Recap

- Last class (January 20, 2004)
  - Duopoly models
  - Multistage games with observed actions
  - Subgame perfect equilibrium
  - Extensive form of a game
  - Two-stage prisoner's dilemma
- Today (January 22, 2004)
  - Finitely repeated games
  - Infinitely repeated games
    - Prisoner’s dilemma
    - Friedman’s Theorem
    - Repeated Cournot game

Repeated games

- Let \( G = \{ A^1, ..., A^n; \pi^1, ..., \pi^n \} \) denote a static game of complete information in which players 1, ..., \( n \) simultaneously choose their actions \( a^1, ..., a^n \) from action spaces \( A^1, ..., A^n \) and receive payoffs \( \pi^1(a^1, ..., a^n), ..., \pi^n(a^1, ..., a^n) \). We call \( G \) the stage game of the repeated game.
- Given a stage game \( G \), let \( G(T) \) denote the finitely repeated game in which \( G \) is played \( T \) times, with the outcomes of all preceding plays observed before the next play begins. The payoffs for \( G(T) \) are the (discounted) sum of the payoffs from the \( T \) stage games.
Repeated games

- **Result:** If the stage game $G$ has a unique Nash equilibrium then for any finite $T$, the repeated game $G(T)$ has a unique subgame-perfect outcome: the Nash equilibrium of $G$ is played in every stage.

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Example
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<table>
<thead>
<tr>
<th>Player 1</th>
<th>L</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>1, 1</td>
<td>5, 0</td>
<td>0, 0</td>
</tr>
<tr>
<td>M</td>
<td>0, 5</td>
<td>4, 4</td>
<td>0, 0</td>
</tr>
<tr>
<td>R</td>
<td>0, 0</td>
<td>0, 0</td>
<td>3, 3</td>
</tr>
</tbody>
</table>

- The stage game is played twice
- The first-stage outcome is observed before the second stage begins
Example

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
</tr>
<tr>
<td>Player 1</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>1,1</td>
</tr>
<tr>
<td>M</td>
<td>0,5</td>
</tr>
<tr>
<td>R</td>
<td>0,0</td>
</tr>
</tbody>
</table>

- Partial strategy for stage 2:
  - Play R in stage 2 if stage 1 outcome is (M,M); otherwise, play L in stage 2.

Example

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
</tr>
<tr>
<td>Player 1</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>2,2</td>
</tr>
<tr>
<td>M</td>
<td>1,6</td>
</tr>
<tr>
<td>R</td>
<td>1,1</td>
</tr>
</tbody>
</table>

- Modified stage 1 game
- Subgame perfect equilibria:
  
  \[[(L,L),(L,L)] \quad [(M,M),(R,R)] \quad [(R,R),(L,L)]\]
Observation

- Let $G$ be a static game of complete information with multiple Nash equilibria. There may be subgame-perfect outcomes of the repeated game $G(T)$ in which for any $t < T$, the outcome in stage $t$ is not a Nash equilibrium of $G$.

Definitions

- In the finitely repeated game $G(T)$, a player’s strategy specifies the player’s actions in each stage, for each possible history of play through the previous stages.

- In the finitely repeated game $G(T)$, a subgame beginning at stage $t+1$ is the repeated game in which $G$ is played $T-t$ times, denoted by $G(T-t)$. 
Example

All possible outcomes (histories) at the end of stage 1:
(L,L) (L,M) (L,R) (M,L) (M,M) (M,R) (R,L) (R,M) (R,R)


Play M in the first stage; Play L in the second stage unless the first stage outcome is (M,M)

Infinitely Repeated Prisoner’s Dilemma

The game is repeated infinitely

For each t, the outcomes of the previous t-1 stage games are observed

Payoffs?
Discounted payoffs

- Let $\delta$ be the value today of a dollar to be received one stage later
  - E.g., $\delta = 1/(1+r)$ where $r$ is the interest rate per stage
  - Given the discount factor $\delta$ the present value of the infinite sequence of payoffs $\pi_1, \pi_2, \pi_3, \ldots$ is
    \[ \pi_1 + \delta \pi_2 + \delta^2 \pi_3 + \ldots = \sum_{t=1}^{\infty} \delta^{t-1} \pi_t. \]

Discounted payoffs

- Suppose after each stage is played, the game continues to the next stage with probability $1-p$ and stops with probability $p$.
- Expected present value of next stage’s payoff $(1-p)\pi/(1+r)$.
- Expected present value of the payoff two stages later $(1-p)^2 \pi/(1+r)^2$.
- Let $\delta = (1-p)/(1+r)$
- $\pi_1 + \delta \pi_2 + \delta^2 \pi_3 + \ldots$ reflects the time value of money and the possibility that the game will end
Average payoffs

- \( V = \pi_1 + \delta \pi_2 + \delta^2 \pi_3 + \ldots = \sum_{t=1}^{\infty} \delta^{t-1} \pi_t \)
- If we received an “average” payoff of \( \pi \) in every stage, then
  \( V = \pi + \delta \pi + \delta^2 \pi + \ldots = \pi (1 + \delta + \delta^2 + \ldots) = \pi / (1 - \delta) \)
- \( \pi / (1 - \delta) = \sum_{t=1}^{\infty} \delta^{t-1} \pi_t \).
  \[ \pi = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \pi_t. \]

Example: Payoffs 4 4 4 4 4 ....
Average payoff = 4 Net present value = 4 / (1 - \( \delta \))

Infinitely repeated games

- Given a stage game \( G \), let \( G(\infty, \delta) \) denote the infinitely repeated game in which \( G \) is repeated forever and the players share discount factor \( \delta \). For each \( t \), the outcomes of the \( t \)-1 preceding plays of the stage game are observed before the \( t \)-th stage begins. Each player's payoff in \( G(\infty, \delta) \) is the present value of the player's payoffs from the infinite sequence of stage games.
Infinitely Repeated Prisoner’s Dilemma

<table>
<thead>
<tr>
<th>Prisoner 1</th>
<th>C (cooperate)</th>
<th>D (defect)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>4, 4</td>
<td>0, 5</td>
</tr>
<tr>
<td>D</td>
<td>5, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

Strategy:
Play C in the first stage. In the tth stage, if the outcome of all t-1 preceding stages has been (C,C), then play C; otherwise, play D.

Definitions

In an infinitely repeated game $G(\infty, \delta)$, a player’s strategy specifies the player’s actions in each stage, for each possible history of play through the previous stages.

In the infinitely repeated game $G(\infty, \delta)$, each subgame beginning at stage t+1 is is identical to the original game $G(\infty, \delta)$. 
Trigger strategies for Prisoner’s Dilemma

Assuming player 1 adopts the trigger strategy, what is the best response of player 2?

Player 2 best response in stage t+1:
- If the outcome in stage t is (D,D)
  - Play D forever
- If the outcomes of stages 1,...,t are (C,C)
  - Play D \rightarrow receive 5 in this stage, switch to (D,D) forever after \rightarrow 5 + \delta.1 + \delta^2.1 + \delta^3.1 + \ldots = 5 + \delta/(1-\delta)
  - Play C \rightarrow receive 4 in this stage, and face the exact same game (same choices) in stage t+2!

Let V be the payoff of player 2 from making the optimal choice in the subgame starting in stage t+1, given that the outcomes in the previous stages have been (C,C)
- Play C \rightarrow V=4+ \delta V \rightarrow V = 4/(1-\delta)
- Play D \rightarrow V= 5+ \delta/(1-\delta)

Play C if 4/(1-\delta) \geq 5+ \delta/(1-\delta) \rightarrow \delta \geq 1/4
Trigger strategies for Prisoner’s Dilemma

- Two types of subgames:
  (i) Subgames where the outcomes of all previous stages have been (C,C)
      The trigger strategies are Nash equilibrium for this class of subgames, as well as for the original game.
  (ii) Subgames where the outcome of at least one earlier stage differs from (C,C)
      Player’s strategies are to repeat (D,D) forever, which is also a Nash equilibrium for the original game.

Observation

- Even if the stage game G has a unique Nash equilibrium, there may be subgame-perfect outcomes of the infinitely repeated game in which no stage’s outcome is a Nash equilibrium of G.
Feasible payoffs in the stage game

- The payoffs \((\pi^1, \pi^2, \ldots, \pi^n)\) are feasible in the stage game \(G\) if they are a convex combination of the pure-strategy payoffs of \(G\).

Example

- What are the pure-strategy payoffs?
  - \((4,4)\) \((0,5)\) \((5,0)\) \((1,1)\)
Feasible payoffs in the Prisoner’s Dilemma

Recap

- Last class (January 22, 2004)
  - Finitely repeated games
  - Infinitely repeated games
    - Prisoner’s dilemma
    - Friedman’s Theorem
- Today (January 27, 2004)
  - Proof of Friedman’s theorem
  - Repeated Cournot game
  - Wage setting
Friedman’s Theorem

Let G be a finite static game of complete information. Let \((e^1, e^2, \ldots, e^n)\) denote the payoffs from a Nash equilibrium of G and let \((x^1, x^2, \ldots, x^n)\) denote any other feasible payoffs from G. If \(x^j > e^j\) for every player \(j\) and if \(\delta\) is sufficiently close to 1, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game \(G(\infty, \delta)\) that achieves \((x^1, x^2, \ldots, x^n)\) as the average payoff.

Feasible payoffs in the Prisoner’s Dilemma

- Player 1
  - Payoffs:
    - (0,5)
    - (5,0)
    - (4,4)
    - (1,1)

- Player 2
  - Payoffs:
    - (0,5)
    - (5,0)
    - (4,4)
    - (1,1)
Proof of Friedman’s Theorem

- Let \((a_{e1}, a_{e2}, \ldots, a_{en})\) be the Nash equilibrium of \(G\) that yields the equilibrium payoffs \((e^1, e^2, \ldots, e^n)\).
- Let \((a_{x1}, a_{x2}, \ldots, a_{xn})\) be the collection of actions that yields the equilibrium payoffs \((x^1, x^2, \ldots, x^n)\).
- Trigger strategy for player \(i\):
  - Play \(a_{xi}\) in the first stage. In the \(t^{th}\) stage, if the outcome of all \(t-1\) preceding stages has been \((a_{x1}, a_{x2}, \ldots, a_{xn})\) then play \(a_{xi}\); otherwise, play \(a_{ei}\).
- Show that the trigger strategies induce a NE
- Show that the equilibrium is subgame perfect

Proof of Friedman’s Theorem (cont.)

- Suppose all players other than player \(i\) use the trigger strategy.
- Best response of player \(i\) in stage \(t\):
  - If the outcome of the previous stage differs from \((a_{x1}, a_{x2}, \ldots, a_{xn})\)
    - Play \(a_{ei}\) forever
  - If the outcomes of all previous stages are \((a_{x1}, a_{x2}, \ldots, a_{xn})\)

\[
\max_{a_i \in A_i} \pi^i(a_{x1}, \ldots, a_{x,i-1}, a_i, a_{x,i+1}, \ldots, a_{xn}) = d^i
\]

\[
d^i \geq \pi^i(a_{x1}, \ldots, a_{x,i-1}, a_{xi}, a_{x,i+1}, \ldots, a_{xn}) > \pi^i(a_{e1}, \ldots, a_{en}) = e^i
\]
Proof of Friedman’s Theorem (cont.)

- If player i deviates in stage t by choosing $a_{di}$:
  - Payoff in stage t: $d_i$
  - Payoff in future stages:
    - $\delta e_i + \delta^2 e_i + \ldots = \delta e_i/(1-\delta)$
  - Total (discounted) payoff: $V_i = d_i + \delta e_i/(1-\delta)$

- If player i plays $a_{xi}$ in stage t:
  - Receive a payoff $x_i$ in this stage, face the same game in the next stage.
    - $V_i = x_i + \delta V_i \rightarrow V_i = x_i/(1-\delta)$.
  - Playing $x_i$ is optimal if and only if
    - $x_i/(1-\delta) \geq d_i + \delta e_i/(1-\delta) \implies \delta \geq (d_i - x_i)/(d_i - e_i)$

Proof of Friedman’s Theorem (cont.)

- It is Nash equilibrium for all players to play the trigger strategy if and only if
  - $\delta \geq \max_i (d_i - x_i)/(d_i - e_i)$

- Subgame perfectness:
  - If the outcome of the previous stage differs from
    - $(a_{x1}, a_{x2}, \ldots, a_{xn})$
      - Play $a_{ei}$ forever
  - If the outcomes of all previous stages are $(a_{x1}, a_{x2}, \ldots, a_{xn})$
    - Play the trigger strategy
Repeated Cournot Game

- Cournot stage game
  - Two competing firms, selling a homogeneous good
  - The marginal cost of producing each unit of the good: $c$
  - The market price, $P$ is determined by (inverse) market demand:
    - $P = a-Q$ if $a>Q$, $P=0$ otherwise.
  - Each firm decides on the quantity to sell (market share): $q_1$ and $q_2$
  - $Q = q_1 + q_2$ total market demand
  - Both firms seek to maximize profits
- Unique NE of the stage game: $q^C = (a-c)/3$  $Q = 2(a-c)/3$
- Monopoly quantity: $q^M = (a-c)/2$

Repeated Cournot Game (cont.)

- The stage game is repeated infinitely many times
- The firms have discount factor $\delta$
- Trigger strategy
  - Produce half the monopoly quantity, $q^M/2$, in the first stage. In the $t^{th}$ stage, produce $q^M/2$ if both firms have produced $q^M/2$ in all previous stages; otherwise, produce $q^C$.
- Show that the trigger strategy induces a subgame perfect NE.
Repeated Cournot Game (cont.)

- Profit of one firm
  - If both produce $q^M/2$: \[(a-c)^2/8 = \pi^M/2\]
  - If both produce $q^C$: \[(a-c)^2/9 = \pi^C\]
- Best response of firm $i$:
  - If the last stage outcome is other than $(q^M/2, q^M/2)$
    - Play $q^C$ forever
  - If all previous stages’ outcomes are $(q^M/2, q^M/2)$
    - Deviate
      \[
      \max (a-q_i-q^M/2-c) q_i \rightarrow q_i = 3(a-c)/8 \quad \pi^D = 9(a-c)^2/64
      \]
      \[
      V = \pi^D + \pi^C \delta /(1- \delta)
      \]
    - Play $q^M/2$
      \[
      V = \pi^M/2 + \delta V \rightarrow V = \pi^M /2(1- \delta)
      \]
  - Playing the trigger strategy is NE iff
    \[
    \pi^M /2(1- \delta) \geq \pi^D + \pi^C \delta /(1- \delta) \rightarrow \delta \geq 9/17
    \]
Repeated Cournot game

- Trigger strategy
  - Produce half the monopoly quantity, $q^M/2$, in the first stage. In the $t^{th}$ stage, produce $q^M/2$ if both firms have produced $q^M/2$ in all previous stages; otherwise, produce $q^C$.
  - Playing the trigger strategy is SPNE iff $\delta \geq 9/17$

What if $\delta < 9/17$?

Repeated Cournot Game (cont.)

- Trigger strategy
  - Produce $q^*$, in the first stage. In the $t^{th}$ stage, produce $q^*$ if both firms have produced $q^*$ in all previous stages; otherwise, produce $q^C$.

- Profit of one firm
  - If both produce $q^*$: $(a-2q^*-c) q^* = \pi^*$
  - If both produce $q^C$: $(a-c)^2/9 = \pi^C$
  - If firm $j$ produces $q^*$ and firm $i$ deviates:
    $\max (a- q_i- q^*-c) q_i \rightarrow q_i = (a- q^*-c)/2$
    $\pi^D = (a- q^*-c)^2/4$
Repeated Cournot Game (cont.)

- Best response of firm i:
  - If the last stage outcome is other than \((q^*, q^*)\)
    - Play \(q^C\) forever
  - If all previous stages’ outcomes are \((q^*, q^*)\)
    - Deviate: \(V^i = \pi_D + \pi_C \delta /(1- \delta)\)
    - Play \(q^*: \quad V^i = \pi^* + \delta V^i \rightarrow V^i = \pi^*/(1- \delta)\)
    - Playing the trigger strategy is NE iff
      \[\pi^*/(1- \delta) \geq \pi_D + \pi_C \delta /(1- \delta)\]
  
Substitute and solve for \(q^*\):
\[q^* = (9-5\delta)(a-c)/3(9-\delta)\]

Recall: \(q^C=(a-c)/3\quad q^M=(a-c)/2\)

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Example: Wage setting

- Stage game
  - One firm, one worker
  - The firm offers the worker a wage, \(w\)
  - The worker accepts or rejects the firm’s offer
    - Reject: the worker becomes self-employed at wage \(w_0\)
    - Accept: Work (disutility \(e\)), or Shirk (disutility 0)
      - If the worker works (supplies effort): Output is high=\(y\)
      - If the worker shirks: Output is high with probability \(p\), and low=0 with probability 1-\(p\)
  - The firm does not observe the worker’s effort decision
  - The output of the worker is observed by both parties
Example: Wage setting (cont.)

- Payoffs (Firm, Worker)
  - Work (Supply effort)
    - High output: (y-w, w-e)
  - Shirk
    - High output: (y-w, w)
    - Low output: (-w, w)

- What is the subgame-perfect equilibrium in this stage game?
  - For any \( w \geq w_0 \), worker accepts employment and shirks
  - Firm offers \( w=0 \) (or any other \( w<w_0 \))

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Example: Wage setting (cont.)

- Strategies
  - Firm: Offer \( w=w^* \) in the first stage.
    In stage \( t \),
    - offer \( w=w^* \) if the history of play is high-wage, high-output (all previous offers have been \( w^* \), all previous offers have been accepted, and all previous outputs have been high)
    - otherwise, offer \( w=0 \)
  - Worker:
    - If \( w>w_0 \), accept the firm’s offer and supply effort if the history of play, including the current offer, is high-wage, high-output (shirk otherwise)
    - If \( w<w_0 \), choose self-employment
Example: Wage setting (cont.)

- Suppose firm offers $w^* \geq w_0$
  - Worker accepts
  - Work (Supply effort)
    \[ V_e = (w^*-e) + \delta V_e \rightarrow V_e = \frac{(w^*-e)}{(1-\delta)} \]
  - Shirk
    \[ V_s = w^* + \delta(pV_s + (1-p) w_0/(1-\delta)) \rightarrow \]
    \[ V_s = \frac{[(1-\delta)w^* + \delta(1-p) w_0]}{(1-\delta)(1-p)} \]
  - Worker should supply effort if $V_e \geq V_s$
    \[ w^* \geq w_0 + e + e(1-\delta)/\delta(1-p) \]

If $p=0$: \[ \frac{(w^*-e)}{(1-\delta)} \geq w^* + \frac{w_0 \delta}{(1-\delta)} \]

Example: Wage setting (cont.)

- When is it the best response for the firm to offer $w^*$?
  - From worker’s best response
    \[ w^* \geq w_0 + e + e(1-\delta)/\delta(1-p) \] (1)
  - $y \geq w^*$
    \[ \rightarrow y \geq w_0 + e + e(1-\delta)/\delta(1-p) \] (2)

The strategies induce a NE if (1) and (2) hold.

Is this a SPNE?
Example: Wage setting (cont.)

- What are the subgames?
  - Subgames beginning after a high-wage, high-output history
  - Subgames beginning after all other histories