A Connection between Coherent and Natural Risk Statistics

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Abstract
Recently Heyde, Kou and Peng [2] proposed the notion of a natural risk statistic associated with a finite sample that relaxes the sub-additivity assumption in the classical coherent risk statistic. In this note we establish a close connection between these two statistics and, as a by-product, obtain an alternative proof of the main result (Theorem 1) in [2] which states that a risk statistic is natural if and only if it can be represented as a supremum over a family of convex combinations of order statistics.

1 Introduction
A coherent risk statistic [1] corresponding to (data) vectors $x \in \mathbb{R}^n$ is a function $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies the following axioms

(A1) positive homogeneity: $\rho(\lambda x) = \lambda \rho(x)$ for all $\lambda \geq 0$ and $x \in \mathbb{R}^n$,

(A2) translation equivariance: $\rho(x + ae) = \rho(x) + a$ for all $a \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

(A3) monotonicity: $\rho(x) \leq \rho(y)$ for all $x, y \in \mathbb{R}^n$ satisfying $x_i \leq y_i$ for all $i = 1, \ldots, n$, and

(A4) subadditivity (or convexity): $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in \mathbb{R}^n$.

The above axioms imply the well-known representation theorem [1] that a risk statistic $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ is coherent if and only if

$$\rho(x) = \sup_{w \in W} \{ \langle w, x \rangle \} \quad \forall \ x \in \mathbb{R}^n$$

where $W \subseteq \{ w \in \mathbb{R}^n : \langle e, w \rangle = 1, \ w \geq 0 \}$ is a closed, convex set. The above result can be derived using conjugate duality [3] as shown by Ruszczyski and Shapiro [4] who prove the following result (stated here in $\mathbb{R}^n$).

1 Recall that a positively homogenous function is subadditive if and only if it is convex.
Theorem 1. (Theorem 2.2 in [4]) If \( f : \mathbb{R}^n \to \mathbb{R} \) is a lower semicontinuous, convex function then
\[
 f(x) = \sup_{w \in W} \{ (w, x) - f^*(w) \} \quad \forall x \in \mathbb{R}^n,
\]
where \( W = \text{dom} f^* \) is a closed convex set. Moreover

(a) \( f \) is monotonic if and only if \( w \geq 0 \) for all \( w \in W \);

(b) \( f \) is translation equivariant if and only if \( \langle e, w \rangle = 1 \) for all \( w \in W \);

(c) \( f \) is positively homogenous if and only if \( f(\lambda w) = \lambda f(w) \) for all \( w \in W \).

Since a real-valued coherent risk statistic is lower semicontinuous and convex, Proposition 1 immediately implies the representation (1).

Motivated by the criticism of the subadditivity axiom (A4), Heyde, Kou and Peng [2] proposed natural risk statistics. A natural risk statistic corresponding to (data) vectors \( x \in \mathbb{R}^n \) is a function \( \rho : \mathbb{R}^n \to \mathbb{R} \) that satisfies axioms (A1)-(A3) along with

(A4') subadditivity or convexity over \( S \): \( \rho(x + y) \leq \rho(x) + \rho(y) \) for all \( x, y \in S \), and

(A5) permutation invariance: \( \rho((x_1, \ldots, x_n)) = \rho((x_{i_1}, \ldots, x_{i_n})) \) for any permutation \( \{i_1, \ldots, i_n\} \) of \( \{1, \ldots, n\} \), for all \( x \in \mathbb{R}^n \).

In this note we show the following result.

Theorem 2. A function \( \hat{\rho} : \mathbb{R}^n \to \mathbb{R} \) is a natural risk statistic if and only if there exists a coherent risk statistic \( \rho : \mathbb{R}^n \to \mathbb{R} \) such that \( \hat{\rho}(x) = \rho(\Pi(x)) \) for all \( x \in \mathbb{R}^n \).

Combining Theorem 2 with representation (1) we obtain the main result of [2].

Corollary 3. A function \( \hat{\rho} : \mathbb{R}^n \to \mathbb{R} \) is a natural risk statistic if and only if it can be represented as
\[
\hat{\rho}(x) = \sup_{w \in W} \{ (w, \Pi(x)) \} \quad \forall x \in \mathbb{R}^n, \tag{2}
\]
for some closed convex set \( W \subseteq \{ w \in \mathbb{R}^n : \langle e, w \rangle = 1, \ w \geq 0 \} \).

Remark 1. [2, Theorem 1] did not specify that the set \( W \) in representation (2) is closed and convex.

2 Proof of Theorem 2

The “If” part:

Suppose \( \rho \) is coherent we have to verify conditions (A1)-(A3), (A4') and (A5) for \( \hat{\rho}(\cdot) := \rho(\Pi(\cdot)) \). It is easy to see that the mapping \( \Pi \) is permutation invariant and satisfies \( \Pi(\lambda x + a) = \lambda \Pi(x) + a \) for any \( \lambda \geq 0, a \in \mathbb{R} \) and \( x \in \mathbb{R}^n \). Moreover \( \Pi \) also preserves monotonicity. To see this let \( (y_{(1)}, y_{(2)}, \ldots, y_{(n)}) = (y_{i_1}, y_{i_2}, \ldots, y_{i_n}) \) where \( \{i_1, i_2, \ldots, i_n\} \) is a permutation of \( \{1, 2, \ldots, n\} \). Then for any \( x \leq y \), we have
\[
(\Pi(y))_k = y_{(k)} = \max\{y_{i_j} : j = 1, \ldots, k\} \geq \max\{x_{i_j} : j = 1, \ldots, k\} \geq x_{(k)} = (\Pi(x))_k \quad \forall k = 1, \ldots, n.
\]
Thus the conditions (A1)-(A3) and (A5) for \( \hat{\rho} \) are immediate from the conditions on \( \rho \). Condition (A4') follows from the facts: \( \Pi(x) = x \) for all \( x \in S \); \( x, y \in S \) implies \( x + y \in S \); and \( \rho \) is subadditive.

The “only if” part:

Given a natural risk statistic \( \hat{\rho} \), we need to demonstrate the existence of another function \( \rho \) that satisfies (A1)-(A4) and \( \hat{\rho}(x) = \rho(\Pi(x)) \) for all \( x \in \mathbb{R}^n \). For the last condition, it is sufficient to show
\[
(\text{A6}) \ \hat{\rho}(x) = \rho(\Pi(x)) = \rho(x) \quad \text{for all } x \in S.
\]
Indeed suppose (A6) is true but $\hat{\rho}(x) \neq \rho(\Pi(x))$ for some $x \notin S$. Then, since $y := \Pi(x) \in S$ we have (by (A6)) $\hat{\rho}(x) \neq \rho(y)$ which contradicts the permutation invariance (A5) property of $\hat{\rho}$ (since $y$ is just a permutation of $x$). Next we construct a function $\rho$ satisfying (A1)-(A3) and (A6).

Given $\hat{\rho}$ consider a function $\bar{\rho} : \mathbb{R}^n \rightarrow \mathbb{R}$ defined as

$$\bar{\rho}(x) := \hat{\rho}(x) + \delta_S(x) \quad \forall x \in \mathbb{R}^n.$$  

The constructed function $\bar{\rho}$ has the following properties.

Convexity: If either $x$ or $y$ is not in $S$ then the convexity condition

$$\bar{\rho}(\lambda x + (1 - \lambda)y) \leq \lambda \bar{\rho}(x) + (1 - \lambda)\bar{\rho}(y)$$

for any $\lambda \in (0, 1)$ holds vacuously. If both $x, y \in S$ then $\lambda x + (1 - \lambda)y \in S$ for all $\lambda \in [0, 1]$. Thus the convexity condition follows from the convexity of $\hat{\rho}$ over $S$.

Translation equivariance: Note that for any $a \in R$ we have $x \in S$ if and only if $x + ea \in S$. Thus if $x \in S$

$$\bar{\rho}(x + ae) = \hat{\rho}(x + ae) = \hat{\rho}(x) + a = \bar{\rho}(x) + a,$$

where the second equality follows from translation equivariance of $\hat{\rho}$, and if $x \notin S$

$$\bar{\rho}(x + ae) = +\infty = \bar{\rho}(x) + a.$$

Positive homogeneity: For any $\lambda > 0$ we have that $x \in S$ if and only if $\lambda x \in S$. Thus if $x \in S$

$$\bar{\rho}(\lambda x) = \hat{\rho}(\lambda x) = \lambda \hat{\rho}(x) = \lambda \bar{\rho}(x),$$

where the second equality follows from positive homogeneity of $\hat{\rho}$, and if $x \notin S$

$$\bar{\rho}(\lambda x) = +\infty = \lambda \bar{\rho}(x).$$

Remark 2. Note that $\bar{\rho}$ is not necessarily monotonic since $\bar{\rho}((1, 1, \ldots, 1, 0)) = +\infty \leq \bar{\rho}((1, 1, \ldots, 1, 1))$. However it is monotonic over $S$.

It follows from Theorem 1 that $\bar{\rho}$ can be represented as

$$\bar{\rho}(x) = \sup_{w \in W'} \{ \langle w, x \rangle \}$$

for some closed convex set $W' \subseteq \{ w \in \mathbb{R}^n : \langle e, w \rangle = 1 \}$. Consider now the following function

$$\rho(x) := \sup_{w \in W} \{ \langle w, x \rangle \} \quad \forall x \in \mathbb{R}^n,$$  \hspace{1cm} (3)

where $W := W' \cap \{ w \in \mathbb{R}^n : w \geq 0 \}$.

Let us first verify that $\rho$, as constructed above, satisfies condition (A6), i.e., $\rho(x) = \hat{\rho}(x)$ for all $x \in S$. First, consider $x \in \text{int}(S)$ and let $y^i = x - \epsilon e^i$ for $i = 1, \ldots, n$. Then, for $\epsilon > 0$ small enough, $y^i \in S$ for all $i = 1, \ldots, n$. Since $\bar{\rho}$ is monotonic over $S$ we have $\rho(x) \geq \bar{\rho}(y^i) \geq \langle w, y^i \rangle$ for all $w \in W'$, for all $i = 1, \ldots, n$. Thus we can include the redundant inequalities $\langle w, x \rangle \geq \langle w, y^i \rangle$ or equivalently $w_i \geq 0$ for all $i = 1, \ldots, n$ in the representation of $\bar{\rho}$ and get

$$\hat{\rho}(x) = \bar{\rho}(x) = \sup_{w \in W} \{ \langle w, x \rangle \} = \rho(x).$$

Next, consider $x \in \text{bd}(S)$. Consider a sequence $\{x^k\} \in \text{int}(S)$ converging to $x$. By continuity of $\hat{\rho}$ (which follows from conditions (A1)-(A3)) we have

$$\hat{\rho}(x) = \lim_{k \rightarrow \infty} \hat{\rho}(x^k) = \lim_{k \rightarrow \infty} \sup_{w \in W} \{ \langle w, x^k \rangle \} = \sup_{w \in W} \lim_{k \rightarrow \infty} \{ \langle w, x^k \rangle \} = \sup_{w \in W} \{ \langle w, x \rangle \} = \rho(x),$$

where we are able to interchange lim and sup since $W$ is a compact set.

Since $\rho(x)$ is finite for $x \in S$ the set $W$ is non-empty. Moreover since $W$ is compact the supremum in (3) is achieved for all $x \in \mathbb{R}^n$, and so $\rho$ is real-valued everywhere. Since it is the maximum of a collection of linear functions it is convex (thus satisfies (A4)). Finally, by Theorem 1 $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies conditions (A1)-(A3). This completes the proof.
References


