Recap

- Last class (January 15, 2004)
  - Examples of games with continuous action sets
    - Tragedy of the commons
  - Duopoly models: Cournot and Bertrand
  - Comparison of duopoly models with Monopoly
- Today (January 20, 2004)
  - Duopoly models
    - Stackelberg - Comparison with Cournot, Bertrand, and Monopoly
  - Multistage games with observed actions
  - Subgame perfect equilibrium
  - Extensive form of a game

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Stackelberg Model

- Two competing firms, selling a homogeneous good
- The *marginal cost* of producing each unit of the good: $c_1$ and $c_2$
- Firm 1 moves first and decides on the quantity to sell: $q_1$
- Firm 2 moves next and after seeing $q_1$, decides on the quantity to sell: $q_2$
- $Q = q_1 + q_2$ total market demand
- The market price, $P$ is determined by (inverse) market demand:
  - $P = a - bQ$ if $a > bQ$, $P = 0$ otherwise.
- Both firms seek to maximize profits
Stackelberg Model

- Q_j: the space of feasible q_j’s, j=1,2
- Strategies of firm 2:
  \( s^2: Q_1 \rightarrow Q_2 \)
- Strategies of firm 1: \( q_1 \in Q_1 \)
- Outcomes and payoffs in pure strategies
  \((q_1, q_2) = (q_1, s^2(q_1))\)
  \(\pi^j(q_1, q_2) = [a-b(q_1+q_2)-c_j] q_j\)

Stackelberg Model: Strategy of Firm 2

- Suppose firm 1 produces \(q_1\)
- Firm 2’s profits, if it produces \(q_2\) are:
  \(\pi_2 = (P-c)q_2 = [a-b(q_1+q_2)]q_2 - c_2q_2\)
  = (Residual) revenue – Cost
- First order conditions:
  \(d \pi_2/dq_2 = a - 2bq_2 - bq_1 - c_2 = RMR - MC = 0 \rightarrow\)
  \(q_2 = (a-c_2)/2b - q_1/2 = R^2(q_1)\)

\[ s^2 = R^2(q_1) \] Strategy of firm 2
Stackelberg Model: Firm 1’s decision

- Firm 1’s profits, if it produces $q_1$ are:
  \[ \pi_1 = (P-c)q_1 = [a-b(q_1 + q_2)]q_1 - c_1 q_1 \]
- We know that from the best response of Firm 2:
  \[ q_2 = (a-c_2)/2b - q_1/2 \]
- Substitute $q_2$ into $\pi_1$:
  \[ \pi_1 = [a-b(q_1 + (a-c_2)/2b - q_1/2)]q_1 - c_1 q_1 \]
  \[ \quad = [(a+c_2)/2-(b/2)q_1-c_1]q_1 \]
- From FOC:
  \[ d\pi_1/dq_1 = (a+c_2)/2-bq_1-c_1 = 0 \rightarrow q_1 = (a-2c_1+c_2)/2b \]

Stackelberg Equilibrium

- We have Firm 1’s profits, if it produces $q_1$:
  \[ q_1 = (a-2c_1+c_2)/2b \]
  And firm 2’s best response
  \[ q_2 = (a-c_2)/2b - q_1/2 \]
- Therefore:
  \[ q_2 = (a+2c_1-3c_2)/4b \]
- If $c_1 = c_2 = c$
  \[ q_1 = (a-c)/2b \]
  \[ q_2 = (a-c)/4b \]
  \[ Q = 3(a-c)/4b \]
Cournot vs. Stackelberg vs. Bertrand

<table>
<thead>
<tr>
<th></th>
<th>Bertrand</th>
<th>Stackelberg</th>
<th>Cournot</th>
<th>Monopoly</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>c</td>
<td>(a+3c)/4</td>
<td>(a+2c)/3</td>
<td>(a+c)/2</td>
</tr>
<tr>
<td>Quantity</td>
<td>(a-c)/b</td>
<td>3(a-c)/4b</td>
<td>(a-c)/3b</td>
<td>(a-c)/2b</td>
</tr>
<tr>
<td></td>
<td></td>
<td>((a-c)/2b+(a-c)/4b)</td>
<td>2(a-c)/3b</td>
<td></td>
</tr>
<tr>
<td>Total Firm Profits</td>
<td>0</td>
<td>3(a-c)^2/16b</td>
<td>2(a-c)^2/9b</td>
<td>(a-c)^2/4b</td>
</tr>
</tbody>
</table>

Example: Stackelberg Competition

- \( P = 130-(q_1+q_2) \), so \( a=130, \ b=1 \)
- \( c_1 = c_2 = c = 10 \)
- Firm 2: \( q_2=(a-c_2)/2b - q_1/2 = 60 - q_1/2 \)
- Firm 1:
  - Residual demand: \( a-b(q_1+q_2) = 70-q_1/2 \)
  - \( \text{RMR} = (a+ c_2)/2-bq_1 = 70-q_1 \)
  - Set \( \text{RMR}=\text{MC} \)
    - \( 70-q_1 = 10 \rightarrow q_1 = 60 \)
- Market price and demand
  - \( Q=90 \quad P=40 \)
Stackelberg Competition: Firm 1 strategy

\[ P = 130 - Q \]

Residual demand: \[ P = 70 - \frac{q}{2} \]

\[ RMR = 70 - q \]

\[ MC = 10 \]

Consumer surplus = 4050

Firm profits = 2700

Deadweight loss = 450
### Monopoly vs. Cournot vs. Bertrand vs. Stackelberg

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<th>Monopoly</th>
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<tr>
<td>Price</td>
<td>10</td>
<td>40</td>
<td>50</td>
<td>70</td>
</tr>
<tr>
<td>Quantity</td>
<td>120</td>
<td>90 (60+30)</td>
<td>80</td>
<td>60</td>
</tr>
<tr>
<td>Total Firm Profits</td>
<td>0</td>
<td>2700 (1800+900)</td>
<td>3200</td>
<td>3600</td>
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</tbody>
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- Firm profits and prices:  
  Bertrand ≤ Stackelberg ≤ Cournot ≤ Monopoly

### Monopoly vs. Cournot vs. Bertrand vs. Stackelberg

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<tr>
<td>Consumer surplus</td>
<td>7200</td>
<td>4050</td>
<td>3200</td>
<td>1800</td>
</tr>
<tr>
<td>Deadweight loss</td>
<td>0</td>
<td>450</td>
<td>800</td>
<td>1800</td>
</tr>
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Multi-Stage Games with Observed Actions

These games have “stages” such that
- In each stage $k$, every player knows all the actions (including those by Nature) that were taken at any previous stage
- Players move simultaneously in each stage $k$
  - Some players may be limited to action set “do nothing” in some stages
  - Each player moves at most once within a given stage
- No information set contained in stage $k$ provides any knowledge of play in that stage

Stackelberg game

- Stage 1
  - Firm 1 chooses its quantity $q_1$; Firm 2 does nothing
- Stage 2
  - Firm 2, knowing $q_1$, chooses its own quantity $q_2$;
    Firm 1 does nothing
Multi-Stage Games with Observed Actions

$h^k$ : History at the start of stage $k$

$$h^k = (a^0, a^1, ..., a^{k-1}), \ k = 1, ..., K$$

$A^i(h^k)$ : Set of actions available to player $i$ in stage $k$ given history $h^k$

$s^i$ : Pure strategy for player $i$ that specifies an action $a \in A^i(h^k)$ for each $k$ and each history $h^k$

Finite games of perfect information

- A multistage game has *perfect information* if
  - for every stage $k$ and history $h^k$, exactly one player has a nontrivial action set, and all other players have one-element action set “do nothing”
  - each player knows all previous moves when making a decision
- In a *finite game of perfect information*, the number of stages and the number of actions at any stage are finite.
- **Theorem (Zemelo 1913; Kuhn 1953)**: A finite game of perfect information has a pure-strategy Nash equilibrium
Backward induction

Determine the optimal action(s) in the final stage K for each history $h^K$

For each stage $j=K-1,...,1$
- Determine the optimal action(s) in stage $j$ for each possible $h^j$ given the optimal actions determined for stages $j+1,...,K$.

The strategy profile constructed by backward induction is a Nash Equilibrium. Each player’s actions are optimal at every possible history.

Example: Stackelberg competition

- $P = 130 - (q_1 + q_2), \quad c_1 = c_2 = c = 10$

Backward induction
- Firm 2 strategy: $s^2(q_1) = q_2 = 60 - q_1/2$
- Firm 1 strategy: $q_1 = 60$
- The outcome (60,30) is a Nash equilibrium (Stackelberg outcome)

Is (60,30) the unique equilibrium in this game?

Cournot equilibrium (40,40) is also an equilibrium for the Stackelberg game! $s^2(q_1) = 40 \quad q_1 = 40$
Classroom exercise: Strategic investment

- Duopoly: Firm 1 and Firm 2
- Each firm has unit cost 2
- By paying \( f \), Firm 1 can install new technology and reduce its unit cost to zero
- Once Firm 1’s investment decision is observed, both firms simultaneously choose output levels \( q_1 \) and \( q_2 \) as in Cournot competition
- \( P = 14 - Q \)

Recall: Cournot best response

\[
q_1 = \frac{(a - c_1)}{2b} - \frac{q_2}{2} \\
q_2 = \frac{(a - c_2)}{2b} - \frac{q_1}{2}
\]

Subgame-perfect equilibrium

- A strategy profile \( s \) of a multistage game with observed actions is a subgame-perfect equilibrium if, for every \( h^k \), the restriction \( s| h^k \) is a Nash equilibrium of subgame \( G(h^k) \).

- \( G(h^k) \): game from stage \( k \) on with history \( h^k \)
- For each player \( j \), \( s'| h^k \) is the restriction of \( s' \) to the histories consistent with \( h^k \)
Classroom exercise: Strategic investment

- Firm 1 does not invest
  \[ q_1 = \frac{(a-c_1)}{2b} - \frac{q_2}{2} = 6 - \frac{q_2}{2} \]
  \[ q_2 = \frac{(a-c_2)}{2b} - \frac{q_1}{2} = 6 - \frac{q_1}{2} \rightarrow (q_1, q_2) = (4,4) \]
  Payoffs: (16,16)

- Firm 1 does invest
  \[ q_1 = \frac{(a-c_1)}{2b} - \frac{q_2}{2} = 7 - \frac{q_2}{2} \]
  \[ q_2 = \frac{(a-c_2)}{2b} - \frac{q_1}{2} = 6 - \frac{q_1}{2} \rightarrow (q_1, q_2) = (16/3, 10/3) \]
  Payoffs: (256/9-f, 100/9)

- Firm 1 choice:
  Invest if \( \frac{256}{9} - f > 16 \), i.e., if \( f < \frac{112}{9} \)

Example: Stackelberg competition

- \( P = 130 - (q_1 + q_2) \), \( c_1 = c_2 = c = 10 \)

  By backward induction the outcome (60,30) is a subgame-perfect equilibrium.

  The outcome (40,40) is NOT subgame perfect, because the strategy \( s^2(q_1) = 40 \) does not induce a Nash equilibrium in stage 2 for player 2, for histories other than \( q_1 = 40 \).