Two-stage stochastic integer programming: A brief introduction

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Abstract

Stochastic integer programming problems combine the difficulty of stochastic programming with integer programming. In this article, we briefly review some of the challenges in solving two-stage stochastic integer programming problems, and discuss the research progress towards these challenges.

1 Introduction

A standard form for a two-stage stochastic integer program (SIP) is as follows:

$$\begin{array}{ll} \min\limits_{x} & c^\top x + \mathbb{E}_P[Q(x,\omega)] \\ \text{s.t.} & Ax = b, \\ & x \in \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1} \end{array} \tag{1}$$

where

$$\begin{array}{rll} Q(x,\omega) := & \min\limits_{y} & q^\top y \\ & \text{s.t.} & Wy = h - Tx, \\ & & y \in \mathbb{R}_+^{n_2-p_2} \times \mathbb{Z}_+^{p_2}. \end{array} \tag{2}$$

Above, n_1, n_2, p_1, p_2 are nonnegative integers with $p_1 \le n_1$ and $p_2 \le n_2$, x represents the first-stage decisions and y represents the second-stage decisions, and ω represents the uncertain data for the second-stage (the parameters (q, h, T) are actual realization of the random data) with known distribution P. Throughout we assume that the matrix W is deterministic. Much of the structural results in stochastic integer programming are for this case. Problem (1) seeks a first-stage decision that minimizes first stage costs and the *expected* cost of second-stage (recourse) decisions. Note that both first- and second-stage variables are restricted to be mixed-integer ($\mathbb{R}$ and $\mathbb{Z}$ denotes reals and integers respectively). If the second stage variables are continuous (i.e. $p_2 = 0$) then problem (1) involves minimizing a convex objective subject to mixed-integer constraints. Much of the theory and algorithms for two-stage stochastic linear programs (that do not rely on convexity of the first stage constraints) are then applicable. By two-stage stochastic integer programs we refer to problems where $p_2 > 0$.

A variety of applications in energy planning [14], manufacturing [9], logistics [19], etc. can be formulated as two-stage stochastic integer programs of the form (1). In this article we briefly review some important progress in theory and algorithms for solving (1). There has been significant development in extensions of the two stage stochastic integer programming framework, e.g. to multistage setting [23] and involving risk averse objectives [28]. However this article is limited to models of the form (1).

2 Structure

In this section we discuss the structure of the expected value function

$$\mathbb{E}_P[Q(x, \omega)]$$

where Q is given by (2). Consider first the value function of a deterministic mixed integer program (MIP) as a function of the objective and right-hand-side vectors

$$\begin{aligned} \Phi(q, t) := \min_y \quad & q^\top y \\ \text{s.t.} \quad & Wy = t \\ & y \in \mathbb{R}_+^{n-p} \times \mathbb{Z}_+^p, \end{aligned} \tag{3}$$

where W is a $m \times n$ matrix, and t is an m-vector. We make the following assumptions.

(A1) For every $t \in \mathbb{R}^m$, there exists $y \in \mathbb{R}_+^{n-p} \times \mathbb{Z}_+^p$ such that $Wy = t$.

(A2) The entries of W are integers.

Assumption (A1) guarantees that the MIP (3) is feasible for all $t \in \mathbb{R}^m$, and assumption (A2) guarantees that (3) has an optimal solution if it is feasible and bounded. It is sufficient to assume that W is rational, however we assume integrality for the ease of exposition. Let us denote $\mathcal{Q} := \{q : \Pi(q) \neq \emptyset\}$ where $\Pi(q) := \{\pi \in \mathbb{R}^m : \ W^\top \pi \leq q\}$. Note that $\mathcal{Q}$ is a closed convex set in $\mathbb{R}^n$.

Theorem 2.1

(a) The value function $\Phi(\cdot, \cdot)$ is lower semicontinuous over $\mathcal{Q} \times \mathbb{R}^m$.

(b) Given $t \in \mathbb{R}^m$, the function $\Phi(\cdot, t)$ is continuous over $\mathcal{Q}$; and given $q \in \mathcal{Q}$ the function $\Phi(q, \cdot)$ is lower semicontinuous over $\mathbb{R}^m$.

(c) Given $q \in \mathcal{Q}$ the function $\Phi(q, \cdot)$ is continuous over $\mathbb{R}^m$ except over (at most) a countable union of hyper-planes.

(d) Given $q \in \mathcal{Q}$ the function $\Phi(q, \cdot)$ is subadditive over $\mathbb{R}^m$, i.e. $\Phi(q, t^1 + t^2) \leq \Phi(q, t^1) + \Phi(q, t^2)$.

(e) Given $t \in \mathbb{R}^m$, the function $\Phi(\cdot, t)$ is concave over $\mathcal{Q}$.

Next we consider the structure of $\mathbb{E}_P[Q(x, \omega)]$. This function is given by the expectation of the MIP value function $Q(x, \omega) = \Phi(q, h - Tx)$ with respect to the distribution P of the data $\omega = (q, h, T)$.

Theorem 2.2 *In addition to assumptions (A1) and (A2), assume that the random set* $\Pi(q)$ *is non-empty with probability one,* $\mathbb{E}[||q||||h||] < +\infty$ *and* $\mathbb{E}[||q||||T||] < +\infty$*. Then the expected value function* $\mathbb{E}_P[Q(x, \omega)]$ *is well defined, real-valued and lower semicontinuous on* $\mathbb{R}^{n_1}$*.*

The above result establishes sufficient conditions for the expected value function to be well defined and lower semicontinuous. Apart from this continuity property, very little can be said of the structure of expected value functions of MIPs in general. Figure 1 illustrates the objective function $c^\top x + \mathbb{E}_P[Q(x, \omega)]$ of a small instance of (1) with $n_1 = 2$. We see that the objective function has none of the desirable properties for optimization. It is non-convex and even discontinuous.

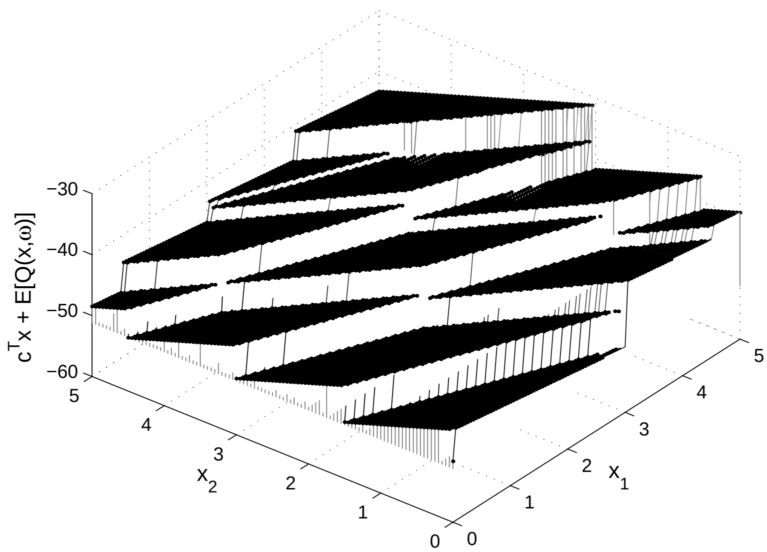

Figure 1: Objective function of a small SIP

When the distribution of the ω is absolutely continuous, i.e., it has a density function, then it can be further shown that $\mathbb{E}_P[Q(x, \omega)]$ is continuous on $\mathbb{R}^{n_1}$, however it is in general nonconvex.

3 Challenges and progress

There are three levels of difficulty in solving stochastic integer programs of the form (1).

1. *Evaluating the second-stage cost for a fixed first-stage decision and a particular realization of the uncertain parameters.* Note that this involves solving an instance of the second-stage problem (2) which may be an NP-hard integer program and involve significant computational difficulties.

2. *Evaluating the expected second-stage cost for a fixed first-stage decision.* If the uncertain parameters have continuous distribution, this involves integrating the value function $Q(x, \cdot)$ of an integer program, and is in general impossible. If the uncertain parameters have a discrete distribution, this involves solving a (typically huge) number of similar integer programs.

3. *Optimizing the expected second-stage cost.* As seen in Section 2, the value function of an integer program is non-convex and often discontinuous. Consequently, the expected second-stage cost function $\mathbb{E}[Q(\cdot, \omega)]$ is non-convex in x. The optimization of such a complex objective function poses severe difficulties.

In the following, we briefly mention some of the theoretical and algorithmic progress towards addressing the afore-mentioned difficulties in two-stage stochastic integer programming.

3.1 Difficulty 1

It is typically assumed that a *single* evaluation of the second-stage problem is somehow tractable. Without this assumption, very little progress in optimizing the expected value of this integer program is possible. There has been some work in obtaining approximate solutions to SIPs through approximate solutions to the integer second-stage problem, e.g. in [9].

3.2 Difficulty 2

Let us now consider the difficulty of evaluating the expected value of the second-stage integer program $\mathbb{E}[Q(x, \omega)]$ for a given first-stage decision x. As mentioned earlier, if the distribution of the uncertain parameters is continuous or if, in case of discrete distributions, the number of possible realizations is extremely large, then it is practically impossible to evaluate $\mathbb{E}[Q(x, \omega)]$ exactly. In this case, one has to resort to approximating the underlying probability distribution by a manageable distribution.

For example, if the underlying distribution is continuous one may approximate it via discretization. Theoretical stability results for SIPs (see [24, 25]) suggest that if the approximate distribution is not too "far" from the true distribution, then the optimal solution to the SIP involving the approximate distribution is close to the true optimal solution.

Alternatively, one may use statistical estimates of the expected value function via Monte Carlo sampling. This can be done in one of two ways. In *interior sampling* approaches, the estimation of $\mathbb{E}[Q(x, \omega)]$ is carried *within* the algorithm used to optimize this function. For example, in the *stochastic branch and bound algorithm* [21], the feasible domain of the first-stage variables x is recursively partitioned into subsets, and statistical upper and lower bounds on the objective function $c^\top x + \mathbb{E}[Q(x, \omega)]$ over these subsets are obtained via sampling. These bounds are used to discard inferior subsets of the feasible domain, and further partition the promising subsets to eventually isolate a subset containing an approximate optimal solution. In *exterior sampling* approaches, the sampling and optimization are decoupled. A Monte Carlo sample of the uncertain parameters is generated, and the expectation objective in the problem is replaced by a sample average. The resulting approximation of the problem is then solved, and its solution serves as a candidate solution to the true problem. By repeating the sampling-optimization procedure several times, it is possible to obtain statistical confidence intervals on the obtained candidate solutions. It can be shown that the number of samples needed to get a fairly accurate solution with high probability is not too large. Discussion of theoretical and algorithmic issues pertaining to the above approach in the context of SIP can be found in [16, 1].

Regardless of how the underlying distribution is approximated, an evaluation of the expected second-stage objective value (under the approximate distribution) requires solving many similar integer programs. Owing to the absence of a computationally useful duality theory for integer programming, it is very difficult to take advantage of the similarities in the different second-stage IPs. When the second-stage variables are pure integer, several proposals for using Gröbner basis and other test set based methods from computational algebra for exploiting IP problem similarity have been put forth [11, 27, 33]. For the case of mixed-integer subproblems, if a cutting plane method is used, then under some conditions it is possible to transform a cut (or a valid inequality) derived for one of the second-stage subproblems into a cut for another subproblem by exploiting similarity [6, 12].

3.3 Difficulty 3

Much of the development in SIP has been towards the difficulty of optimizing $f(x) := c^\top x + \mathbb{E}[Q(x,\omega)]$, i.e., the sum of the first-stage and the expected second-stage costs. We classify these developments as follows.

Convex approximations of the value function

Consider an SIP with *simple integer recourse* with only right hand side uncertainty, i.e. q, T are deterministic, the second-stage problem has a special structure such that the value function Q is separable and is given by

$$Q(x,\omega) = \sum_{i=1}^{m} \phi_i(h_i - T_i x)$$

and each separable piece ϕ_i is of the form

$$\phi(t) = q^+ \lceil t \rceil_+ + q^- \lfloor t \rfloor_-, \quad t \in \mathbb{R},$$

where $\lceil t \rceil_+ = \max\{0, \lceil t \rceil\}$ and $\lfloor t \rfloor_- = \max\{0, -\lfloor t \rfloor\}$. In this case, a single evaluation of $f(x)$ is easy, however owing to the non-convex nature of ϕ, the function $f(x)$ is difficult to optimize. Fortunately, it has been shown [10] that an expectation of the continuous counterpart of the simple integer recourse function $Q(x,\omega)$ (i.e. where the recourse variables are continuous instead of being discrete valued) under a particular class of distributions of ω provides the tightest convex under-estimator for $\mathbb{E}[Q(x,\omega)]$ over its entire domain. Recently, similar results for constructing convex approximations of general integer recourse functions (SIPs involving pure integer second-stage variables) by perturbing the underlying distribution have been obtained [34]. These convex approximating functions are amenable for optimization and can be used to provide strong lower bounds within some of the algorithms for optimizing $f(x)$

discussed next. An open problem is the refinement of these approximations using additional constraints on the first stage variables, e.g. bounds on x.

Stage-wise decomposition algorithms

This class of algorithms adopt the natural viewpoint of optimizing the objective function $f(x) := c^\top x + \mathbb{E}[Q(x,\omega)]$ over the set of feasible first-stage decisions (say denoted by X). Note that $\mathbb{E}[Q(x,\omega)]$ is not available in closed-form, nor is it suited for direct optimization. Typical algorithms in this class progress in the following manner. In an iteration i, the algorithm builds and/or refines a computationally tractable approximation (typically an under-estimator) $\hat{Q}_i(x)$ of $\mathbb{E}[Q(x,\omega)]$. The under-estimating function $c^\top x + \hat{Q}_i(x)$ is optimized with respect to the first-stage variables (this optimization problem is often referred to as the *master problem*) to obtain a lower bound on the true optimal objective value as well as to provide a candidate first-stage solution x^i. Corresponding to the candidate solution, the second-stage expected value function $\mathbb{E}[Q(x^i,\omega)]$ is evaluated. Assuming that the distribution of ω is discrete, this step involves independent solution of the second-stage problems for each realization of ω, allowing for a computationally convenient decomposition. The value $c^\top x^i + \mathbb{E}[Q(x^i,\omega)]$ provides an upper bound on the optimal objective value. The evaluation of $\mathbb{E}[Q(x^i,\omega)]$ also provides information on how the approximation $\hat{Q}_i$ is to be updated/refined to $\hat{Q}_{i+1}$ for the master problem of iteration $i+1$. The process continues until the bounds have converged. The details of the various stage-wise decomposition algorithms differ mainly in how the approximation $\hat{Q}_i$ is constructed and updated.

For SIPs with binary first-stage variables and mixed-integer second-stage variables, the *integer L-shaped method* [18] approximates the second-stage value function by linear "cuts" that are exact at the binary solution where the cut is generated and are under-estimates at other binary solutions. The optimization of the master problem, i.e. optimizing $c^\top x + \hat{Q}_i(x)$ with respect to the first-stage binary variables, is carried out via a branch-and-bound strategy. As soon as a new first-stage binary solution is encountered in the branch-and-bound search, the second-stage subproblems are solved to generate a new cut and to refine the approximation $\hat{Q}_i$. The integer L-shaped method requires that the second-stage integer problems (corresponding to the current candidate first-stage solution x^i) are all solved to optimality before a valid cut can be generated. Recall that typical integer programming algorithms progress by solving a sequence of intermediate linear programming problems. Using disjunctive programming techniques, it is possible to derive cuts from the solutions to these intermediate LPs that are valid under-estimators of $\mathbb{E}[Q(x,\omega)]$ at all binary first-stage solutions [12, 29]. This avoids solving difficult integer second-stage problems to optimality in all iterations of the algorithm, providing significant computational advantage.

For SIPs where the first-stage variables are not necessarily all binary, dual functions from the second-stage integer program can, in principle, be used to construct cuts to build the approximation $\hat{Q}_i$ [8]. Owing to the non-convex nature of IP dual functions, the cuts are no longer linear, resulting in a non-convex master problem. If the second-stage variables are pure integer (and the first-stage variables are mixed-integer), then it can be shown that $\mathbb{E}[Q(x,\omega)]$ is piece-wise constant over subsets that form a partitioning of the feasible region of x [27]. Optimization of $c^\top x + \mathbb{E}[Q(x,\omega)]$ over such a subset is easy. This leads immediately to a scheme where the subsets are enumerated, and the one over which the objective function value is least is chosen. By exploiting certain monotonicity properties, the subsets can be enumerated efficiently within a branch-and-bound strategy [2]. Additional properties of the MIP value function $Q(x, \omega)$, such as sub-additivity, can be used to improve the method [17].

Scenario-wise decomposition

Assuming the distribution of ω is discrete, i.e. the random parameter takes one of a finite set of values (scenarios) $\{\omega_1, \ldots, \omega_S\}$ having probabilities $\{p_1, \ldots, p_S\}$, the two-stage SIP can be re-formulated as follows

$$\begin{array}{ll} \min & \sum_{s=1}^{S} p_s(c^\top x_s + q_s^\top y_s) \\ \text{s.t.} & Ax_s = b \quad s = 1, \ldots, S, \\ & T_s x_s + W_s y_s = h_s \quad s = 1, \ldots, S, \\ & x_s \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1} \quad s = 1, \ldots, S, \\ & y_s \in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2} \quad s = 1, \ldots, S, \\ & x_1 = x_2 = \cdots = x_S. \end{array}$$

Note that copies of the first-stage variable have been introduced for each scenario. The last constraint, known as the *non-anticipativity* constraint, guarantee that the first-stage variables are identical across the different scenarios. Consider the Lagrangian dual problem obtained by relaxing the non-anticipativity constraints through the introduction of Lagrange multipliers. Note that for a given set of multipliers, the problem is separable by scenarios, thus the dual function can be evaluated in a decomposed manner. Optimization of the dual function can be performed using standard non-smooth optimization techniques. However, owing to the non-convexities, there exists a duality gap, and one needs to resort to a branch-and-bound strategy to prove optimality [7].

Cuts for deterministic equivalent MIP

If the number of scenarios (assuming a finite distribution setting) is not astronomical then a possible approach is to directly solve the deterministic equivalent MIP,

$$\begin{array}{ll} \min & c^\top x + \sum_{s=1}^{S} p_s q_s^\top y_s \\ \text{s.t.} & Ax = b \\ & T_s x + W_s y_s = h_s \quad s = 1, \ldots, S, \\ & x \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1} \quad s = 1, \ldots, S, \\ & y_s \in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2} \quad s = 1, \ldots, S, \end{array}$$

using an off-the-shelf solver such as CPLEX or EXPRESS.

One of the most important features of these solvers is the generation of cutting planes by analyzing the polyhedral structure of the problem. Such cuts significantly improve the linear programming relaxation of the problem, and expedite linear programming based branch-and-bound search. Typically such cuts are generated by analyzing a sing row of the constraint system and do not combine information from multiple rows. In a stochastic integer program the constraint system is repeated with small changes for each scenario. Thus it is possible to effectively combine cuts from multiple rows corresponding to different scenarios [30]. Such cuts have been shown to significantly improve performance over single row cuts. An open issue is that such multi-row cuts links second stage variables across multiple scenarios, and hence destroys decomposability. Algorithmic techniques that can bypass this issue pose an important challenge.

4 Concluding Remarks

This article offers a very limited view of some of the theoretical and algorithmic concepts in SIP. The concepts alluded to here have been significantly enriched and extended in recent years. We have not discussed the large number of important developments in application-specific areas of SIP (see, e.g., [31] for a bibliography of applications of SIP), and also omitted the important progress made in approximation algorithms for SIP. We hope that this simple introduction will pique the readers interest towards further exploration of SIP.

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