QUANTITY FLEXIBILITY - \((w, \delta)\)

Supplier sell at \(w\) per unit. Demand is distributed with cdf \(F(x)\) and pdf \(f(x)\). Supplier’s unit cost is \(c_S\), retailer’s unit cost is \(c_R\), and \(c = c_S + c_R\). Unit selling price is \(p\).

Centralized supply chain (CSC):

\[
\Pi(q) = p\cdot S(q) - cq = p\left(q - \int_0^q F(x)dx\right) - cq
\]

\[
\frac{\partial \Pi(q)}{\partial q} = p - p\cdot F(q) - c = p(1 - F(q)) - c = 0 \Rightarrow c = p(1 - F(q)) \tag{1}
\]

Quantity Flexibility (QF): Refund the retailer \(w + c_R\) for units unsold up to \(\min\{I, \delta q\}\)

\(S(q)\): expected number of units sold by the retailer.

\(D \geq q \Rightarrow \text{sell } q\)
\(D < q \Rightarrow \text{sell } D\)

\[
S(q) = \underbrace{(1 - F(q))\cdot q}_{Pr(D \geq q)} + \int_0^q xf(x)dx \tag{2}
\]

Let us compute the term \(\int_0^q xf(x)dx\).

\[
\frac{d}{dx}xF(x) = F(x) + xf(x)
\]

\[
\int_0^q \left[ \frac{d}{dx}xF(x) \right] dx = q\cdot F(q) = \int_0^q F(x)dx + \int_0^q xf(x)dx
\]

\[
\Rightarrow \int_0^q xf(x)dx = q\cdot F(q) - \int_0^q F(x)dx \tag{3}
\]
From (2) and (3):

\[ S(q) = q - \int_0^q F(x)dx \]  \hspace{1cm} (4)

What is the retailer’s expected compensation?

\[ \delta q \]

\[ 0 \hspace{1cm} (1-\delta)q \hspace{1cm} q \]

\[ D < (1-\delta)q \Rightarrow \text{leftover inventory} > \delta q \Rightarrow \text{compensated for } \delta q \]
\[ (1-\delta)q < D < q \Rightarrow \text{leftover inventory} < \delta q \Rightarrow \text{compensated for } q - D \]
\[ D > q \Rightarrow \text{no inventory left.} \]

Expected # of units for which the retailer gets compensation:

\[ I_R = F((1-\delta)q)\delta q + \int_{(1-\delta)q}^q (q-x)f(x)dx \]

Since

\[ \int_{(1-\delta)q}^q (q-x)f(x)dx = q \int_{(1-\delta)q}^q f(x)dx - \int_{(1-\delta)q}^q xf(x)dx \]

and

\[ q \int_{(1-\delta)q}^q f(x)dx = q [F(q) - F((1-\delta)q)] \]

we have

\[ I_R = F((1-\delta)q)\delta q + qF(q) - qF((1-\delta)q) - \int_{(1-\delta)q}^q xf(x)dx \]
Let’s compute $\int_{(1-\delta)q}^{q} xf(x)dx$.

$$
\int_{(1-\delta)q}^{q} xf(x)dx = \int_{0}^{q} xf(x)dx - \int_{0}^{(1-\delta)q} xf(x)dx
$$

$$
= qF(q) - \int_{0}^{q} F(x)dx - \left[(1-\delta)qF((1-\delta)q) - \int_{0}^{(1-\delta)q} F(x)dx \right]
$$

$$
= qF(q) - (1-\delta)qF((1-\delta)q) - \int_{(1-\delta)q}^{q} F(x)dx
$$

Hence, inserting $\int_{(1-\delta)q}^{q} xf(x)dx$ from Equation 5 into $I_R$, we get

$$
I_R = F((1-\delta)q)\delta q + qF(q) - qF((1-\delta)q) - qF(q) + (1-\delta)qF((1-\delta)q) + \int_{(1-\delta)q}^{q} F(x)dx
$$

$$
\Rightarrow I_R = \int_{(1-\delta)q}^{q} F(x)dx.
$$

(6)

Retailer’s profit function:

$$
\Pi_R(q) = p \cdot S(q) - (w + c_R)q + (w + c_R)\int_{(1-\delta)q}^{q} F(x)dx
$$

$$
\frac{\partial \Pi_R(q)}{\partial q} = p \cdot \frac{\partial S(q)}{\partial q} - (w + c_R) + (w + c_R)(F(q) - F((1-\delta)q)(1-\delta))
$$

$$
= p(1 - F(q^*)) - (w + c_R) [1 - F(q) + F((1-\delta)q)(1-\delta)] = 0
$$

$$
p(1 - F(q)) = (w + c_R) [1 - F(q) + F((1-\delta)q)(1-\delta)]
$$

For $q^*$ to be optimal for the retailer, it needs to satisfy the retailers FOC, i.e.,

$$
\Rightarrow w = \frac{p(1 - F(q^*))}{[1 - F(q^*) + F((1-\delta)q^*)(1-\delta)]} c_R
$$

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Can we claim that if the supplier chooses $w$ in this way, the retailer will choose $q^*$? This is the case only if the retailer’s profit function is concave.

\[
\frac{\partial^2 \Pi_R(q)}{\partial q^2} = -p \cdot f(q) + (w + c_R)f(q) - (w + c_R)(1 - \delta)^2 f((1 - \delta)q)
\]

\[
\frac{\partial F((1 - \delta)q)}{\partial q} = f((1 - \delta)q) \cdot (1 - \delta)
\]

\[
\Rightarrow \frac{\partial^2 \Pi_R(q)}{\partial q^2} = - \left( p - (w + c_R) \right) f(q) - (w + c_R)(1 - \delta)^2 f((1 - \delta)q) < 0 \quad \geq 0
\]

as long as $w + c_R \not\geq p$, i.e., $w \not\geq p - c_R$

\[
(w + c_R) \not\geq 0, \quad \text{i.e.,} \quad w \geq -c_R
\]

What do you think happens to retailer’s profit as $\delta \uparrow$? As $\delta \uparrow$, the retailer gets protection for more units but at a higher cost!

Supplier’s profit:

\[
\Pi_s = (w - c_s)q - (w + c_R) \int_{(1 - \delta)q}^{q} F(x) dx
\]

\[
\delta = 0 \Rightarrow w = \frac{p(1 - F(q))}{1 - F(q) + F(q)} - c_R = p(1 - F(q)) - c_R
\]

From equation 1 we have $p(1 - F(q^*)) = c$, i.e., if $\delta = 0$, we have $w = c_s$, $\Pi_s = 0$ and $\Pi_R = \Pi$.

Compliance

If the retailer orders $q$, he will certainly not accept more than $q$ units from the supplier. However, what if the supplier provides less than $q$ units? This might happen due to shortages or other problems at the supplier, or the supplier might simply find it more profitable to deliver a different quantity given the contractual terms and her profit maximization objective. In voluntary compliance, the supplier delivers an amount (not to exceed the retailer’s order quantity) to maximize her profits. In forced compliance, the supplier delivers exactly
the amount ordered by the retailer. Is there a difference in quantities delivered under these two regimes, say, if the retailer does not order the “correct” quantity?

Look at the supplier’s FOC. Is \( q^* \) the best response for the supplier?

\[
\frac{\partial \Pi_S(q)}{\partial q} = (w - c_s) - (w + c_R) [F(q) - F((1 - \delta)q)(1 - \delta)] \leftarrow \text{add and subtract} (w + c_R) \\
= (w - c_s) + (w + c_R) [1 - F(q) + F((1 - \delta)q)(1 - \delta)] - (w + c_R)
\]

Recall \( w = \frac{p(1 - F(q)) - c_R}{\Delta} \Rightarrow \Delta = \frac{p(1 - F(q))}{w + c_R} \)

\[
\Rightarrow \frac{\partial \Pi_s}{\partial q} = -(c_s + c_R) + p(1 - F(q)) = -c + p(1 - F(q)) = 0
\]

\[
\Rightarrow c = p(1 - F(q)) \text{ same as the FOC of CSC!}
\]

Does this mean that \( q^* \) is optimal/best response for the supplier? Check 2\textsuperscript{nd} order conditions at \( q^* \).

\[
\frac{\partial^2 \Pi_S(q)}{\partial q^2} = -(w + c_R) [f(q) - (1 - \delta)^2 f((1 - \delta)q)]
\]

If \( (1 - \delta)^2 f((1 - \delta)q) > f(q) \) then \( \frac{\partial^2 \Pi_s}{\partial q^2} > 0 \), i.e., the function is concave and \( q^* \) is a minimizer! This will hold if \( \delta \) is small or \( f(q) \) is small.

Hence, under voluntary compliance, even if the wholesale price is \( w(\delta) \), coordination may not be achieved, depending on \( \delta \) and \( f(q) \). Clearly it is achieved under forced compliance.