Recap

- Last class (January 13, 2004)
  - Dominant and dominated actions
  - Best response
  - Nash equilibrium
  - Mixed strategies
  - Pareto dominance
- Today (January 15, 2004)
  - Examples of games with continuous action sets
  - Duopoly models: Cournot and Bertrand

Duopoly models

- Two competing firms, selling a homogeneous good
- The marginal cost of producing each unit of the good: $c_1$ and $c_2$
- The market price, $P$ is determined by (inverse) market demand:
  - $P=a-bQ$ if $a>bQ$, $P=0$ otherwise.
- Both firms seek to maximize profits
- Cournot: Firms set quantities simultaneously
- Bertrand: Firm set prices simultaneously
- Stackelberg: Firms set quantities, firm 1 followed by firm 2
Cournot Competition

- The market price, $P$ is determined by (inverse) market demand:
  - $P=a-bQ$ if $a>bQ$, $P=0$ otherwise.
- Each firm decides on the quantity to sell (market share): $q_1$ and $q_2$
- $Q=q_1+q_2$ total market demand
- Both firms seek to maximize profits

Cournot Competition: Best response of Firm 1

- Suppose firm 2 produces $q_2$
- Firm 1’s profits, if it produces $q_1$ are:
  \[ \pi_1 = (P-c_1)q_1 = [a-b(q_1+q_2)]q_1 - c_1q_1 = (\text{Residual revenue} - \text{Cost}) \]
- How to choose $q_1$ to maximize $\pi_1$?
- First note that $\pi_1$ is concave: $\frac{d^2\pi_1}{dq_1^2} = -2b < 0$
- First order conditions (FOC):
  \[ \frac{d\pi_1}{dq_1} = a - 2bq_1 - bq_2 - c_1 = 0 \rightarrow q_1 = \frac{(a-c_1)}{2b} - \frac{q_2}{2} = R_1(q_2) \]
Cournot Competition: Best response of Firm 2

1. Suppose firm 1 produces $q_1$
2. Firm 2’s profits, if it produces $q_2$ are:
   \[ \pi_2 = (P-c_2)q_2 = [a-b(q_1+q_2)]q_2 - c_2q_2 \]
   \[ = \text{(Residual) revenue – Cost} \]

3. First order conditions:
   \[ \frac{d\pi_2}{dq_2} = a - 2bq_2 - bq_1 - c_2 = \]
   \[ = \text{RMR} - \text{MC} = 0 \rightarrow \]
   \[ q_2 = \frac{(a-c_2)}{2b} - \frac{q_1}{2} = R_2(q_1) \]

Example: Cournot Competition

1. $P = 130-(q_1+q_2)$, so $a=130$, $b=1$
2. $c_1 = c_2 = c = 10$
3. Suppose Firm 2 thinks that Firm 1 will set $q_1=40$
   - Residual demand of Firm 2: $P = 90-q_2$
   - Residual revenue of Firm 2: $RR = [90-q_2]q_2$
4. Residual marginal revenue:
   \[ \text{RMR} = 90-2q_2 \]
5. Setting $\text{RMR} = \text{MC} = 10$
   \[ 90-2q_2 = 10 \rightarrow q_2 = 40 \]
Cournot Competition: Graphical solution

\[ P = 130 - Q \]

Residual demand: \[ P = 90 - q \]

\[ RMR = 90 - 2q \]

MC = 10

Cournot Equilibrium

\[ q_1 = (a-c_1)/2b - q_2/2 \]
\[ q_2 = (a-c_2)/2b - q_1/2 \]

Solving together for \( q_1 \) and \( q_2 \):

\[ q^C_1 = (a-2c_1+c_2)/3b \quad q^C_2 = (a-2c_2+c_1)/3b \]

Market demand and price:

\[ Q^C = q^C_1 + q^C_2 = (2a- c_1 - c_2)/3b \]
\[ P = a - bQ^C = (a+c_1+c_2)/3 \]
Example: Cournot Competition

- $P = 130 - (q_1 + q_2)$, so $a=130$, $b=1$
- $c_1 = c_2 = c = 10$
- The firms’ best response functions:
  
  $q_1 = \frac{(a - bq_2 - c)}{2b} = \frac{(130 - q_2 - 10)}{2} = 60 - \frac{q_2}{2}$
  
  $q_2 = \frac{(a - bq_1 - c)}{2b} = \frac{(130 - q_1 - 10)}{2} = 60 - \frac{q_1}{2}$

- Solving for $q_1$ and $q_2$:
  
  $q_1 = q_2 = 40 \quad Q=80 \quad P = 50$

- Firms’ profits:
  
  $\pi_1 = \pi_2 = (50 - 10) \times 40 = 1600$

Cournot Competition: Graphical solution

- $q_1 = R_1(q_2) = 60 - \frac{q_2}{2}$
- $q_2 = R_2(q_1) = 60 - \frac{q_1}{2}$

Diagram of Cournot equilibrium with $R_1(q_2)$ and $R_2(q_1)$ lines intersecting at $(q_1, q_2) = (40, 40)$. The equilibrium price $P = 50$ and quantity $Q = 80$. Firms' profits $\pi_1 = \pi_2 = 1600$. 
Cournot Equilibrium with N firms

\[ \max_{q_i} \pi_i(q_i, q_{-i}) = [a - bq_i - b \sum_{j \neq i} q_j]q_i - c_i q_i \]

First order conditions:
\[ a - 2bq_i - b \sum_{j \neq i} q_j - c_i = 0 \quad \forall i = 1, \ldots, N \]

Substitute \( Q = \sum q_i; \)
\[ a - bq_i - bQ - c_i = 0 \quad \forall i = 1, \ldots, N \]

Sum over N:
\[ Na - bQ - bNQ - \sum c_i = 0 \]

\[
Q^C = \frac{Na}{(N+1)b} - \frac{\sum c_i}{(N+1)b} \\
p^C = \frac{a}{N+1} + \frac{\sum c_i}{N+1}
\]

If each firm has the same cost \( c_i = c \):
\[ q_i^C = \frac{Q^C}{N} = \frac{a - c}{(N+1)b} \quad p^C = \frac{a + Nc}{N+1} \]
Bertrand Equilibrium Model

- Firms set prices rather than quantities
  - \( P = a - bQ \)
- Customers buy from the firm with the cheapest price
- The market is split evenly if firms offer the same price

Best response

- Firm 1’s profit function:
  \[ \pi(P_1) = (P_1 - c_1) q_1 \]
- To ensure \( q_1 > 0 \) (recall: \( P = a - bQ \) and \( Q = (a - P)/b \))
  \( P_1 \leq a \)
- To ensure nonnegative profits
  \( P_1 \geq c_1 \)
- Firm 1 should choose
  \( c_1 \leq P_1 \leq a \)
- Similarly, firm 2 should choose
  \( c_2 \leq P_2 \leq a \)
Best response (cont.)

- Firm i’s demand depends on the relationship between $P_1$ and $P_2$
  \[ q_i = \begin{cases} 
  0, & \text{if } P_i > P_j \\
  \frac{a - P_i}{b}, & \text{if } P_i < P_j \\
  \frac{a - P}{2b}, & \text{if } P_i = P_j = P 
  \end{cases} \]
  \( i = 1, 2 \)

- Firm 1 should choose $c_1 \leq P_1 \leq P_2$ (if possible)
- Firm 2 should choose $c_2 \leq P_2 \leq P_1$ (if possible)

Bertrand equilibrium

- For both firms to sell positive quantities profitably
  \( c_1 \leq P_1 \leq P_2 \) and \( c_2 \leq P_2 \leq P_1 \)

- Suppose $c = c_1 = c_2$
  \[ P = c \qquad q_1 = q_2 = \frac{(a-c)}{2b} \]

- Suppose $c_1 < c_2$
  \[ P_1 = c_2 - \varepsilon \quad P_2 \geq c_2 \]
  \[ q_1 = \frac{(a - c_2 + \varepsilon)}{b} \quad q_2 = 0 \]
Example

- \( P = 130-(q_1+q_2) \) (a=130, b=1)
- \( c_1 = c_2 = c = 10 \)
- \( P=10 \)
- \( q_1= q_2= (a-P)/2b = 60 \)  \( Q=120 \)
- Firms’ profits:
  \[ \pi_1 = \pi_2 = 0 \]

Quantity-setting monopolist

- Single firm (monopolist), selling a single good
- The *marginal cost* of producing each unit of the good: \( c \)
- The firm decides on the quantity to sell: \( Q \) (market demand)
- The market price, \( P \) is determined by (inverse) market demand:
  - \( P=a-bQ \) if \( a>bQ \), \( P=0 \) otherwise.
- The firm seeks to maximize profits
Quantity-setting monopolist

- The firm’s profits, if it produces Q are:
  \[ \pi = (P-c)Q = (a-bQ)Q - cQ \]
  \[ = \text{Revenue} - \text{Cost} \]
- How to choose Q to maximize \( \pi \)?
- First note that \( \pi \) is concave: \( d^2\pi/dQ^2 = -2b < 0 \)
- First order conditions (FOC):
  \[ d\pi/dQ = a - 2bQ - c \]
  \[ = \text{Marginal revenue} - \text{Marginal cost} \]
  \[ = 0 \rightarrow Q = (a-c)/2b \]
  \[ P = (a+c)/2 \]

Example

- \( P = 130-Q \) (a=130, b=1)
- \( c = 10 \)
- \( Q = (a-c)/2b = 60 \)
  \[ P = (a+c)/2 = 70 \]
- Monopolist’s profits:
  \[ \pi = (70-10)60 = 3600 \]
### Monopoly vs. Cournot vs. Bertrand

<table>
<thead>
<tr>
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<th>Competitive</th>
<th>Bertrand</th>
<th>Cournot</th>
<th>Monopoly</th>
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<tr>
<td>Price</td>
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- Firm profits and prices:  
  Competitive ≤ Bertrand ≤ Cournot ≤ Monopoly

### Monopoly vs. Cournot vs. Bertrand

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<tbody>
<tr>
<td>Price</td>
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<tr>
<td>Quantity</td>
<td>(\frac{a-c}{b})</td>
<td>(\frac{a-c}{b})</td>
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<td>Total Firm Profits</td>
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<td>0</td>
<td>(\frac{2(a-c)}{9b})</td>
<td>(\frac{(a-c)^2}{4b})</td>
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- Firm profits and prices:  
  Competitive ≤ Bertrand ≤ Cournot ≤ Monopoly
Cournot competition

\[ P = 130 - Q \]

\[ MC = 10 \]

Consumer surplus = 3200
Firm profits = 3200
Deadweight loss = 800

Bertrand competition

\[ P = 130 - Q \]

\[ MC = 10 \]

Consumer surplus = 7200
Monopoly

\[ P = 130 - Q \]

\[ MC = 10 \]

Consumer surplus = 1800

Deadweight loss = 1800

Firm profits = 3600

Monopoly vs. Cournot vs. Bertrand

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<td>Consumer surplus</td>
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