Recap

- Last Tuesday (February 4, 2003)
  - Subgames in extensive form games
  - Games of incomplete information
    - Cournot competition under incomplete information
    - Battle of the sexes under incomplete information
- Today (February 6, 2003)
  - Battle of the sexes under incomplete information
  - First-price sealed-bid auctions
  - Equilibrium recap
  - One-card poker game

Example: Battle of the sexes

Bob

Alice	Ballet	Football
Ballet	$2+t_A$, 1	0, 0
Football	0, 0	1, $2+t_B$

- t_A is privately known to Alice and t_B is privately known to Bob
- t_A and t_B are independent draws from a uniform distribution on [0,x]

Example: Battle of the sexes

Bob

Alice	Ballet	Football
Ballet	$2+t_A$, 1	0, 0
Football	0, 0	1, $2+t_B$

- Strategies
  - Alice: Play ballet if t_A exceeds a critical value c_A; otherwise, play football
  - Bob: Play football if t_B exceeds a critical value c_B; otherwise, play ballet

Example: Battle of the sexes

- Alice's expected payoffs
  - Alice plays Ballet: $(c_B/x)(2+t_A)$
  - Alice plays Football: $1- c_B/x$
  - Play ballet if $(c_B/x)(2+t_A)> 1- c_B/x \rightarrow t_A>x/c_B-3 = c_A$
- Bob's expected payoffs
  - Bob plays Ballet: $1-c_A/x$
  - Bob plays Football: $(c_A/x)(2+ t_B)$
  - Play football if $(c_A/x)(2+t_B)> 1- c_A/x \rightarrow t_B>x/c_A-3 = c_B$

Example: Battle of the sexes

- $t_A > x/c_B-3 = c_A$
- $t_B > x/c_A-3 = c_B$
- $c_A = c_B = c$
- $c^2+3c-x=0$
  $c = (-3+\sqrt{(9+4x)})/2$
- What is the probability that Alice plays ballet?
  $1- c_A/x = 1 - [-3+\sqrt{(9+4x)}]/2x$
- What is the probability that Bob plays football?
  $1- c_B/x = 1 - [-3+\sqrt{(9+4x)}]/2x$

In the limit ($x\rightarrow 0$), these probabilities approach 2/3!

First-Price Sealed-Bid Auction

- Two bidders, one good
- Bidder i's valuation for the good is v_i, is known only by bidder i. Valuations are independently and uniformly distributed on [0,1].
- Each bidder i submits a nonnegative bid b_i. The higher bidder wins and pays his bid. Other bidder pays and receives nothing.
- In case of a tie, the winner is determined by a coin flip
- Bidder i's payoff, if wins and pays p, is v_i- p
- Bidders are risk-neutral
- All of this information is common knowledge

First-Price Sealed-Bid Auction

- Action spaces
  - $A_1 = A_2 = [0,\infty)$
- Type spaces
  - $T_1 = T_2 = [0,1]$
- Beliefs
  - $p_1(t_2|t_1)= p_1(t_2)$
  - $p_2(t_1|\ t_2)= p_2(t_1)$
- Player i's (expected) payoff function

$$\pi_i(b_1,b_2;v_1,v_2)=\begin{cases} v_i-b_i\ , & \text{if } b_i>b_j \\ (v_i-b_i)/2, & \text{if } b_i=b_j \\ 0, & \text{if } b_i>b_j \end{cases}$$

First-Price Sealed-Bid Auction

- Strategy for player i: $b_i(v_i)$
- Strategies $(b_1(v_1), b_2(v_2))$ are a Bayesian Nash equilibrium if for each v_i in [0,1], $b_i(v_i)$ solves

$\max\ (v_i- b_i)\text{Prob}\{b_i> b_j(v_j)\}+ (v_i- b_i)\text{Prob}\{b_i= b_j(v_j)\}/2$

- Consider a linear equilibrium
  $b_i(v_i) = a_i + c_i v_i \quad i=1,2$
- Assuming player j adopts the strategy $b_j(v_j) = a_j+c_j v_j$, player i's best response:

$\max\ (v_i- b_i)\text{Prob}\{b_i> b_j(v_j)\}= (v_i- b_i)\text{Prob}\{b_i> a_j+c_j v_j\}$

First-Price Sealed-Bid Auction

- Assuming player j adopts the strategy $b_j(v_j) = a_j+c_j v_j$, player i's best response:
  $\max\ (v_i- b_i)\text{Prob}\{b_i> a_j+c_j v_j\}$
  s.t. $b_i \le \min\{a_j+c_j\ ,\ v_i\}$

$\text{Prob}\{b_i> a_j+c_j v_j\} = \text{Prob}\{v_j<(b_i-a_j)/c_j\} = (b_i-a_j)/c_j$

$\max\ (v_i- b_i)(b_i-a_j)/c_j$
s.t. $b_i \le \min\{a_j+c_j\ ,\ v_i\}$

From F.O.C.: $b_i=v_i$, if $v_i\le a_j$, $b_i= (v_i+a_j)/2$, otherwise

First-Price Sealed-Bid Auction

- Player i's best response
  $b_i= v_i$, if $v_i\le a_j$, $b_i= (v_i+a_j)/2$, otherwise
- Can a_j be
  - Between 0 and 1?
  - Greater than or equal to 1?
    - $b_j(v_j) = a_j + c_j v_j \ge 1$
  - Less than or equal to zero?
    - $b_i(v_i)= (v_i+a_j)/2$

We have $a_j\le 0$, $b_i = a_i+c_i v_i = a_j/2+1/2(v_i) \rightarrow$
$a_i= a_j/2\ c_i=1/2$

First-Price Sealed-Bid Auction

- Player i's best response
  $a_j\le 0$, $a_i+c_i v_i=a_j/2+1/2(v_i) \rightarrow a_i= a_j/2\ c_i=1/2$
- Player j's best response
  $a_i\le 0$, $a_j+c_j v_j=a_i/2+1/2(v_j) \rightarrow a_j= a_i/2\ c_j=1/2$

We have
$a_i = a_j = 0$ and $c_i= c_j=1/2$ and $b_i(v_i)= v_i/2\ i=1,2$

Equilibrium recap

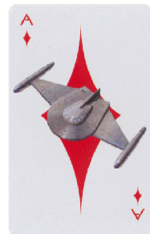
- Static games of complete information
  - Nash equilibrium
- Dynamic games of complete information
  - Subgame-perfect Nash equilibrium
- Static games of incomplete information (Bayesian games)
  - Bayesian Nash equilibrium
- Dynamic games of incomplete information
  - Perfect Bayesian equilibrium

Example: One-card poker

- Two players
- One deck of cards, half aces, half kings
- Pay $a to play
- Each player is dealt a card face down
- After seeing his/her card, each player (simultaneously)
  - Action B: bets b, or
  - Action P: passes
- Payoffs
  - (B,P) or (P,B) → betting player gets the pot
  - (B,B) or (P,P) → higher card gets the pot; in case of a tie, the pot is split

Example: One-card poker

- A player is dealt an ace or king with equal probability



Record the results

	Player 1	Player 2	Payoffs
Game 1	Ace-Bet	King-Bet	(a+b, −(a+b))
Game 2	King-Pass	King-Bet	(-a, a)
Game 3	King-Pass	King-Pass	(0, 0)
etc			