n+1} - a^2_{n+1}, \ldots, a^{k-1} + a^k_{n+1} - a^{k-1}_{n+1}, a^k\}$. Then

$$\begin{aligned}
\Delta(A_{k+1}) &= \Delta(A_k) \circ a^{k+1} \\
&= \min\{\Delta(A_k) + a^{k+1}_{n+1} - a^k_{n+1}, \max\{\Delta(A_k), a^{k+1}\}\} \\
&= \min\{\Delta(A_k) + a^{k+1}_{n+1} - a^k_{n+1}, a^{k+1}\} \\
&= \min\{a^1 + a^{k+1}_{n+1} - a^1_{n+1}, \ldots, a^{k-1} + a^k_{n+1} - a^{k-1}_{n+1}, \\
&\qquad a^k + a^{k+1}_{n+1} - a^k_{n+1}, a^{k+1}\},
\end{aligned}$$

where the third equality follows from the fact that $\Delta(A_k) \le a^k \le a^{k+1}$. Thus the claim holds. □

Lemma 1 *If $A = \{a^1, \ldots, a^A\}$ and $B = \{b^1, \ldots, b^B\}$ are nested sets such that $A \cup B = \{a^1, \ldots, a^A, b^1, \ldots, b^B\}$ is nested, then*

$$\Delta(A \cup B) \succeq \Delta(A) \circ \Delta(B).$$

Proof: Since $\Delta(A \cup B)_{n+1} = (\Delta(A) \circ \Delta(B))_{n+1}$, it is sufficient to show that $\Delta(A \cup B) \le \Delta(A) \circ \Delta(B)$. We have

$$\begin{aligned}
\Delta(A \cup B) &= \min\{a^1 + b^B_{n+1} - a^1_{n+1}, \ldots, a^A + b^B_{n+1} - a^A_{n+1}, \\
&\qquad b^1 + b^B_{n+1} - b^1_{n+1}, \ldots, b^B\} \\
&= \min\{a^1 + a^A_{n+1} - a^1_{n+1} + (b^B_{n+1} - a^A_{n+1}), \ldots, a^A + (b^B_{n+1} - a^A_{n+1}), \\
&\qquad \min\{b^1 + b^B_{n+1} - b^1_{n+1}, \ldots, b^B\}\} \\
&= \min\{\Delta(A) + b^B_{n+1} - a^A_{n+1}, \Delta(B)\} \\
&\le \min\{\Delta(A) + b^B_{n+1} - a^A_{n+1}, \max\{\Delta(A), \Delta(B)\}\} \\
&= \Delta(A) \circ \Delta(B).
\end{aligned}$$

□

Lemma 2 *If $a, b, c, d \in \mathbb{R}^{n+1}$ are such that $a \succeq c$, $b \succeq d$, $a_{n+1} = c_{n+1}$ and $b_{n+1} = d_{n+1}$, then $a \circ b \succeq c \circ d$.*

Proof: Without loss of generality, assume that $d_{n+1} = b_{n+1} \ge a_{n+1} = c_{n+1}$. Then

$$(a \circ b)_{n+1} = b_{n+1} = d_{n+1} = (c \circ d)_{n+1}.$$

Since $a \succeq c$ and $b \succeq d$, we have $\max(a_j, b_j) \le \max(c_j, d_j)$ for all $j = 1, \ldots, n$; and since $b_{n+1} = d_{n+1}$, $a_{n+1} = c_{n+1}$ and $a_j \le c_j$ for all $j = 1, \ldots, n$, we have $a_j + b_{n+1} - a_{n+1} = a_j + d_{n+1} - c_{n+1} \le c_j + d_{n+1} - c_{n+1}$ for all $j = 1, \ldots, n$. Thus

$$(a \circ b)_j = \min\{a_j + b_{n+1} - a_{n+1}, \max(a_j, b_j)\} \le \min\{c_j + d_{n+1} - c_{n+1}, \max(c_j, d_j)\} = (c \circ d)_j \text{ for all } j = 1, \ldots, n.$$

The claim then follows from the definition of $\succeq$. □

Theorem 3 *If A is nested, then $\Delta(A) \succeq \Phi(A)$ for any $\Phi(A)$.*

Proof: The proof is by induction on $|A|$. Note that the claim holds trivially for nested sets A such that $|A| \le 2$. Assume that the claim holds for all nested sets A such that $|A| \le k$.

Consider a nested set A such that $|A| = k + 1$. Given $\Phi(A)$, obtained by an arbitrary sequence of pairings of the vectors in A, we can write

$$\Phi(A) = \Phi(A^1) \circ \Phi(A^2)$$

for some $A^1, A^2 \subset A$ such that $A^1 \cap A^2 = \emptyset$ and $A^1 \cup A^2 = A$. Note that $|A^1| \le k$ and $|A^2| \le k$. Thus by our induction hypothesis $\Delta(A^1) \succeq \Phi(A^1)$ and $\Delta(A^2) \succeq \Phi(A^2)$. We also notice that $\Delta(A^2)_{n+1} = \Phi(A^2)_{n+1}$ and $\Delta(A^1)_{n+1} = \Phi(A^1)_{n+1}$. Then

$$\begin{aligned}\Phi(A) &\preceq \Delta(A^1) \circ \Delta(A^2) \\ &\preceq \Delta(A^1 \cup A^2) = \Delta(A),\end{aligned}$$

where the first statement follows from Lemma 2 and the second statement follows from Lemma 1. □

Lemma 3 *If $A = \{a^1, \dots, a^K\}$ is nested and $B \subset A$ is such that $a^K \in B$, then $\Delta(A) \succeq \Delta(B)$.*

Proof: Since $\Delta(B)_{n+1} = \Delta(A)_{n+1}$, it is sufficient to show that $\Delta(A) \le \Delta(B)$. Let $A \setminus B = \{a^{i_1}, \dots, a^{i_l}\}$ and $B = \{a^{j_1}, \dots, a^{j_m}, a^K\}$. Then

$$\begin{aligned}\Delta(A) &= \min\{a^{i_1} + a^K_{n+1} - a^{i_1}_{n+1}, \dots, a^{i_l} + a^K_{n+1} - a^{i_l}_{n+1}, \\ &\qquad a^{j_1} + a^K_{n+1} - a^{j_1}_{n+1}, \dots, a^{j_m} + a^K_{n+1} - a^{j_m}_{n+1}, a^K\} \\ &= \min\{a^{i_1} + a^K_{n+1} - a^{i_1}_{n+1}, \dots, a^{i_l} + a^K_{n+1} - a^{i_l}_{n+1}, \Delta(B)\} \\ &\le \Delta(B).\end{aligned}$$

□

Combining Theorem 3 and Lemma 3, we obtain

Theorem 4 *Let $A = \{a^1, \dots, a^K\}$ be nested. All the non-dominated inequalities obtained by pairings of the vectors in A are contained in the set $\cup_{k=1}^K \{\Delta(A_k)\}$.*

Hence there are at most K non-dominated inequalities.

Now we give sufficient conditions for the inequalities in $\cup_{k=1}^K \Delta(A_k)$ to be facet-defining for a particular class of nested systems. Let $A = \{a^1, \dots, a^K\} \in \mathbb{R}^{n+1}$ be a nested set such that $a^i \ge 0$ for all $i = 1, \dots, K$, and consider the mixed 0-1 set (with one continuous variable):

$$X = \Big\{(x, y) \in \{0,1\}^n \times \mathbb{R}_+ : \sum_{j=1}^n a^i_j x_j + y \ge a^i_{n+1}, \; i = 1, \dots, K\Big\}.$$

Without loss of generality, we assume that $a^i_j \le a^i_{n+1}$ for all $j = 1, \dots, n$ and $i = 1, \dots, K$, since otherwise the coefficients can be strengthened to $a^i_j = a^i_{n+1}$. Let $A_i = \{a^1, \dots, a^i\}$ for $i = 1, \dots, K$, and $\Delta^i = \Delta(A_i)$.

Theorem 5 *Given $i \in \{1, \dots, K\}$, the sequential pairing inequality*

$$\sum_{j=1}^n \Delta^i_j x_j + y \ge a^i_{n+1}$$

is facet-defining for conv(X) if, for all $k \in \{i, i+1, \dots, K\}$,

(a) there exists $j^* \in \{1, \dots, n\}$ such that $\Delta^i_{j^*} + a^k_{n+1} - a^i_{n+1} \le a^k_{j^*}$, and

(b) $\sum_{j \in Z(i)} a^k_j \ge a^k_{n+1} - a^i_{n+1}$ where $Z(i) = \{j \in \{1, \dots, n\} : a^i_j = 0\}$.

Proof: The proof is constructive and the details are given in the Appendix. □

4 The disjoint case

A set $A = \{a^1, \ldots, a^K\} \subset \mathbb{R}^{n+1}$ satisfying

1. $a^k \geq 0$ for all $k = 1, \ldots, K$,
2. for any two vectors a^l and a^m, $a^l_j a^m_j = 0$ for $j = 1, \ldots, n$, and
3. $a^1_{n+1} \leq a^2_{n+1} \leq \cdots \leq a^K_{n+1}$.

is said to be *disjoint*. Here we consider mixed integer systems where the coefficients of the integer variables are disjoint. An example is the deterministic equivalent formulation of a two-stage stochastic program with integer second stage variables [3]

$$\begin{array}{lll} \min & c^T y + \sum_{s=1}^S p_s q_s^T x_s & \\ & y \in Y \subseteq \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1} & \\ & T_s y + W_s x_s \geq h_s & s = 1, \ldots, S \\ & x_s \in \mathbb{Z}_+^{n_2} & s = 1, \ldots, S. \end{array} \tag{5}$$

In (5), there are two sets of decision variables. The first-stage variables y are decided prior to a scenario s of realizations of the uncertain problem parameters (q_s, T_s, W_s, h_s). The second-stage decisions x_s constitute "recourse" actions corresponding to the scenario s realized. A scenario s occurs with probability p_s, and the objective is to minimize the sum of first-stage and expected second-stage costs. Note that the second-stage variables constitute a disjoint system.

Theorem 6 *If $A = \{a^1, \ldots, a^K\}$ is disjoint, then*

$$\Delta(A) = a^1 + \sum_{i=2}^{K} \min\{a^i_{n+1} - a^{i-1}_{n+1}, a^i\}.$$

Proof: The proof follows directly from Definitions 1 and 2, and the definition of a disjoint set. □

Lemma 4 *Let $A = \{a^1, \ldots, a^A\}$ and $B = \{b^1, \ldots, b^B\}$ be disjoint sets such that $A \cup B$ is disjoint and $a^A \leq b^B$. Then there exists $C \subseteq A \cup B$ with $b^B \in C$ such that*

$$\Delta(C) \succeq \Delta(A) \circ \Delta(B).$$

Proof: Since $\Delta(C)_{n+1} = b^B_{n+1} = (\Delta(A) \circ \Delta(B))_{n+1}$, it is sufficient to show that $\Delta(C) \leq \Delta(A) \circ \Delta(B)$. From Lemma 6, we have

$$\begin{aligned} \Delta(A) \circ \Delta(B) &= (a^1 + \sum_{i=2}^{A} \min\{a^i_{n+1} - a^{i-1}_{n+1}, a^i\}) \circ (b^1 + \sum_{i=2}^{B} \min\{b^i_{n+1} - b^{i-1}_{n+1}, b^i\}) \\ &= (a^1 + \sum_{i=2}^{A} \min\{a^i_{n+1} - a^{i-1}_{n+1}, a^i\}) \\ &\quad + \underbrace{\min\{b^B_{n+1} - a^A_{n+1}, (b^1 + \sum_{i=2}^{B} \min\{b^i_{n+1} - b^{i-1}_{n+1}, b^i\})\}}_{(d^{AB})}. \end{aligned}$$

Let $i^* = \min\{i \in \{1, \ldots, B\} : \; b^i_{n+1} \geq a^A_{n+1}\}$ and $C = A \cup \{b^{i^*}, \ldots, b^B\}$. Note that C is disjoint. Then

$$\begin{aligned} \Delta(C) &= a^1 + \sum_{i=2}^{A} \min\{a^i_{n+1} - a^{i-1}_{n+1}, a^i\} \\ &\quad + \underbrace{\min\{b^{i^*}_{n+1} - a^A_{n+1}, b^{i^*}\} + \sum_{i=i^*+1}^{B} \min\{b^i_{n+1} - b^{i-1}_{n+1}, b^i\}}_{(d^C)} \end{aligned}$$

By letting $b^0_{n+1} = -\infty$, we can write

$$d^{AB}_j = \min\{b^B_{n+1} - a^A_{n+1}, \sum_{i=1}^{B} \min\{b^i_{n+1} - b^{i-1}_{n+1}, b^i\}\}.$$

Let $J_{i^*} = \{j \in \{1, \ldots, n\} : \ b^{i^*}_j > 0\}$. Then

$$d^{AB}_j = \begin{cases} \min\{b^B_{n+1} - a^A_{n+1}, b^{i^*}_{n+1} - b^{i^*-1}_{n+1}, b^{i^*}_j\} & \text{if } j \in J_{i^*} \\ \min\{b^B_{n+1} - a^A_{n+1}, \sum_{i=1, i\neq i^*}^{B} \min\{b^i_{n+1} - b^{i-1}_{n+1}, b^i_j\}\} & \text{if } j \notin J_{i^*}, \end{cases}$$

$$d^C_j = \begin{cases} \min\{b^{i^*}_{n+1} - a^A_{n+1}, b^{i^*}_j\} & \text{if } j \in J_{i^*} \\ \sum_{i=i^*+1}^{B} \min\{b^i_{n+1} - b^{i-1}_{n+1}, b^i_j\}\} & \text{if } j \notin J_{i^*}, \end{cases}$$

By definition of i^*, we have $b^{i^*}_{n+1} - a^A_{n+1} \leq \min\{b^{i^*}_{n+1} - b^{i^*-1}_{n+1}, b^B_{n+1} - a^A_{n+1}\}$. Clearly, $\sum_{i=i^*+1}^{B} \min\{b^i_{n+1} - b^{i-1}_{n+1}, b^i_j\} \leq \sum_{i=1, i\neq i^*}^{B} \min\{b^i_{n+1} - b^{i-1}_{n+1}, b^i_j\}$. Moreover,

$$\begin{aligned} \sum_{i=i^*+1}^{B} \min\{b^i_{n+1} - b^{i-1}_{n+1}, b^i_j\} &\leq \sum_{i=i^*+1}^{B} (b^i_{n+1} - b^{i-1}_{n+1}) \\ &= b^B_{n+1} - b^{i^*}_{n+1} \leq b^B_{n+1} - a^A_{n+1}. \end{aligned}$$

Consequently, $d^C \leq d^{AB}$, and therefore $\Delta(C) \leq \Delta(A) \circ \Delta(B)$. □

As before, we let $\Phi(A) \in \mathbb{R}^{n+1}$ be a vector obtained by an arbitrary sequence of pairings of the vectors in A.

Theorem 7 *If $A = \{a^1, \ldots, a^K\}$ is disjoint, then for any $\Phi(A)$, there exists $\widehat{A} \subseteq A$ with $a^K \in \widehat{A}$ such that*

$$\Delta(\widehat{A}) \succeq \Phi(A).$$

Proof: The proof is by induction on $|A|$. The claim holds trivially for any disjoint set A such that $|A| \leq 2$. Assume that the claim holds for any disjoint set A with $|A| \leq k$.

Consider a disjoint set A such that $|A| = k+1$. Given $\Phi(A)$ obtained by an arbitrary sequence of pairings of the vectors in A, we can write

$$\Phi(A) = \Phi(A_1) \circ \Phi(A_2)$$

for some $A_1, A_2 \subset A$ such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$. Note that $|A_1| \leq k$ and $|A_2| \leq k$. Thus by our induction hypothesis, there exists $\widehat{A}_1 \subseteq A_1$ and $\widehat{A}_2 \subseteq A_2$, such that $\Delta(\widehat{A}_1) \succeq \Phi(A_1)$ and $\Delta(\widehat{A}_2) \succeq \Phi(A_2)$. Then from Lemma 2,

$$\Phi(A) \preceq \Delta(\widehat{A}_1) \circ \Delta(\widehat{A}_2).$$

By Lemma 4, there exists a subset $\widehat{A} \subseteq (\widehat{A}_1 \cup \widehat{A}_2) \subseteq A$ such that

$$\Phi(A) \preceq \Delta(\widehat{A}_1) \circ \Delta(\widehat{A}_2) \preceq \Delta(\widehat{A}).$$

□

As a consequence of Theorem 7, among all inequalities obtained by pairings of the vectors in a disjoint set A, it is sufficient to consider the inequalities corresponding to the $2^K - 1$ vectors in $C = \{\Delta(\widehat{A}) : \ \widehat{A} \subseteq A, \widehat{A} \neq \emptyset\}$.

Even though it suffices to consider the inequalities defined by the set C, the number of such inequalities is exponential in K. Here we present a polynomial time separation algorithm for finding a most violated inequality in C if one exists. The algorithm is based on solving shortest path problems on a directed graph G with nodes $\mathcal{N} = \{0, 1, \ldots, K\}$ and arcs (i, j) for all i and $j > i$. Given a point x^*, the separation problem of determining whether there exists any violated pairing inequalities can be reduced to finding a shortest path from node 0 to node k for $1 \leq k \leq K$ where the length of arc (i,j) is given by $\sum_{r=1}^{n} \min\{a_r^j, a_{n+1}^j - a_{n+1}^i\}x_r^*$ for $i > 0$ and $\sum_{r=1}^{n} a_r^j x_r^*$ for $i = 0$. This is true because a path $P = (0, i_1, i_2, \ldots, i_k)$ in G corresponds to a matrix $\widehat{A} = (a^{i_1}, a^{i_2}, \ldots, a^{i_k})$ since the length of the path is equal to the left-hand side of the inequality $\Delta(\widehat{A})$. Note that by Lemma 6, the left-hand side of the inequality $\Delta(\widehat{A})$ is $\sum_{r=1}^{n} a_r^{i_1} x_r^* + \sum_{j=2}^{k} \sum_{r=1}^{n} \min\{a_r^{i_j}, a_{n+1}^{i_j} - a_{n+1}^{i_{j-1}}\}x_r^*$, which is exactly the length of P. Therefore, there is a violated inequality with right-hand side a_{n+1}^k if and only if the length of a shortest path from 0 to k is less than a_{n+1}^k. Using Dijkstra's algorithm the separation problem can be solved in $O(K^2)$ time and we can find as many as K violated inequalities from the shortest paths from 0 to k for $k = 1, \ldots, K$.

Now we give sufficient conditions for the inequalities in C to be facet-defining for a certain class of disjoint systems. Let $A = \{a^1, \ldots, a^K\} \in \mathbb{R}^{n+1}$ be a disjoint set, and consider the mixed 0-1 set

$$X = \Big\{(x,y) \in \{0,1\}^n \times \mathbb{R}_+ : \sum_{j=1}^{n} a_j^i x_j + y \geq a_{n+1}^i,\ i = 1, \ldots, K\Big\},$$

with one continuous variable. Without loss of generality, as in the nested set case, we assume that $a_j^i \leq a_{n+1}^i$ for all $j = 1, \ldots, n$ and $i = 1, \ldots, K$. We also assume that

$$\sum_{j=1}^{n} a_j^i \geq a_{n+1}^i, \quad i = 1, \ldots, K, \tag{6}$$

since otherwise, we can replace y by $y + (a_{n+1}^i - \sum_{j=1}^{n} a_j^i)$. Consider $\widehat{A} = \{a^{q_1}, \ldots, a^{q_Q}\} \subseteq A$. Let $\mathcal{Q} = \{q_1, \ldots, q_Q\}$ and, for brevity, let $q = q_1$, $Q = q_Q$. Define $\widehat{\Delta} = \Delta(\widehat{A})$, where the jth element $\widehat{\Delta}_j$ is given by $\widehat{\Delta}_j = \min\{a_{n+1}^{r(j)} - a_{n+1}^{c(r(j))}, a_j^{r(j)}\}$, with $r(j) = \{i \in \{1, \ldots, n\} : a_j^i > 0\}$ for $j = 1, \ldots, n$ and $c(i) = \text{argmax}\{k \in \mathcal{Q} : k < i\}$ for all $i \in \mathcal{Q}$.

Theorem 8 *Given $\widehat{A} \subseteq A$ and the corresponding index set $\mathcal{Q}$, the sequential pairing inequality*

$$\sum_{j=1}^{n} \widehat{\Delta}_j x_j + y \geq a_{n+1}^Q$$

is facet-defining for conv(X) if

(a) $\max\{a_j^i : j \in \{1, \ldots, n\}\} \geq \max\{a_j^q : j \in \{1, \ldots, n\}\} + a_{n+1}^i - a_{n+1}^q$, *for all* $i \in \mathcal{Q}$.

(b) $\sum_{j=1}^{n} a_j^i \geq a_{n+1}^i - a_{n+1}^Q + a_k^i$, *for all* $k \in \{1, \ldots, n\}$ *and* $i \in \{Q+1, \ldots, K\}$.

Proof: The proof is constructive and the details are given in the Appendix. □

5 Applications

Dynamic knapsack sets: Consider the set X given by (4) with $a \in \mathbb{R}_+^n$ and $d \in \mathbb{R}_+^n$. Let $d_{ij} = \sum_{k=i}^{j} d_k$, Loparic et al. [6] proved that the inequality

$$y + \sum_{j=1}^{i} \min\{a_j, d_{ji}\}x_j \geq d_{1i} \tag{7}$$

is valid for $\text{conv}(X)$ for $i = 1, \ldots, n$, and facet-defining when $i = n$. Dynamic knapsack sets are nested. Applying the pairing sequence Δ to the inequalities (4) gives the inequalities (7). $i = n$ corresponds to $i = K$ in Theorem 5, and the inequality corresponding to $i = n$ satisfies the facet-defining conditions (a) and (b) in Theorem 5. We also notice that conditions (a) and (b) in Theorem 5 provide more facet-defining inequalities for dynamic knapsack sets.

Mixed vertex packing: The mixed vertex packing problem (MVPP) is a generalization of the vertex packing problem having both binary and bounded continuous variables. Let N denote the index set of binary variables, M denote the index set of continuous variables and $N(k) = \{i \in N : (k,i) \in E \cup F\}$, where $E \subseteq \{(i,j) : i,j \in N\}$ is defined as the *binary edge set* and $F \subseteq \{(i,k) : i \in N, k \in M\}$ is defined as the *mixed edge set*. The feasible solution set of MVPP is

$$
\begin{aligned}
X_{\mathrm{MVP}} = \Big\{ & (x,y) \in \{0,1\}^n \times \mathbb{R}^m : \\
& x_i + x_j \le 1, \qquad (i,j) \in E \qquad (8)\\
& a_{ik}x_i + y_k \le u_k, \qquad (i,k) \in F \qquad (9)\\
& 0 \le y_k \le u_k, \qquad k \in M \Big\}.
\end{aligned}
$$

For each $k \in M$, let $T = \{i_1, i_2, \ldots, i_t\} \subset N(k)$ such that $a_{i_{j-1}k} < a_{i_jk}$ for $j = 2, 3, \ldots, t$. Atamtürk et al. [1] showed that the *star* inequality

$$\sum_{i \in T} \bar{a}_{ik}x_i + y_k \le u_k, \tag{10}$$

where $\bar{a}_{i_1k} = a_{i_1k}$ and $\bar{a}_{i_jk} = a_{i_jk} - a_{i_{j-1}k}$ for $j = 2, \ldots, t$, is valid for X_{MVP}. Note that the *mixed edge set* inequalities form a disjoint set with respect to the binary variables.

We now show that the pairing scheme can generate all of the *star* inequalities. By complementing the binary variables for the *mixed edge set* inequalities (9) corresponding to edge $(i,k) \in F, i \in T$, we have

$$a_{ik}\bar{x}_i - y_k \ge a_{ik} - u_k, \ (i,k) \in F, \ i \in T \tag{11}$$

where $\bar{x}_i = 1 - x_i$. Applying the pairing sequence Δ to (11), we obtain

$$\sum_{i \in T} \bar{a}_{ik}\bar{x}_i - y_k \ge a_{i_tk} - u_k$$

with $\bar{a}_{i_1k} = a_{i_1k}$ and $\bar{a}_{i_jk} = a_{i_jk} - a_{i_{j-1}k}$ for $j = 2, \ldots, t$. That is,

$$\sum_{i \in T} \bar{a}_{ik}(1 - x_i) - y_k \ge a_{i_tk} - u_k,$$

which is exactly the *star* inequality (10). It is also shown in [1] that the *star* inequality is facet-defining for $conv(X_{\mathrm{MVP}})$ if $a_{i_tk} = \max_{j \in N(k)} a_{jk}$ and $N(i) = \emptyset$ for all $i \in T$. If $a_{i_tk} = \max_{j \in N(k)} a_{jk}$, then facet-defining conditions (a) and (b) in Theorem 8 are also satisfied by the equivalent formulation (11). The condition (b) is trivially true since $a_{i_tk} = \max_{j \in N(k)} a_{jk}$ corresponds to $Q = K$ for the disjoint case in Theorem 8 and condition (a) is also satisfied since the inequalities in condition (a) always hold at equality.

Deterministic lot-sizing: The deterministic uncapacitated lot-sizing problem is to minimize total production and inventory holding cost while satisfying demand over a finite discrete-time planning horizon. Let y_i be the production in period i, $x_i \in \{0,1\}$ indicate if there is a production set-up in period i, d_i be the demand in period $i \in \{1, \ldots, n\}$, and $d_{st} = \sum_{i=s}^{t} d_i$. The feasible solution set of the lot-sizing problem is

$$X_{\mathrm{LS}} = \Big\{ (x,y) \in \{0,1\}^n \times \mathbb{R}^n_+ : \ \sum_{j=1}^{i} y_j \ge d_{1i}, \ 0 \le y_i \le d_{in}x_i, \ \ i = 1, \ldots, n \Big\}.$$

Barany et al. [2] described the convex hull of X_{LS} by introducing the (ℓ, S) inequalities

$$\sum_{i \in S} y_i + \sum_{i \in L \setminus S} d_{i\ell}x_i \ge d_{1\ell} \tag{12}$$

for $1 \le \ell \le n$, $L = \{1, \ldots, \ell\}$ and $S \subseteq L$.

We now show that the pairing scheme can generate all of the (ℓ, S) inequalities. For given ℓ and S, we use the constraints $\sum_{j\leq k} y_j \geq d_{1k}$ for each $k \leq \ell$ and $y_j \leq d_{jn}x_j$ for each $j \in \{1,\dots,n\}$ to obtain the inequalities

$$\sum_{j\in S_k} y_j + \sum_{j\in L_k\setminus S_k} d_{jn}x_j \geq d_{1k} \text{ for each } k \leq \ell, \tag{13}$$

where $L_k = \{1,2,\dots,k\}$ and $S_k = S \cap L_k$. The family of inequalities (13) is nested (note here $1 \in S_k$ for each $k \leq \ell$). By Lemma 2, sequential pairing provides the (ℓ, S) inequality in (12) since we have $\Delta_j^\ell = \min\{d_{1\ell} - d_{11}, \dots, d_{1\ell} - d_{1(j-1)}, d_{jn} + d_{1\ell} - d_{1j}, \dots, d_{jn}\} = d_{1\ell} - d_{1(j-1)} = d_{j\ell}$ corresponding to each $j \in L \setminus S$.

Stochastic lot-sizing: The stochastic uncapacitated lot-sizing problem is the stochastic programming extension of the deterministic formulation. Instead of deterministic cost and demand information for each time period, the problem parameters are random and evolve as discrete time stochastic processes with a finite probability space. A scenario tree is used to model this information where each node i in stage t of the tree represents a possible state of the system. For each node i, let $\mathcal{T}(i) = (\mathcal{V}(i), \mathcal{E}(i))$ be the subtree containing all descendants of node i, $\mathcal{L}(i)$ be the leaf nodes of the subtree $\mathcal{T}(i)$, $\mathcal{P}(i,j)$ be the set of nodes on the path from node i to node j and $d_{ij} = \sum_{k\in\mathcal{P}(i,j)} d_k$, where d_i represents the demand in period $t(i)$ for node i. For brevity, let $\mathcal{T} = \mathcal{T}(0), \mathcal{V} = \mathcal{V}(0), \mathcal{L} = \mathcal{L}(0)$ and $\mathcal{P}(i) = \mathcal{P}(0,i)$.

Let y_i be the production and x_i be the indicator variable for a production set-up in period $t(i)$ corresponding to the state defined by node i. The feasible solution set of the stochastic lot-sizing problem [4] is

$$X_{\text{SLS}} = \Big\{(x,y) \in \{0,1\}^n \times \mathbb{R}^n_+ : \sum_{j\in\mathcal{P}(i)} y_j \geq d_{0i},\ 0 \leq y_i \leq M_i x_i,\ \ i \in \mathcal{V}\Big\},$$

where $M_i = \max_{j\in\mathcal{L}(i)} d_{ij}$ is an upper bound on y_i.

Guan et al. [4] developed a family of valid inequalities for X_{SLS} called the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities. Consider a set of nodes $\mathcal{Q} = \{1,2,\dots,Q\} \subset \mathcal{V}$, such that $d_{01} \leq d_{02} \leq \dots \leq d_{0Q}$ and $\{m, m+1, \dots, n-1, n\} \subseteq \mathcal{Q}(i)$ if $m < n$ and $m,n \in \mathcal{Q}(i)$, where $\mathcal{Q}(i) = \mathcal{Q} \cap \mathcal{V}(i)$. Let $\mathcal{V}_{\mathcal{Q}} = \cup_{i\in\mathcal{Q}}\mathcal{P}(i)$ and for each $i \in \mathcal{V}_{\mathcal{Q}}$ let

$$\begin{aligned}
\overline{D}_{\mathcal{Q}}(i) &= \max\{d_{0j} :\ j \in \mathcal{Q}(i)\},\\
\widetilde{D}_{\mathcal{Q}}(i) &= \begin{cases} 0, \ \text{ if } \{j : j \in \mathcal{Q}\setminus\mathcal{Q}(i) \text{ such that } d_{0j} \leq \overline{D}_{\mathcal{Q}}(i)\} = \emptyset \\ \max\{d_{0j} :\ j \in \mathcal{Q}\setminus\mathcal{Q}(i) \text{ such that } d_{0j} \leq \overline{D}_{\mathcal{Q}}(i)\}, \ \text{ otherwise,}\end{cases}\\
M_{\mathcal{Q}}(i) &= \max\{d_{ij} : j \in \mathcal{Q}(i)\}, \text{ and}\\
\delta_{\mathcal{Q}}(i) &= \min\Big\{\overline{D}_{\mathcal{Q}}(i) - \widetilde{D}_{\mathcal{Q}}(i), M_{\mathcal{Q}}(i)\Big\}.
\end{aligned}$$

Then, given $S_{\mathcal{Q}} \subseteq \mathcal{V}_{\mathcal{Q}}$ and $\overline{S}_{\mathcal{Q}} = \mathcal{V}_{\mathcal{Q}} \setminus S_{\mathcal{Q}}$, the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality

$$\sum_{i\in S_{\mathcal{Q}}} y_i + \sum_{i\in\overline{S}_{\mathcal{Q}}} \delta_{\mathcal{Q}}(i)x_i \geq M_{\mathcal{Q}}(0) \tag{14}$$

is valid for X_{SLS}.

We can use sequential pairing to generate all $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities. Given a $(\mathcal{Q}, S_{\mathcal{Q}})$ tuple, first, we can use sequential pairing, as in the deterministic lot-sizing case, to generate (ℓ, S) inequalities corresponding to $\mathcal{P}(i)$ for each $i \in \mathcal{Q}$ as

$$\sum_{j\in\mathcal{P}(i)\cap S_{\mathcal{Q}}} x_j + \sum_{j\in\mathcal{P}(i)\cap\overline{S}_{\mathcal{Q}}} d_{ji}y_j \geq d_{0i}. \tag{15}$$

Then, we use sequential pairing of the inequalities (15) for $i = 1$ to Q to obtain

$$\sum_{i\in S_{\mathcal{Q}}} x_i + \sum_{i\in\overline{S}_{\mathcal{Q}}} \delta_{\mathcal{Q}}(i)y_i \geq d_{0Q} = M_{\mathcal{Q}}(0). \tag{16}$$

To see that sequential pairing leads to the correct coefficients in (16), note that this claim is clearly true for $|\mathcal{Q}| = 1$ since this case is exactly that of an (ℓ, S) inequality for the deterministic lot-sizing problem.

Assuming that the claim is true for $|\mathcal{Q}| = k$, we have

$$\sum_{i \in S_{\mathcal{Q}} \cap \mathcal{V}_{\mathcal{Q}_k}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}} \cap \mathcal{V}_{\mathcal{Q}_k}} \delta_{\mathcal{Q}_k}(i) y_i \geq d_{0k},$$

where $\mathcal{Q}_k = \{1, 2, \ldots, k\}$. By pairing the above inequality with the (ℓ, S) inequality

$$\sum_{j \in \mathcal{P}(k+1) \cap S_{\mathcal{Q}}} x_j + \sum_{j \in \mathcal{P}(k+1) \cap \overline{S}_{\mathcal{Q}}} d_{j,k+1} y_j \geq d_{0,k+1}$$

corresponding to $i = k+1$, the resulting coefficients corresponding to each $j \in \overline{S}_{\mathcal{Q}}$ are as follows.

(i) The coefficient corresponding to each $i \in \mathcal{V}_{\mathcal{Q}_k} \setminus \mathcal{P}(k+1)$ remains unchanged and $\delta_{\mathcal{Q}_k}(i) = \delta_{\mathcal{Q}_{k+1}}(i)$.

(ii) The coefficient corresponding to each $i \in \mathcal{P}(k+1) \setminus \mathcal{V}_{\mathcal{Q}_k}$ is equal to $\min\{d_{0,k+1} - d_{0k}, d_{i,k+1}\}$, which is $\delta_{\mathcal{Q}_{k+1}}(i)$.

(iii) The coefficient corresponding to each $i \in \mathcal{P}(k+1) \cap \mathcal{V}_{\mathcal{Q}_k}$ is equal to $\delta_{\mathcal{Q}_k}(i) + d_{0,k+1} - d_{0k} = \delta_{\mathcal{Q}_{k+1}}(i)$ since $M_{\mathcal{Q}_{k+1}}(i) = M_{\mathcal{Q}_k}(i) + d_{0,k+1} - d_{0k}$, $\widetilde{D}_{\mathcal{Q}_{k+1}}(i) = \widetilde{D}_{\mathcal{Q}_k}(i)$ and $\overline{D}_{\mathcal{Q}_{k+1}}(i) = \overline{D}_{\mathcal{Q}_k}(i) + d_{0,k+1} - d_{0k}$.

Thus we have the correct coefficients in (16).

6 Computational Experiments

In this section we provide some numerical results to demonstrate the computational effectiveness of the pairing scheme on randomly generated instances of mixed-integer programs with nested and disjoint sets of constraints. All computations have been carried out on a Linux workstation with dual 2.4 GHz Intel Xeon processors and 2 GB RAM using CPLEX 8.1.

For the nested case, we generated random instances of the model

$$\begin{array}{lll} \min & \sum_{j=1}^{mn} c_j x_j + \sum_{k=1}^{p} h_k y_k & \\ & \sum_{j=1}^{in} a_j^i x_j + \sum_{k=1}^{p} g_k^i y_k \geq b_i & i = 1, \ldots, m \\ & x_j \in \{0,1\} & j = 1, \ldots, mn \\ & y_k \geq 0 & k = 1, \ldots, p. \end{array}$$

This model has n additional binary variables in each successive row, with a total of mn binary variables and p continuous variables. The constraint coefficients and the right-hand sides were generated such that these form a nested system and were uniformly distributed within the interval $[50, 75]$ and $[50, 100]$, respectively. The objective function coefficients were uniformly distributed within the interval $[10, 100]$. In Table 1 we present computational results for $p \in \{1, 2, 3\}$, $n \in \{1, 2, 3\}$ and $m \in \{10, 20, 40\}$. For each combination of m, n, and p, we tested five instances and report the average objective function value in the column labelled "OptVal." The row labelled "LP" provides the average optimal objective value of the linear programming relaxation without any cuts; the row labelled "LP+CUTS"(LPC) provides the average optimal objective value after adding all inequalities obtained through pairing as cuts, which can be done since the total number of cuts is small and equal to the number of rows; and the row labelled "IP" provides the optimal value of the corresponding integer programming problem. The column labelled "Gap" provides the percentage LP relaxation gap, computed as (IP-LP)/LP $\times$ 100% and (IP-LPC)/LPC $\times$ 100%. We observe that the cuts yield significant improvements. In 13 of the 27 cases, the gap is reduced to 0% from over 10%. In all but three of the cases, the gap is reduced by more than half.

Table 1: Computational Results for the Nested Case

			$m = 10$		$m = 20$		$m = 40$	
p	n		OptVal	Gap	OptVal	Gap	OptVal	Gap
		LP	100.59	19.58%	51.10	13.97%	24.43	22.18%
1	1	LP+CUTS	114.41	8.53%	59.40	0.00%	31.18	0.67%
		IP	125.08		59.40		31.39	
		LP	65.88	23.63%	48.87	13.25%	21.21	15.99%
1	2	LP+CUTS	77.80	9.82%	56.33	0.00%	25.02	0.89%
		IP	86.27		56.33		25.25	
		LP	39.71	19.83%	48.19	14.50%	21.43	14.56%
1	3	LP+CUTS	43.27	12.64%	56.36	0.00%	24.95	0.51%
		IP	49.53		56.36		25.08	
		LP	23.00	4.61%	31.47	10.38%	65.86	13.05%
2	1	LP+CUTS	24.11	0.00%	35.12	0.00%	75.75	0.00%
		IP	24.11		35.12		75.75	
		LP	22.62	9.50%	31.45	11.65%	58.21	15.77%
2	2	LP+CUTS	24.99	0.00%	35.60	0.00%	66.96	3.11%
		IP	24.99		35.60		69.11	
		LP	22.02	7.92%	31.42	13.90%	56.89	15.39%
2	3	LP+CUTS	23.92	0.00%	36.49	0.00%	63.95	4.87%
		IP	23.92		36.49		67.23	
		LP	20.28	19.45%	21.99	30.57%	69.13	14.96%
3	1	LP+CUTS	24.18	3.95%	28.03	11.52%	81.29	0.00%
		IP	25.18		31.68		81.29	
		LP	17.05	28.54%	20.39	27.60%	64.66	13.35%
3	2	LP+CUTS	20.47	14.20%	22.81	18.99%	74.62	0.00%
		IP	23.86		28.16		74.62	
		LP	18.99	25.28%	20.06	29.40%	64.13	11.93%
3	3	LP+CUTS	22.52	11.41%	22.74	19.96%	72.82	0.00%
		IP	25.42		28.41		72.82	

For the disjoint case, we generated random instances of the model

$$\begin{array}{lll} \min & \sum_{i=1}^{m}\sum_{j=1}^{n} c_j^i x_j^i + \sum_{k=1}^{p} h_k y_k & \\ & \sum_{j=1}^{n} a_j^i x_j^i + \sum_{k=1}^{p} g_k y_k \geq b_i & i = 1, \ldots, m \\ & x_j^i \in \{0,1\} & j = 1, \ldots, n, i = 1, \ldots, m \\ & y_k \geq 0 & k = 1, \ldots, p. \end{array}$$

Each row of this model has n independent binary variables giving rise to a disjoint system involving a total of mn binary variables. A total of p continuous variables couple the binary variables together. The constraint coefficients and the right-hand sides were generated uniformly within the interval $[40, 120]$ and $[100, 125]$ respectively. The objective function coefficients were uniformly distributed within the interval $[10, 100]$ for the continuous variables and within the interval $[10/m, 100/m]$ for the binary variables. In Table 2, we present computational results corresponding to $p \in \{1, 2, 3\}$, $n \in \{1, 2, 3\}$ and $m \in \{10, 20, 40\}$. As before, we report averages over five random instances for each combination of m, n and p. In this case, we use the shortest path separation routine described in Section 4 to add only violated cuts. The average number of cuts added is reported in the row labelled "# CUTS." Once again, we observe that the cuts yield significant improvements. In 6 of the 27 cases, the gap is reduced to 0%. In 19 of the 27 cases, the gap is reduced by more than half. The number of cuts ranges from 30, on average for 10 rows, to 491, on average for 40 rows.

Table 2: Computational Results for the Disjoint Case

			$m = 10$		$m = 20$		$m = 40$	
p	n		OptVal	Gap	OptVal	Gap	OptVal	Gap
		LP	1082.24	11.62%	703.83	9.03%	874.35	12.57%
1	1	LP+CUTS	1099.88	10.17%	729.58	5.71%	928.66	7.14%
		IP	1224.47		773.73		1000.08	
		# CUTS	45		163		925	
		LP	585.63	38.48%	538.49	25.20%	702.46	29.22%
1	2	LP+CUTS	793.18	16.68%	655.58	8.93%	946.30	4.65%
		IP	952.00		719.89		992.45	
		# CUTS	32		162		1317	
1	3	LP	410.88	26.45%	446.83	23.08%	559.81	24.67%
1	3	LP+CUTS	480.88	13.92%	452.76	22.05%	619.81	16.59%
		IP	558.66		580.87		743.13	
		# CUTS	10		27		124	
		LP	685.99	8.36%	388.66	5.96%	497.25	7.69%
2	1	LP+CUTS	693.98	7.29%	400.19	3.17%	522.35	3.03%
		IP	748.58		413.29		538.67	
		# CUTS	34		74		341	
		LP	507.76	33.72%	356.65	14.23%	464.42	13.56%
2	2	LP+CUTS	710.69	7.22%	415.34	0.12%	529.88	1.38%
		IP	766.03		415.83		537.30	
		# CUTS	39		119		530	
		LP	400.18	20.50%	339.61	12.71%	437.98	13.60%
2	3	LP+CUTS	448.58	10.89%	357.13	8.21%	470.21	7.25%
		IP	503.38		389.06		506.95	
		# CUTS	20		63		347	
		LP	533.31	2.84%	285.17	2.09%	387.16	4.80%
3	1	LP+CUTS	542.35	1.19%	291.25	0.00%	406.67	0.00%
		IP	548.88		291.25		406.67	
		# CUTS	25		34		173	
		LP	433.80	21.02%	280	3.86%	375.75	6.84%
3	2	LP+CUTS	540.33	1.63%	291.23	0.00%	403.35	0.00%
		IP	549.27		291.23		403.35	
		# CUTS	45		59		289	
		LP	420.45	14.19%	279.87	4.38%	360.44	9.80%
3	3	LP+CUTS	459.61	6.19%	292.69	0.00%	399.61	0.00%
		IP	489.95		292.69		399.61	
		# CUTS	24		67		375	

7 Conclusions

We have developed a new and very simple way of pairwise combining linear inequalities for MIPs to obtain new linear inequalities. These new inequalities can be useful in tightening the LP relaxation for general MIPs. The order in which the inequalities are combined can have a significant impact on the results. For some structured systems, we provided combination orders that are optimal in the sense that no other combination order cannot dominate the set of inequalities given by the optimal order. These structures arise in multi-period MIPs. We discussed applications of these structures to deterministic and stochastic lot-sizing problems. One of our goals is to apply the procedure to general multi-period stochastic MIPs. To do this we need to generalize the structures considered in this paper to scenario trees. We are currently developing these results.

Acknowledgements

This research has been supported by the National Science Foundation awards DMI-0100020, DMI-0121495, and DMI-0133943. The authors thank Alper Atamtürk and Oktay Günluk for helpful comments.

Appendix

Theorem 5 *Given $i \in \{1, \dots, K\}$, the sequential pairing inequality*

$$\sum_{j=1}^{n} \Delta_j^i x_j + y \geq a_{n+1}^i \tag{17}$$

is facet-defining for conv(X) if there exists $j^* \in \{1, \dots, n\}$ such that for all $k \in \{i, i+1, \dots, K\}$,

(a) $\Delta_{j^*}^i + a_{n+1}^k - a_{n+1}^i \leq a_{j^*}^k$, *and*

(b) $\sum_{j \in Z(i)} a_j^k \geq a_{n+1}^k - a_{n+1}^i$ *where* $Z(i) = \{j \in \{1, \dots, n\} : a_j^i = 0\}$.

Proof: We construct $\dim(X) = n+1$ linearly independent vectors belonging to X that satisfy (17) at equality.

We construct a vector corresponding to each of the $n+1$ variables. Let e^y and e^{x_j} be unit vectors in $\mathbb{R}^{n+1}$ corresponding to the coordinates y and x_j for $j = 1, \dots, n$. The constructed vectors are denoted by $\{u^j\}_{j=0}^n$ and are constructed as follows.

(i) Vector u^0 corresponds to variable y and is given by

$$u^0 = a_{n+1}^i e^y + \sum_{r \in Z(i)} e^{x_r}.$$

(ii) Vector u^{j^*} corresponds to variable x_{j^*} and is given by

$$u^{j^*} = [a_{n+1}^i - \Delta_{j^*}^i] e^y + e^{x_{j^*}}.$$

(iii) For each x_j where $j \in \{1, \dots, n\} \setminus \{Z(i) \cup \{j^*\}\}$, the corresponding vector u^j is given by

$$u^j = [a_{n+1}^i - \Delta_j^i] e^y + e^{x_j} + \sum_{r \in Z(i)} e^{x_r}.$$

Note that there are $n - |Z(i)| - 1$ such vectors.

(iv) For each x_j where $j \in Z(i)$, the corresponding vector u^j is given by

$$u^j = [a_{n+1}^i - \Delta_{j^*}^i] e^y + e^{x_j} + e^{x_{j^*}}.$$

Feasibility: We need to show that $\{u^j\}_{j=0}^n$ satisfies

$$a^k u \geq a_{n+1}^k \qquad k = 1, \dots, K, \tag{18}$$

$$u_0 \geq 0,\ u_j \in \{0, 1\} \qquad j = 1, \dots, n, \tag{19}$$

where $a_0^k = 1$ for all $k = 1, \dots, K$.

(i) The vector u^0 clearly satisfies (19) since $a_{n+1}^i \geq 0$. The left-hand-side of (18) is

$$\begin{aligned} a^k u^0 &= a_{n+1}^i + \textstyle\sum_{j \in Z(i)} a_j^k \\ &\geq \begin{cases} a_{n+1}^k & \text{if } k < i \\ a_{n+1}^i + a_{n+1}^k - a_{n+1}^i = a_{n+1}^k & \text{if } k \geq i, \end{cases} \end{aligned}$$

where the inequality for the case $k < i$ follows from the fact that $a_j^k \geq 0$ for all k, j, and $a_{n+1}^k \leq a_{n+1}^i$ for all $k < i$; and the inequality for the case $k \geq i$ follows from condition (b) of the Theorem. Thus u^0 satisfies (18).

(ii) The vector u^{j^*} clearly satisfies (19) since $\Delta^i_{j^*} \le a^i_{j^*} \le a^i_{n+1}$. The left-hand-side of (18) corresponding to $k < i$ is

$$\begin{aligned} a^k u^{j^*} &= a^i_{n+1} - \Delta^i_{j^*} + a^k_{j^*} \\ &\ge a^i_{n+1} - (a^k_{j*} + a^i_{n+1} - a^k_{n+1}) + a^k_{j^*} = a^k_{n+1}, \end{aligned}$$

where the inequality follows from the fact that $\Delta^i_{j*} \le a^k_{j*} + a^i_{n+1} - a^k_{n+1}$ for all $k = 1, \ldots, i$. The left-hand-side of (18) corresponding to $k \ge i$ is

$$\begin{aligned} a^k u^{j^*} &= a^i_{n+1} - \Delta^i_{j^*} + a^k_{j^*} \\ &\ge a^k_{n+1} - a^i_{n+1} + a^i_{n+1} = a^k_{n+1}, \end{aligned}$$

where the inequality follows from condition (a). Thus u^{j^*} satisfies (18).

(iii) For a given $j \in \{1, \ldots, n\} \setminus \{Z(i) \cup \{j^*\}\}$, the vector u^j clearly satisfies (19) since $\Delta^i_j \le a^i_j \le a^i_{n+1}$. The left-hand-side of (18) corresponding to $k < i$ is

$$\begin{aligned} a^k u^j &= a^i_{n+1} - \Delta^i_j + a^k_j + \textstyle\sum_{r \in Z(i)} a^k_r \\ &= a^i_{n+1} - \Delta^i_j + a^k_j \\ &\ge a^i_{n+1} - (a^k_j + a^i_{n+1} - a^k_{n+1}) + a^k_j = a^k_{n+1}, \end{aligned}$$

where the second line follows from the nested property $a^k_r \le a^i_r$ for all $k = 1, \ldots, i$, $r = 1, \ldots, n$, and $a^i_r = 0$ for all $r \in Z(i)$; and the third line follows from the fact that $\Delta^i_j \le a^k_j + a^i_{n+1} - a^k_{n+1}$ for all $k = 1, \ldots, i$. The left-hand-side of (18) corresponding to $k \ge i$ is

$$\begin{aligned} a^k u^j &= a^i_{n+1} - \Delta^i_j + a^k_j + \textstyle\sum_{r \in Z(i)} a^k_r \\ &\ge a^i_{n+1} - \Delta^i_j + a^k_j + a^k_{n+1} - a^i_{n+1} \\ &= -\Delta^i_j + a^k_j + a^k_{n+1} \\ &\ge a^k_{n+1}, \end{aligned}$$

where the second line follows from condition (b), and the last line follows from the fact that $a^k_j \ge a^i_j \ge \Delta^i_j$ for all $k = i, i+1, \ldots, K$. Thus u^j satisfies (18).

(iv) For a given $j \in Z(i)$ the vector u^j clearly satisfies (19) since $\Delta^i_{j^*} \le a^i_{j^*} \le a^i_{n+1}$. The vector u^j also satisfies (18) since $u^j \ge u^{j*}$ and u^{j^*} satisfies (18).

Tightness: It is easily verified that the vectors $\{u^j\}_{j=0}^n$ satisfy the inequality (17) as an equality.

Linear independence: To verify the linear independence of the $n+1$ vectors $\{u^j\}_{j=0}^n$, observe that we can obtain $n+1$ unit vectors from $\{u^j\}_{j=0}^n$ as follows:

$e^{x_j} = u^j - u^{j^*}$ for all $j \in Z(i)$.

$e^y = u^0 - \sum_{j \in Z(i)} e^{x_j}$.

$e^{x_{j^*}} = u^{j^*} - [a^i_{n+1} - \Delta^i_{j^*}]e^y$.

$e^{x_j} = u^j - [a^i_{n+1} - \Delta^i_j]e^y - \sum_{r \in Z(i)} e^{x_r}$ for all $j \in \{1, \ldots, n\} \setminus \{Z(i) \cup \{j^*\}\}$.

□

Theorem 8 *Given $\widehat{A} \subseteq A$ and the corresponding index set $\mathcal{Q}$, the sequential pairing inequality*

$$\sum_{j=1}^n \widehat{\Delta}_j x_j + y \ge a^Q_{n+1} \tag{20}$$

is facet-defining for conv(X) if

(a) $\max\{a_j^i : j \in \{1,\ldots,n\}\} \geq \max\{a_j^q : j \in \{1,\ldots,n\}\} + a_{n+1}^i - a_{n+1}^q$, *for all* $i \in \mathcal{Q}$.

(b) $\sum_{j=1}^n a_j^i \geq a_{n+1}^i - a_{n+1}^Q + a_k^i$, *for all* $k \in \{1,\ldots,n\}$ *and* $i \in \{Q+1,\ldots,K\}$.

Proof: We construct $\dim(X) = n+1$ linearly independent vectors belong to X that satisfy (20) at equality.

We construct a vector corresponding to each of the $n+1$ variables. Denote $s(i) = \text{argmax}\{a_j^i : j \in \{1,\ldots,n\}\}$ for all $i \in \mathcal{Q}$. Let e^y be the unit vector in $\mathbb{R}^{n+1}$ corresponding to the coordinate y and e^{x_j} be the unit vector in $\mathbb{R}^{n+1}$ corresponding to the coordinate x_j for $j = 1,\ldots,n$. Let $Z(\mathcal{Q}) = \{j \in \{1,\ldots,n\} : \exists\, i \in \mathcal{Q}$ such that $a_j^i > 0\}$ and $\overline{Z}(\mathcal{Q}) = \{1,\ldots,n\} \setminus Z(\mathcal{Q})$. We construct the following $n+1$ vectors, denoted by $\{u^j\}_{j=0}^n$.

(i) Vector u^0 corresponds to variable y and is given by

$$u^0 = a_{n+1}^Q e^y + \sum_{i \in \overline{Z}(\mathcal{Q})} e^{x_i}.$$

(ii) For each $j \in \overline{Z}(\mathcal{Q})$, the corresponding vector u^j is given by

$$u^j = u^0 - e^{x_j}.$$

(iii) For each $j \in Z(\mathcal{Q})$, the corresponding vector u^j is given by

$$u^j = (a_{n+1}^{r(j)} - \widehat{\Delta}_j)e^y + \sum_{i \in \overline{Z}(\mathcal{Q})} e^{x_i} + e^{x_j} + \sum_{i \in \mathcal{Q},\ i > r(j)} e^{x_{s(i)}}.$$

Feasibility: We need to show that $\{u^j\}_{j=0}^n$ satisfies

$$a^k u \geq a_{n+1}^k \qquad k = 1,\ldots,K, \tag{21}$$

$$u_0 \geq 0,\ u_j \in \{0,1\} \qquad j = 1,\ldots,n, \tag{22}$$

where $a_0^k = 1$ for all $k = 1,\ldots,K$.

(i) The feasibility of u^0 is based on (6).

(ii) The feasibility of u^j for each $j \in \overline{Z}(\mathcal{Q})$ is based on condition (b).

(iii) For a given $j \in Z(\mathcal{Q})$, the vector u^j satisfies (22) since $\widehat{\Delta}_j \leq a_j^{r(j)} \leq a_{n+1}^{r(j)}$. The left-hand of (21) corresponding to $i \in \{1,\ldots,K\} \setminus \mathcal{Q}$ is

$$a_{n+1}^{r(j)} - \widehat{\Delta}_j + \textstyle\sum_{j=1}^n a_j^i u_j^j \quad \geq \quad \textstyle\sum_{j=1}^n a_j^i \geq a_{n+1}^i,$$

where the first inequality follows from $\widehat{\Delta}_j \leq a_j^{r(j)} \leq a_{n+1}^{r(j)}$ and the second inequality follows from (6).

The left-hand side of (21) corresponding to $i \in \mathcal{Q}$ and $i = r(j)$ is

$$a_{n+1}^{r(j)} - \widehat{\Delta}_j + \sum_{j=1}^n a_j^i u_j^j = a_{n+1}^{r(j)} - \widehat{\Delta}_j + a_j^{r(j)} \geq a_{n+1}^{r(j)} = a_{n+1}^i,$$

where the inequality follows from the definition of $\widehat{\Delta}_j$.

The left-hand side of (21) corresponding to $i \in \mathcal{Q}$ and $i < r(j)$ is

$$\begin{aligned} a_{n+1}^{r(j)} - \widehat{\Delta}_j + \textstyle\sum_{j=1}^n a_j^i u_j^j \quad &\geq \quad a_{n+1}^{r(j)} - \widehat{\Delta}_j \\ &\geq \quad a_{n+1}^{r(j)} - (a_{n+1}^{r(j)} - a_{n+1}^{c(r(j))}) \\ &= \quad a_{n+1}^{c(r(j))} \geq a_{n+1}^i, \end{aligned}$$

where the second inequality follows from the definition of $\widehat{\Delta}_j$.

The left-hand side of (21) corresponding to $i \in \mathcal{Q}, i > r(j)$ and $r(j) = q$ is

$$\begin{aligned} a_{n+1}^{r(j)} - \widehat{\Delta}_j + \textstyle\sum_{j=1}^n a_j^i u_j^j &= a_{n+1}^{r(j)} - \widehat{\Delta}_j + a_{s(i)}^i \\ &\geq a_{n+1}^q - a_j^q + a_{s(i)}^i \\ &\geq a_{n+1}^q - a_j^q + \max\{a_j^q : j \in \{1, \dots, n\}\} + a_{n+1}^i - a_{n+1}^q \\ &\geq a_{n+1}^i, \end{aligned}$$

where the first inequality follows from $r(j) = q$, $\widehat{\Delta}_j \leq a_j^q$ and the second inequality follows from condition (a).

The left-hand side of (21) corresponding to $i \in \mathcal{Q}, i > r(j)$ and $r(j) \neq q$ is

$$\begin{aligned} a_{n+1}^{r(j)} - \widehat{\Delta}_j + \textstyle\sum_{j=1}^n a_j^i u_j^j &= a_{n+1}^{r(j)} - \widehat{\Delta}_j + a_{s(i)}^i \\ &\geq a_{n+1}^{r(j)} - (a_{n+1}^{r(j)} - a_{n+1}^{c(r(j))}) + a_{s(i)}^i \\ &= a_{n+1}^{c(r(j))} + a_{s(i)}^i \\ &\geq a_{n+1}^{c(r(j))} + \max\{a_j^q : j \in \{1, \dots, n\}\} + a_{n+1}^i - a_{n+1}^q \\ &\geq a_{n+1}^i, \end{aligned}$$

where the first inequality follows from the definition of $\widehat{\Delta}_j$, the second inequality follows from condition (a) and the third inequality follows from the fact that $a_{n+1}^{c(r(j))} \geq a_{n+1}^q$.

Tightness:

(i, ii) It is easily verified that u^0 and u^j for each $j \in \overline{Z}(\mathcal{Q})$ satisfy (20) as an equality.

(iii) For a given $j \in Z(\mathcal{Q})$, the left-hand side of (20) corresponding to u^j is

$$\begin{aligned} u_0^j + \textstyle\sum_{i=1}^n \widehat{\Delta}_i u_i^j &= (a_{n+1}^{r(j)} - \widehat{\Delta}_j) + \widehat{\Delta}_j + \textstyle\sum_{i \in \mathcal{Q}, i > r(j)} \widehat{\Delta}_{s(i)} \\ &= a_{n+1}^{r(j)} + \textstyle\sum_{i \in \mathcal{Q}, i > r(j)} (a_{n+1}^i - a_{n+1}^{c(i)}) = a_{n+1}^{Q}, \end{aligned}$$

where the second equality follows from

$$\widehat{\Delta}_{s(i)} = \min\{a_{n+1}^i - a_{n+1}^{c(i)}, a_{s(i)}^i\}, \text{ and}$$

$$a_{s(i)}^i = \max\{a_j^i : j \in \{1, \dots, n\}\} \geq a_{n+1}^i - a_{n+1}^q \geq a_{n+1}^i - a_{n+1}^{c(i)},$$

which follows from (a).

Linear Independence: To verify the linear independence of the $n + 1$ vectors $\{u^j\}_{j=0}^n$, we can obtain the following $n + 1$ vectors from $\{u^j\}_{j=0}^n$ as follows:

$$\begin{aligned} e^y &= u^0 - \sum_{i \in \overline{Z}(\mathcal{Q})} e^{x_i}. \\ e^{x_j} &= u^0 - u^j, \quad \text{for each } \ j \in \overline{Z}(\mathcal{Q}). \\ v^j &= u^j - (a_{n+1}^{r(j)} - \widehat{\Delta}_j) e^y - \sum_{i \in \overline{Z}(\mathcal{Q})} e^{x_i} \\ &= e^{x_j} + \sum_{i \in \mathcal{Q}, i > r(j)} e^{x_{s(i)}}, \text{for each } j \in Z(\mathcal{Q}). \end{aligned}$$

By sorting v^j according to the decreasing sequence of $r(j)$, it can be verified that v^j for each $j \in Z(\mathcal{Q})$ forms a lower triangular. Therefore, these vectors are linearly independent, which implies that original vectors are linearly independent. □

References

[1] A. Atamtürk, G. L. Nemhauser and M. W. P. Savelsbergh. The mixed vertex packing problem. *Mathematical Programming*, 89:35–53, 2000.

[2] I. Barany, T. Van Roy, and L. A. Wolsey. Strong formulations for multi-item capacitated lot sizing. *Management Science*, 30:1255–1262, 1984.

[3] J. R. Birge and F. Louveaux. *Introduction to Stochastic Programming*. Springer, 1997.

[4] Y. Guan, S. Ahmed, G. L. Nemhauser, and A. J. Miller. A branch-and-cut algorithm for the stochastic uncapacitated lot-sizing problem. To appear in *Mathematical Programming*.

[5] O. Günluk and Y. Pochet. Mixing mixed-integer inequalities. *Mathematical Programming*, 90:429–457, 2001.

[6] M. Loparic, H. Marchand and L. A. Wolsey. Dynamic knapsack sets and capacitated lot-sizing. *Mathematical Programming*, 95:53–69, 2003.

[7] G. L. Nemhauser and L. A. Wolsey. *Integer and Combinatorial Optimization*. John Wiley & Sons, 1988.

[8] G. L. Nemhauser and L. A. Wolsey. A recursive procedure for generating all cuts for 0-1 mixed integer programs. *Mathematical Programming*, 46:379–390, 1990.