Globalized Robust Counterparts

♣ **We are about to reconsider two of the basic assumptions on the "decision environment" we made, namely, that**

A.2. The decision maker is fully responsible for consequences of the decisions to be made when, and only when, the actual data is within the prespecified uncertainty set $\mathcal{U}$.

A.3. The constraints are *hard* — we cannot tolerate violations of constraints, even small ones, when the data is in $\mathcal{U}$.

E.g., the mail traffic (or shopping) around Christmas is much higher than during the rest of the year. When following A.2-3 literally, when designing the corresponding service capacities, we should

— either completely ignore Christmas and orient ourselves towards the most-of-the-year load on the system,

— or design the system as if the Christmas load could happen every day.

• **Both these extremes hardly are wise.**

♠ **What we want now is to "immunize" against data uncertainty in the case when**
— the data are allowed to run out of the uncertainty set, and
— we allow for *controlled* **violation of the constraints when it happens.**

♠ **Pursuing this goal, we, as always, can focus on a** *single* **uncertainty-affected conic constraint, which now is convenient to write down in the form of**

$$A[\zeta]x + b[\zeta] \in \mathbf{Q} = \left\{ y : \begin{array}{c} P_i y + p_i \in \mathbf{K}_i, \\ 1 \leq i \leq I \end{array} \right\} \subset F \qquad \text{(UCC)}$$

$A[\zeta]$, $b[\zeta]$ **are affine in their arguments.**

♣ **We assume that the "physically possible" perturbations** ζ **run through the set**

$$\mathcal{Z} + \mathcal{L} \subset E = \mathbb{R}^L, \qquad \text{(Pert)}$$

- $\mathcal{Z}$: **closed convex "normal range" of** ζ
- $\mathcal{L}$: **closed convex cone.**

Definition: x **is robust feasible for** (UCC), (Pert) *with global sensitivity* α, **if**

$$\mathrm{dist}(A[\zeta]x + b[\zeta], \mathbf{Q}) \leq \alpha \mathrm{dist}(\zeta, \mathcal{Z}|\mathcal{L}) \;\forall \zeta \in \mathcal{Z} + \mathcal{L}$$

$$\left[\begin{array}{c} \mathrm{dist}(y, \mathbf{Q}) = \min\limits_{y' \in \mathbf{Q}} \|y - y'\|_F; \\ \mathrm{dist}(z, \mathcal{Z}|\mathcal{L}) = \inf\limits_{z'} \{\|z - z'\|_E : z' \in \mathcal{Z}, z - z' \in \mathcal{L}\} \end{array} \right]$$

$$A[\zeta]x + b[\zeta] \in \mathbf{Q} \qquad \text{(UCC)}$$

x is robust feasible for (UCC) with global sensitivity α
$\Leftrightarrow \mathrm{dist}(A[\zeta]x + b[\zeta], \mathbf{Q}) \leq \alpha \mathrm{dist}(\zeta, \mathcal{Z}|\mathcal{L})\ \forall \zeta \in \mathcal{Z} + \mathcal{L}$

Clearly, *if x is robust feasible for* (UCC) *with a whatever global sensitivity, x is robust feasible for* (UCC), *the uncertainty set being $\mathcal{Z}$.*

♠ **Given an uncertain conic problem with certain objective**

$$\mathcal{P} = \left\{ \min_x \left\{ c^T x : A_i[\zeta]x + b_i[\zeta] \in \mathbf{Q}_i,\ i = 1, ..., m \right\}_\zeta \right\}$$

and a *perturbation structure*, **that is, $\mathcal{Z}$, $\mathcal{L}$ and norms used to measure the participating distances, the** *Globalized Robust Counterpart* **of the uncertain problem is the semi-infinite conic problem**

$$\min_x \left\{ c^T x : \begin{array}{r} \mathrm{dist}(A_i[\zeta]x + b_i[\zeta], \mathbf{Q}_i) \leq \alpha_i \mathrm{dist}(\zeta, \mathcal{Z}|\mathcal{L}) \\ \forall \zeta \in \mathcal{Z} + \mathcal{L} \end{array} \right\} \qquad \text{(GRC)}$$

where $\alpha_i \geq 0$ are given parameters.

♡**Alternatively, we can treat x** *and* **α_i as the decision variables and to optimize a new convex objective, depending both on x and α_i, under the same semi-infinite constraints as in (GRC), and, perhaps, additional certain constraints on x and α_i.**

♠ **Sometimes it makes sense to "add some structure" to the perturbations, specifically, to assume that**

$$\zeta = [\zeta^1; ...; \zeta^k] \in \underbrace{[\mathcal{Z}_1 \times ... \times \mathcal{Z}_K]}_{\mathcal{Z}} + \underbrace{[\mathcal{L}_1 \times ... \times \mathcal{L}_K]}_{\mathcal{L}}$$

($\mathcal{Z}_k$ are closed convex sets, $\mathcal{L}_k$ are closed convex cones) and to define the GRC of an uncertain conic constraint

$$A[\zeta]x + b[\zeta] \in \mathbf{Q}$$

as

$$\forall \zeta \in \mathcal{Z} + \mathcal{L} : \mathrm{dist}(A[\zeta]x + b[\zeta], \mathbf{Q}) \leq \sum_{k=1}^{K} \alpha_k \mathrm{dist}(\zeta^k, \mathcal{Z}_k | \mathcal{L}_k).$$

GRC of Scalar Linear Inequality

♣ **Consider the GRC of an uncertain** *scalar linear inequality*

$$\mathrm{dist}(a^T[\zeta]x + b[\zeta], \mathbb{R}_-) \leq \alpha \mathrm{dist}(\zeta, \mathcal{Z}|\mathcal{L}) \ \forall \zeta \in \mathcal{Z} + \mathcal{L} \qquad \text{(ULC)}$$

♠ **Since** $a[\zeta]$, $b[\zeta]$ **are affine in** ζ, **we have**

$$a[\zeta]x + b[\zeta] \equiv \omega^T[x]\zeta + \gamma[x]$$

with $\omega[x]$, $\gamma[x]$ *affine in* x.

Theorem: Semi-infinite inequality (ULC) is equivalent to the pair of semi-infinite inequalities

$$\begin{array}{ll}(a) & \omega^T[x]\zeta + \gamma[x] \leq 0 \ \forall \zeta \in \mathcal{Z} \\ (b) & \omega^T[x]\zeta \leq \alpha \ \forall \zeta \in \mathcal{L}_1 = \{\zeta \in \mathcal{L} : \|\zeta\|_E \leq 1\}.\end{array}$$

In particular, (GRC) is computationally tractable, provided that $\mathcal{Z}$, $\mathcal{L}$ and $\|\cdot\| = \|\cdot\|_E$ are so.

Proof, sufficiency: Let (a), (b) take place, let $\zeta \in \mathcal{Z} + \mathcal{L}$, **and let** $\zeta = \zeta^i + \zeta_i$ **with** $\zeta^i \in \mathcal{Z}$, $\zeta_i \in \mathcal{L}$ **and** $\|\zeta_i\| \to \mathrm{dist}(\zeta, \mathcal{Z}|\mathcal{L})$ **as** $i \to \infty$. **Then**

$$\omega^T[x]\zeta + \gamma[x] \leq \overbrace{[\omega^T[x]\zeta^i + \gamma[x]]}^{\leq 0 \text{ by } (a)} + \overbrace{[\omega^T[x]\zeta_i]}^{\leq \alpha\|\zeta_i\| \text{ by } (b)} \leq \alpha\|\zeta_i\| \\ \to \alpha \mathrm{dist}(\zeta, \mathcal{Z}|\mathcal{L}), \ i \to \infty.$$

Proof, necessity: Let x be such that

$$\mathrm{dist}(\omega^T[x]\zeta+\gamma[x],\mathbb{R}_-)\leq \alpha\mathrm{dist}(\zeta,\mathcal{Z}|\mathcal{L})\ \forall \zeta\in\mathcal{Z}+\mathcal{L} \qquad \text{(ULC)}$$

and let us prove that then

$$\begin{array}{ll}(a) & \omega^T[x]\zeta+\gamma[x]\leq 0\ \forall \zeta\in\mathcal{Z}\\ (b) & \omega^T[x]\zeta\leq\alpha\ \forall\zeta\in\mathcal{L}_1=\{\zeta\in\mathcal{L}:\|\zeta\|\leq 1\},\end{array}$$

(a) **is evident. To prove** (b)**, fix** $\bar{\zeta}\in\mathcal{Z}$**, and let** $\zeta\in\mathcal{L}_1$**. For** $t>0$ **we have**

$$\begin{array}{l}\mathrm{dist}(\bar{\zeta}+t\zeta,\mathcal{Z}|\mathcal{L})\leq t\|\zeta\|\leq t\\ \Rightarrow \omega^T[x][\bar{\zeta}+t\zeta]+\gamma[x]\leq\alpha t\\ \Rightarrow t^{-1}\left[\omega^T[x]\bar{\zeta}+\gamma[x]\right]+\omega^T[x]\zeta\leq\alpha\\ \Rightarrow \omega^T[x]\zeta\leq\alpha.\end{array}$$

♠ **Illustration: $\|\cdot\|_\infty$ Antenna Design via GRC.**

• **$\|\cdot\|_\infty$ Antenna Design is the uncertain LO**

$$\left\{\min_{x,\tau}\{\tau : -\tau\mathbf{1} \le d - D(I+\mathrm{Diag}\{\zeta\})x \le \tau\mathbf{1}\} : \|\zeta\|_\infty \le \rho\right\}$$

$$[D : m\times n]$$

♠ **The** *robust performance* **of a design x is**

$$F_x(\rho) = \max_{\zeta:\|\zeta\|_\infty\le\rho} \|d - D(I+\mathrm{Diag}\{\zeta\})x\|_\infty.$$

This is a convex nondecreasing function of ρ.

♠ **Let us fix an uncertainty level $\bar\rho \ge 0$ and set $\mathcal{Z} = \{\zeta : \|\zeta\|_\infty \le \bar\rho\}$, $\mathcal{L} = \mathbb{R}^n$, $\|\zeta\|_E \equiv \|\zeta\|_\infty$.**

Note: *A pair (x,τ) is robust feasible, with global sensitivity α, for the Antenna Design problem iff $\tau \ge F_x(\bar\rho)$ and*

$$\alpha \ge \alpha(x) := \lim_{\rho\to\infty} \frac{d}{d\rho}F_x(\rho) = \max_{1\le i\le m} \sum\nolimits_{j=1}^n |D_{ij}||x_j|$$

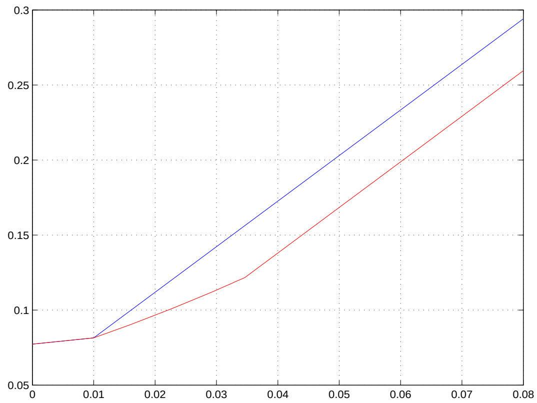
$\Rightarrow$ $F_x(\bar\rho)$, $\alpha(x)$ **imply a "global" upper bound on the robust performance of x:**

$$\forall \rho \ge 0 : F_x(\rho) \le F_x(\bar\rho) + \alpha(x)\max\left[0, \rho - \bar\rho\right].$$

Invoking the easily computable quantity $F_x(0)$, we can improve this bound to

$$F_x(\rho) \le \begin{cases} \frac{\bar\rho-\rho}{\bar\rho}F_x(0) + \frac{\rho}{\bar\rho}F_x(\bar\rho), & 0 \le \rho < \bar\rho \\ F_x(\bar\rho) + \alpha(x)[\rho - \bar\rho], & \rho \ge \bar\rho \end{cases}$$

$$F_x(\rho) \le \begin{cases} \frac{\bar{\rho}-\rho}{\bar{\rho}} F_x(0) + \frac{\rho}{\bar{\rho}} F_x(\bar{\rho}), & 0 \le \rho < \bar{\rho} \\ F_x(\bar{\rho}) + \alpha(x)[\rho - \bar{\rho}], & \rho \ge \bar{\rho} \end{cases} \tag{UB}$$



Robust performance of a design (magenta)
and its upper bound (UB) with $\bar{\rho} = 0.01$

♠ **When solving Antenna Design problem, our ideal goal would be to optimize in x the robust performance $F_x(\rho)$** *for all ρ simultaneously*, **which of course is impossible.**

♠ **With the usual RC approach** *we fix the uncertainty level $\rho = \bar{\rho}$ and optimize the robust performance at this level.* **This makes sense when we know reasonably well the uncertainty level we intend to work with.**

♠ *When the range of possible uncertainty levels is wide, it can be more to the point to look for a "good global upper bound"* (UB) *on the robust performance*.

Example: **Given a "reference uncertainty level" $\bar{\rho}$, we can act as follows:**

• **We solve the RC of the problem, thus finding the best robust performance $\phi(\bar{\rho}) = \min_x F_x(\bar{\rho})$ at the reference uncertainty level $\rho = \bar{\rho}$;**

• **We then** *allow for controlled deterioration of the robust performance at the reference uncertainty level and choose among the corresponding designs the one with the smallest global sensitivity*, **i.e. solve the problems**

$$x_\delta \in \operatorname*{Argmin}_x \{\alpha(x) : F_x(\bar{\rho}) \leq (1+\delta)\phi(\bar{\rho})\} \qquad (P_\delta)$$

for several values of δ. *The larger δ, the worse is the robust performance of x_δ at the uncertainty level $\bar{\rho}$*, **and** *the smaller is the* (upper bound on the) *rate at which the robust performance deteriorates when $\rho \geq \bar{\rho}$ grows.*

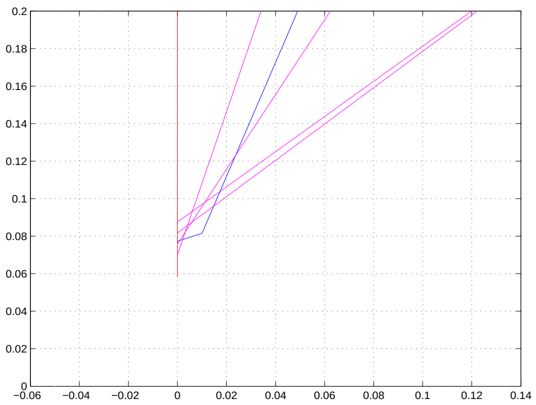
• **The arising tradeoff "robust performance at the reference uncertainty level" vs. "deterioration of robust performance as the uncertainty level grows" can be resolved by the end-user.**

Illustration: Set $\bar{\rho} = 0$. **Here** (P_δ) **is a simple LO program:**

$$x_\delta \in \underset{x}{\text{Argmin}} \left\{ \max_{1\le i\le m} \sum_{j=1}^n |D_{ij}||x_j| : \|d - Dx\|_\infty \le (1+\delta) \min_u \|d - Du\|_\infty, \forall i \right\}$$

♡ **Here are the optimal values in** (P_δ) **(≡ global sensitivities of the designs** x_δ**):**

δ	0	0.2	0.3	0.4	0.5
$\alpha(x_\delta)$	2.6×10^5	3.8134	1.9916	0.9681	0.9379



Red and magenta: upper bounds $F_{x_\delta}(0) + \alpha(x_\delta)\rho$ on $F_{x_\delta}(\rho)$

Blue: upper bound on the robust performance of the robust optimal design associated with $\rho = 0.01$.

Globalized Robust Counterparts of Conic Constraints

♣ **Entity of interest:** semi-infinite conic constraint

$$\forall \zeta \in \mathcal{Z} + \mathcal{L} : \mathrm{dist}(A[\zeta]x + b[\zeta], \mathbf{Q}) \leq \sum_{k=1}^{K} \alpha_k \mathrm{dist}(\zeta^k, \mathcal{Z}_k | \mathcal{L}_k). \tag{GRC}$$

- $\mathbf{Q} = \{y : P_i y + q_i \in \mathbf{K}_i,\ 1 \leq i \leq I\}$
  $[\mathbf{K}_i : \mathbb{R}^{m_i}_+ / \mathbf{L}^{m_i} / \mathbf{S}^{m_i}_+]$
- $\mathcal{Z}_k \subset \mathbb{R}^{L_k}$: closed convex set
- $\mathcal{L}_k \subset \mathbb{R}^{L_k}$: closed convex cone

Note: $A[\zeta]x + b[\zeta] \equiv \sum_{k=1}^{K} \Omega_k[x]\zeta^k + \gamma[x]$ with $\Omega_k[x]$, $\gamma[x]$ affine in x.

♠ **Equivalent reformulation of** (GRC)

- **Recessive cone** $\mathrm{Rec}(X)$ **of a closed convex set** X**:** the set of all directions d such that $\bar{y} + td \in X\ \forall t > 0$ for some (and then for all) $\bar{y} \in X$.

Examples: **A.** X **is bounded** $\Rightarrow \mathrm{Rec}(X) = \{0\}$

B. X **is a cone** $\Rightarrow \mathrm{Rec}(X) = X$

Theorem: $(x, \alpha \geq 0)$ *is feasible for* (GRC) *if and only if*

$$\begin{array}{ll} (a) & A[\zeta]x + b[\zeta] \in \mathbf{Q}\ \forall \zeta \in \mathcal{Z}_1 \times \ldots \times \mathcal{Z}_K \\ (b) & \Psi_k(x) := \sup\limits_{\zeta^k \in \mathcal{L}^k_1} \mathrm{dist}(\Omega_k[x]\zeta^k, \mathrm{Rec}(\mathbf{Q})) \leq \alpha_k,\ k \leq K \\ & [\mathcal{L}^k_1 = \{\zeta^k \in \mathcal{L}^k : \|\zeta^k\|_{(k)} \leq 1\}] \end{array}$$

$$\forall \zeta \in \mathcal{Z} + \mathcal{L}:$$
$$\mathrm{dist}\Big(\sum_{k=1}^{K} \Omega_k[x]\zeta^k + \gamma[x], \mathbf{Q}\Big) \leq \sum_{k=1}^{K} \alpha_k \mathrm{dist}(\zeta^k, \mathcal{Z}_k | \mathcal{L}_k) \qquad \text{(GRC)}$$

$$? \Updownarrow ?$$

$$\begin{array}{ll} (a) & A[\zeta] + b[\zeta] \in \mathbf{Q} \ \forall \zeta \in \mathcal{Z}_1 \times \ldots \times \mathcal{Z}_K \\ (b) & \Psi_k(x) := \sup\limits_{\zeta^k \in \mathcal{L}_1^k} \mathrm{dist}(\Omega_k[x]\zeta^k, \mathrm{Rec}(\mathbf{Q})) \leq \alpha_k, \ k \leq K \\ & [\mathcal{L}_1^k = \{\zeta^k \in \mathcal{L}^k : \|\zeta^k\|_{(k)} \leq 1\}] \end{array}$$

Proof, $\Uparrow$: Let (a), (b) take place, and let $\zeta = [\zeta^1; \ldots; \zeta^K] \in \mathcal{Z} + \mathcal{L}$. Then

$$\begin{array}{ll} (1) & \exists \bar{\zeta}_t \in \mathcal{Z}, \eta_t \in \mathcal{L}, \ t = 1, 2, \ldots : \\ & \zeta = \lim\limits_{t \to \infty} [\bar{\zeta}_t + \eta_t], \ \|\eta_t^k\|_{(k)} \to \mathrm{dist}(\zeta^k, \mathcal{Z}_k | \mathcal{L}_k), \ t \to \infty \\ (2) & \exists y_t^k \in \mathrm{Rec}(\mathbf{Q}) : \|\Omega_k[x]\eta_t^k - y_t^k\|_F \leq \Psi_k(x) \|\eta_t^k\|_{(k)} \ \forall t, k \end{array}$$

Setting $y = \sum_{k=1}^K \Omega_k[x]\zeta^k + \gamma[x]$, **we have**

$$\begin{array}{l} \mathrm{dist}(y, \mathbf{Q}) \\ \leq \Big\| \sum_{k=1}^L \Omega_k[x]\zeta^k + \gamma[x] - \overbrace{\Big[\sum_{k=1}^K [\Omega_k[x]\bar{\zeta}_t^k + y_t^k] + \gamma[x]\Big]}^{\in \mathbf{Q} \text{ by } (a) \ \& \ y_t^k \in \mathrm{Rec}(\mathbf{Q})} \Big\|_F \\ = \Big\| \sum_{k=1}^K \Omega_k[x]\eta_t^k - y_t^k \Big\|_F \leq \sum_{k=1}^K \|\Omega_k[x]\eta_t^k - y_t^k\|_F \\ \leq \sum_{k=1}^K \Psi_k(x) \|\eta_t^k\|_{(k)} \\ \Rightarrow \mathrm{dist}(y, \mathbf{Q}) \leq \sum_{k=1}^K \Psi_k(x) \mathrm{dist}(\zeta^k, \mathcal{Z}_k | \mathcal{L}_k) \\ \quad \leq \sum_{k=1}^K \alpha_k \mathrm{dist}(\zeta^k, \mathcal{Z}_k | \mathcal{L}_k) \end{array}$$

where $\leq$ is due to (b).

$$\forall \zeta \in \mathcal{Z} + \mathcal{L}:$$
$$\mathrm{dist}(\sum_{k=1}^{K} \Omega_k[x]\zeta^k + \gamma[x], \mathbf{Q}) \leq \sum_{k=1}^{K} \alpha_k \mathrm{dist}(\zeta^k, \mathcal{Z}_k | \mathcal{L}_k) \quad \text{(GRC)}$$

$$? \Updownarrow ?$$

$$\begin{array}{ll} (a) & A[\zeta] + b[\zeta] \in \mathbf{Q} \ \forall \zeta \in \mathcal{Z}_1 \times ... \times \mathcal{Z}_K \\ (b) & \Psi_k(x) := \sup\limits_{\zeta^k \in \mathcal{L}_1^k} \mathrm{dist}(\Omega_k[x]\zeta^k, \mathrm{Rec}(\mathbf{Q})) \leq \alpha_k, \ k \leq K \\ & [\mathcal{L}_1^k = \{\zeta^k \in \mathcal{L}^k : \|\zeta^k\|_{(k)} \leq 1\}] \end{array}$$

Proof, $\Downarrow$: Let (x, α) satisfy (GRC), and let us prove that (x, α) satisfies (a), (b). (a) is evident. To prove (b), let us fix $m \leq K$, and let $\bar{\zeta} \in \mathcal{Z}$, $\eta^m \in \mathcal{L}_1^m$. Setting $\eta^k = 0$ for $k \neq m$, let

$$y_t = \sum\nolimits_{k=1}^{K} \Omega_k[x][\bar{\zeta}^k + t\eta^k], \ t = 1, 2, ...$$

and let $z_t \in \mathop{\mathrm{Argmin}}\limits_{z \in \mathbf{Q}} \|y_t - z\|_F$. We have

$$\begin{array}{l} \|y_t - z_t\|_F = \mathrm{dist}(y_t, \mathbf{Q}) \leq \sum\limits_{k=1}^{K} \alpha_k \mathrm{dist}(\bar{\zeta}^k + t\eta^k, \mathcal{Z}_k | \mathcal{L}_k) \\ \leq \alpha_m \|t\eta^m\|_{(m)} \leq t\alpha_m \Rightarrow \\ (*): \quad \|t^{-1}y_t - t^{-1}z_t\|_F \leq \alpha_m \end{array}$$

As $t \to \infty$, we have $t^{-1}y_t \to \Omega_m[x]\eta^m$, whence, by $(*)$, the sequence $\{t^{-1}z_t\}$ is bounded; passing to a subsequence, we can assume that $t^{-1}z_t \to \bar{z}$ as $t \to \infty$.

Summary: given $m \leq K$ and $\eta^m \in \mathcal{L}_1^m$, there exist $\{z_t \in \mathbf{Q}, y_t\}$ such that

$$\begin{array}{c} \bar{z} = \lim_{t\to\infty} t^{-1} z_t, \\ \Omega_m[x]\eta^m = \lim_{t\to\infty} t^{-1} y_t, \ \|t^{-1}(y_t - z_t)\|_F \leq \alpha_m \end{array} \qquad (*)$$

Simple Lemma: *Let U be a closed convex set, and $u_t \in U$, $q_t > 0$ be such that $q_t \to \infty$ and $q_t^{-1} u_t \to \bar{u}$ as $t \to \infty$. Then $\bar{u} \in \mathrm{Rec}(U)$.*

Indeed, with $a \in U$ and $\tau \geq 0$, we have $a + \tau\bar{u} = \lim_{t\to\infty} \overbrace{[[1 - \tau/q_t]a + [\tau/q_t]u_t]}^{\in U}$, whence $a + \tau\bar{u} \in U$ since U is closed.

Applying Simple Lemma, we see that $\bar{z} \in \mathrm{Rec}(\mathbf{Q})$. Therefore $(*)$ implies that

$$\begin{array}{l} \mathrm{dist}(\Omega_m[x]\eta^m, \mathrm{Rec}(\mathbf{Q})) \leq \|\Omega_m[x]\eta^m - \bar{z}\|_F \\ = \lim\limits_{t\to\infty} \|t^{-1}y_t - t^{-1}z_t\|_F \leq \alpha_m. \end{array}$$

Thus,

$$\forall (m \leq K, \eta^m \in \mathcal{L}_1^m) : \mathrm{dist}(\Omega_m[x]\eta^m, \mathrm{Rec}(\mathbf{Q})) \leq \alpha_m,$$

and (b) follows.

♠ **We have reduced the Globalized Robust Counterpart** (GRC) **of an uncertain conic constraint to a pair of semi-infinite conic constraints** (a), (b), **and such a system not necessarily is tractable.** *What to do when* (a) - (b) *is intractable?*

As in the case of RC, assume that $0 \in \mathcal{Z}_k$, $k \leq K$, **and embed** (GRC) **into a single-parametric family of semi-infinite conic constraints**

$$\forall \zeta \in \rho\mathcal{Z} + \mathcal{L}: \quad \mathrm{dist}\Big(\sum_{k=1}^{K} \Omega_k[x]\zeta^k + \gamma[x], \mathbf{Q}\Big) \leq \sum_{k=1}^{K} \alpha_k \mathrm{dist}(\zeta^k, \rho\mathcal{Z}_k | \mathcal{L}_k) \qquad \text{(GRC)}$$

$$\Updownarrow$$

$$\begin{array}{ll} (a_\rho) & A[\zeta]x + b[\zeta] \in \mathbf{Q} \ \forall \zeta \in \rho\mathcal{Z}_1 \times \ldots \times \rho\mathcal{Z}_K \\ (b) & \Psi_k(x) := \sup\limits_{\zeta^k \in \mathcal{L}_1^k} \mathrm{dist}(\Omega_k[x]\zeta^k, \mathrm{Rec}(\mathbf{Q})) \leq \alpha_k, \ k \leq K \\ & [\mathcal{L}_1^k = \{\zeta^k \in \mathcal{L}^k : \|\zeta^k\|_{(k)} \leq 1\}] \end{array}$$

and let us look for *tight tractable approximations* **of** (a_ρ) **and** (b)**.**

$$\begin{array}{rl} (a_\rho) & A[\zeta]x+b[\zeta]\in \mathbf{Q}\,\forall \zeta\in \rho\mathcal{Z}_1\times ...\times \rho\mathcal{Z}_K \\ (b) & \Psi_k(x):=\sup\limits_{\zeta^k\in\mathcal{L}_1^k} \mathrm{dist}(\Omega_k[x]\zeta^k,\mathrm{Rec}(\mathbf{Q}))\leq \alpha_k,\ \forall k \end{array} \tag{GRC$_\rho$}$$

• **We already know what a ϑ-tight safe tractable approximation of (a_ρ) is:** a system $\mathcal{S}_\rho$ of efficiently computable convex constraints on x and additional variables such that

— if $\rho\geq 0$ and x are such that x can be extended to a feasible solution of $\mathcal{S}_\rho$, x is feasible for (a_ρ);

— if $\rho\geq 0$ and x are such that x cannot be extended to a feasible solution of $\mathcal{S}_\rho$, x is not feasible for $(a_{\vartheta\rho})$

• *By definition*, **a** *safe κ-tight tractable approximation of (b) is a collection of efficiently computable convex upper bounds Φ_k on Ψ_k such that $\Phi_k(x)\leq\kappa\Psi_k(x)$ for every x.*

♣ Replacing (a_ρ) with a ϑ-tight s.t.a. $\mathcal{S}_\rho$, and (b) with a κ-tight s.t.a. $\Phi_k(x)\leq\alpha_k$, $k\leq K$, we get a tractable system $\mathcal{S}_\rho^+$ of convex constraints on $x,\alpha_1,...,\alpha_K$ and additional variables such that

— if ρ and (x,α) are such that (x,α) can be extended to a feasible solution of $\mathcal{Z}_\rho^+$, (x,α) is feasible for (GRC$_\rho$).

— if ρ and (x,α) are such that (x,α) cannot be extended to a feasible solution of $\mathcal{Z}_\rho^+$, $(x,\kappa^{-1}\alpha)$ is not feasible for (GRC$_{\vartheta^{-1}\rho}$).

Tight Approximations of Ψ_k

♣ **Situation:** We are given

• Euclidean space F with closed convex cone K^F and norm $\|\cdot\|_F$

• Euclidean space E with closed convex cone K^E and norm $\|\cdot\|_E$

• A linear mapping $x \mapsto Ax : E \to F$.

♠ **Goal:** to build a tight efficiently computable convex upper bound $\Phi[A]$ on the function

$$\Psi[A] = \max_e \left\{ \mathrm{dist}_{\|\cdot\|_F}(Ae, K^F) : e \in K^E, \|e\|_E \le 1 \right\}$$

Fact: The resulting problem admits a kind of *duality*.

♡ **Definition:** Let B be a Euclidean space with a norm $\|\cdot\|$. The conjugate to $\|\cdot\|$ norm is defined as

$$\|z\|_* = \max_{y:\|y\|\le 1} \langle z, y\rangle$$

In fact, $\|\cdot\|_*$ is the smallest norm on B such that

$$\langle x, y\rangle \le \|z\|_* \|y\| \ \forall y, z.$$

Example 1: When B is $\mathbb{R}^n$ with the standard inner product, $p \in [1, \infty]$ and $\|\cdot\| = \|\cdot\|_p$, one has $\|\cdot\|_* = \|\cdot\|_q$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Example 2: When B is $\mathbb{R}^{m\times n}$ with the standard inner product, $p \in [1, \infty]$ and $\|Y\| = \|\sigma(Y)\|_p$, $\sigma(Y)$ being the vector of singular values of Y, one has $\|Z\|_* = \|\sigma(Z)\|_q$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Fact: $(\|\cdot\|_*)_* = \|\cdot\|$.

$$\max_e \left\{ \mathrm{dist}_{\|\cdot\|_F}(Ae, K^F) : e \in K^E, \|e\|_E \leq 1 \right\} \qquad (*)$$

Observe that

$$\mathrm{dist}_{\|\cdot\|_F}(Ae, K^F) = \max_f \left\{ -\langle Ae, f\rangle_F : f \in K_*^F, \|f\|_{F,*} \leq 1 \right\}$$

$\Rightarrow$

$$\max_e \left\{ \mathrm{dist}_{\|\cdot\|_F}(Ae, K^F) : e \in K^E, \|e\|_E \leq 1 \right\}$$

$$= \max_{e,f} \left\{ -\underbrace{\langle Ae, f\rangle_F}_{=\langle A^* f, e\rangle_E} : \left\{ \begin{array}{l} e \in K^E, \|e\|_E \leq 1, \\ f \in K_*^F, \|f\|_{F,*} \leq 1 \end{array} \right. \right\}$$

$$= \max_f \left\{ \mathrm{dist}_{\|\cdot\|_{E,*}}(A^* f, K_*^E) : f \in K_*^F, \|f\|_{F,*} \leq 1 \right\}$$

Thus: *Quantity* $(*)$ *remains intact when one carries out the following transformation of its data:*

- $A \leftarrow A_+ := A^*$
- $E \leftarrow E_+ := F$, $K^E \leftarrow K_+^E := K_*^F$,
  $\|\cdot\|_E \leftarrow \|\cdot\|_E^+ := \|\cdot\|_{F,*}$
- $F \leftarrow F_+ := E$, $K^F \leftarrow K_+^F := K_*^E$,
  $\|\cdot\|_F \leftarrow \|\cdot\|_F^+ := \|\cdot\|_{E,*}$.

Conclusion: *"Good cases" of* $(*)$ — tuples $(E, K^E, \|\cdot\|_E, F, K^F, \|\cdot\|_F)$ allowing for efficient computation (of a tight upper bound on) the quantity $(*)$ as a function of A — *are met in "conjugate pairs" with members of a pair linked to each other by the above transformation.*

$$E, K^E, B_E = \{\|e\|_E \le 1\} \;\&\; F, K^F, B_F = \{\|f\|_F \le 1\}$$
$$\Downarrow$$
$$\Psi[A] = \max_{e \in B_E \cap K^E} \mathrm{dist}_{\|\cdot\|_F}(Ae, K^F)$$

♣ From now on we assume that K^E, B_E, K^F, B_F are computationally tractable, and ask when $\Psi[\cdot]$ is efficiently computable (or admits tight computable upper bound).

0) [Trivial case] $\boxed{\begin{array}{l} K^E = \{0\} \\ \hline K^F = F \end{array}} \Rightarrow \Psi[A] \equiv 0$

1) $\boxed{\begin{array}{l} K^E = E, B_E = \mathrm{Conv}\{e_1, ..., e_N\} \\ \hline K^F = \{0\}, \|f\|_F = \max\limits_{i \le N} |\langle f, f_i \rangle_F| \end{array}}$

$$\Rightarrow \boxed{\begin{array}{l} \Psi[A] = \max\limits_i \mathrm{dist}_{\|\cdot\|_F}(Ae_i, K^F) \\ \hline \Psi[A] = \max\limits_{i,e} \left\{\langle Ae, f_i \rangle_F : \|e\|_E \le 1, e \in K^E\right\} \end{array}}$$

Note: In the GRC context, K^E is the cone $\mathcal{L}$ (or $\mathcal{L}_k$), and K^F is the recessive cone of $\mathbf{Q}$. Therefore **1)** implies, e.g., that *the GRC of uncertain conic constraint is tractable when*

— $\mathcal{L} = E$ *and* $\|\cdot\|_E = \|\cdot\|_1$, *or when*

— $\mathbf{Q}$ *is bounded and* $\|\cdot\|_F = \|\cdot\|_\infty$

2) $K^E = E, K_F = \{0\}$ [self-conjugate case]. In the GRC context this relates to $\mathcal{L} = E$, $\mathbf{Q}$ bounded. Here

$$\Psi[A] = \max_e \{\|Ae\|_F : \|e\|_E \leq 1\} =: \|A\|_{E,F}.$$

♡ We know 3 generic cases when $\|A\|_{E,F}$ is efficiently computable:

- $B_E = \mathrm{Conv}\{e_1, ..., e_N\}$
  $\Rightarrow \|A\|_{E,F} = \max_i \|Ae_i\|_F$
- $B_F = \{f : \langle f, f_i \rangle_F \leq 1, i \leq N\}$
  $\Rightarrow \|A\|_{E,F} = \max\limits_{i,e} \{\langle A^* f_i, e\rangle_E : \|e\|_E \leq 1\}$
- $\|\cdot\|_E = \|\cdot\|_2$, $\|\cdot\|_F = \|\cdot\|_2$
  $\Rightarrow \|A\|_{E,F} = \sqrt{\lambda_{\max}(A^*A)}$

♡ When $\|\cdot\|_E = \|\cdot\|_p$, $\|\cdot\|_F = \|\cdot\|_r$, computing $\|A\|_{E,F}$ is provably NP-hard, provided that $p > r$. However, we have

Theorem [Nesterov] *When $p \geq 2 \geq r$, $\|A\|_{E,F}$ admits the efficiently computable upper bound*

$$\min_{\mu,\nu} \left\{ \frac{\|\mu\|_{\frac{p}{p-2}} + \|\nu\|_{\frac{r}{2-r}}}{2} : \begin{bmatrix} \mathrm{Diag}\{\mu\} & A^T \\ A & \mathrm{Diag}\{\nu\} \end{bmatrix} \succeq 0 \right\},$$

and this bound is tight within the factor $\left[\frac{2\sqrt{3}}{\pi} - \frac{2}{3}\right]^{-1} \approx 2.2936$.

• Depending on p, r, the tightness factor can be improved; e.g., *when $p = \infty, r = 2$, it reduces to* $\sqrt{\pi/2}$.

3)

$$F = \mathbb{R}^m,\ \|\cdot\|_F = \|\cdot\|_\infty,\ K^F = \left\{ u : \begin{array}{r} u_i \geq 0, i \in I_+ \\ u_i \leq 0, i \in I_- \\ u_i = 0, i \in I_0 \end{array} \right\}$$

$$E = \mathbb{R}^n,\ \|\cdot\|_E = \|\cdot\|_1,\ K^E = \left\{ v : \begin{array}{r} v_j \geq 0, j \in J_+ \\ v_j \leq 0, j \in J_- \\ v_j = 0, j \in J_0 \end{array} \right\}$$

$\Rightarrow$

$$\Psi[A] = \max_{u \in U} \max_{e \in K^E, \|e\|_E \leq 1} u^T e,$$

$$U = \{-f_i\}_{i \in I_+} \cup \{f_i\}_{i \in I_-} \cup \{\pm f_i\}_{i \in I_0}$$

[f_i: basic orths in F]

$$\Psi[A] = \max_{v \in V} \mathrm{dist}_{\|\cdot\|_F}(Av, K^F),$$

$$V = \{e_j\}_{j \in J_+} \cup \{-e_j\}_{j \in J_-} \cup \{\pm e_j\}_{j \notin (J_+ \cup J_- \cup J_0)}$$

[e_j: basic orths in E]

4)

$F=\mathbb{R}^m$, $K^F=\mathbf{L}^m$, $\|\cdot\|_F=\|\cdot\|_2$, $K^E=E=\mathbb{R}^n$, $\|\cdot\|_E=\|\cdot\|_2$
$F=\mathbb{R}^n$, $K^F=\{0\}$, $\|\cdot\|_F=\|\cdot\|_2$, $E=\mathbb{R}^m$, $K^E=\mathbf{L}^m$, $\|\cdot\|_E=\|\cdot\|_2$

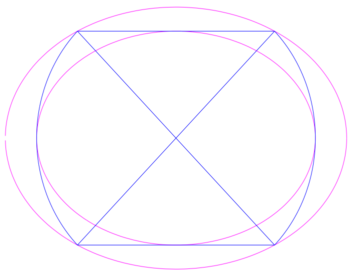
$\Rightarrow$

$\Psi[A]\leq \|A^*\mathrm{Diag}\{\sqrt{\frac{3}{2}}I_{m-1},\frac{\sqrt{3}}{2}\}\|_{2,2}\leq\sqrt{3/2}\Psi[A]$
$\Psi[A]\leq \|A\mathrm{Diag}\{\sqrt{\frac{3}{2}}I_{m-1},\frac{\sqrt{3}}{2}\}\|_{2,2}\leq\sqrt{3/2}\Psi[A]$

Explanation: Assuming $E=\mathbb{R}^m$, $\|\cdot\|_E=\|\cdot\|_2$, $K^E=\mathbf{L}^m$, $F=\mathbb{R}^n$, $K^F=\{0\}$, $\|\cdot\|_F=\|\cdot\|_2$, **we have**

$$\Psi[A]=\max_v\left\{\|Av\|_2:\begin{array}{c}v\in\mathbf{L}^m,\\ \|v\|_2\leq 1\end{array}\right\}=\max_v\left\{\|Av\|_2:v\in B\right\},$$
$$B:=\mathrm{Conv}\{\{v\in\mathbf{L}^m,\|v\|_2\leq1\}\cup\{v\in-\mathbf{L}^m,\|v\|_2\leq1\}\},$$

$\Rightarrow$ $\Psi[A]$ *is the operator norm of* A *induced by the norm* $\|\cdot\|^E$ *with the unit ball* B *in* E *and the Euclidean norm in* F. **The norm** $\|v\|^E$ **can be approximated within the factor** $\sqrt{3/2}$ **by the Euclidean norm** $\sqrt{v_1^2+...+v_{m-1}^2+2v_m^2}$**:**



Blue: set B

and the result follows.

♠ **Illustration: Least Squares Antenna Design via GRC.**

• $\|\cdot\|_2$ **Antenna Design is the uncertain Least Squares problem with instances**

$$\min_{x,\tau}\left\{\tau : \underbrace{[h - H(I+\mathrm{Diag}\{\zeta\})x;\tau] \in \mathbf{Q} \equiv \mathbf{L}^{m+1}}_{\Leftrightarrow \|h-H(I+\mathrm{Diag}\{\zeta\})x\|_2 \le \tau}\right\}$$

• $H : m \times n$ • ζ: data perturbation

♠ **The** *robust performance* **of a design** x **is**

$$F_x(\rho) = \max_{\zeta:\|\zeta\|_\infty \le \rho} \|h - H(I+\mathrm{Diag}\{\zeta\})x\|_2.$$

This is a convex nondecreasing function of ρ**.**

♠ **Let us fix an uncertainty level** $\bar\rho \ge 0$ **and set** $\|\cdot\|_F = \|\cdot\|_2$, $\mathcal{Z} = \{\zeta : \|\zeta\|_\infty \le \bar\rho\}$, $\mathcal{L} = \mathbb{R}^n$, $\|\cdot\|_E \equiv \|\cdot\|_\infty$.

Note: *A pair* (x,τ) *is robust feasible, with global sensitivity* α*, for the Least Squares Antenna Design problem, iff* $\tau \ge F_x(\bar\rho)$ *and*

$$\alpha \ge \alpha(x) := \max_{\|\zeta\|_\infty \le 1} \mathrm{dist}_{\|\cdot\|_2}([D[x]\zeta;0], \mathbf{L}^{m+1})$$
$$D[x] = H\mathrm{Diag}\{x\}$$

Observation: $\mathrm{dist}_{\|\cdot\|_2}([u;0];\mathbf{L}^{m+1}) = 2^{-1/2}\|u\|_2.$

Observation: $\lim\limits_{\rho\to\infty} \frac{d}{d\rho}F_x(\rho) = \lim_{\rho\to\infty} F_x(\rho)/\rho = \max\limits_{\|\zeta\|_\infty \le 1} \|D[x]\zeta\|$

$\Rightarrow \lim\limits_{\rho\to\infty} \frac{d}{d\rho}F_x(\rho) = 2^{1/2}\alpha(x)$

Conclusion: similarly to the LO case, $F_x(0)$, $F_x(\bar{\rho})$ and $\alpha(x)$ produce a *global upper bound* on $F_x(\cdot)$:

$$F_x(\rho) \leq \begin{cases} \frac{\bar{\rho}-\rho}{\bar{\rho}} F_x(0) + \frac{\rho}{\bar{\rho}} F_x(\bar{\rho}), & 0 \leq \rho < \bar{\rho} \\ F_x(r\bar{h}o) + 2^{1/2}\alpha(x)[\rho - \bar{\rho}], & \rho \geq \bar{\rho} \end{cases}$$

$\Rightarrow$ when the uncertainty level $\bar{\rho}$ we should work with is only vaguely known, we can use the GRC methodology to optimize, to some extent, a *global* upper bound on the robust performance, similarly to what we did in the $\|\cdot\|_\infty$ Antenna Design.

Tractability Issues

♣ **In contrast to the LO case, now neither $F_x(\rho)$, nor $\alpha(x)$ is easy to compute. However,** *these quantities admit tight tractable upper bounds:*

- $$F_x(\rho) = \max_{\|\zeta\|_\infty} \| \underbrace{h - H(I + \mathrm{Diag}(\zeta))x}_{h-Hx+D[x]\zeta} \|_2$$
$$= \max_{\eta,t:\|[\eta;t]\|_\infty \le 1} \|\rho D[x]\eta + t[h - Hx]\|_2$$
$$= \|[\rho D[x], h - Hx]\|_{\infty,2}.$$

By Nesterov's Norm Bound Theorem, the efficiently computable quantity

$$\widehat{F}_x(\rho) = \min_{\mu,\nu} \left\{ \frac{\|\mu\|_1 + \nu}{2} : \left[\begin{array}{c|c} \nu I_m & [\rho D[x], h - Hx] \\ \hline [\rho D[x], h - Hx]^T & \mathrm{Diag}\{\mu\} \end{array}\right] \succeq 0 \right\}$$

is a tight within the factor $\sqrt{\pi/2}$ upper bound on $F_x(\rho)$.

Note: same as $F_x(\rho)$, $\widehat{F}_x(\rho)$ is a convex nondecreasing function of $\rho \ge 0$.

- $$\alpha(x) = \max_{\|\zeta\|_\infty \le 1} \mathrm{dist}_{\|\cdot\|_2}([D[x]\zeta; 0], \mathbf{L}^{m+1})$$
$$= 2^{-1/2} \max_{\|\zeta\|_\infty \le 1} \|D[x]\zeta\|_2 = 2^{-1/2}\|D[x]\|_{\infty,2}$$

$\Rightarrow$ the efficiently computable quantity

$$\widehat{\alpha}(x) = \min_{\mu,\nu} \left\{ \frac{\|\mu\|_1+\nu}{2\sqrt{2}} : \left[\begin{array}{c|c} \nu I_m & D[x] \\ \hline D^T[x] & \mathrm{Diag}\{\mu\} \end{array}\right] \succeq 0 \right\}$$

is a tight within the factor $\sqrt{\pi/2}$ upper bound on $\alpha(x)$.

Note: $\widehat{\alpha}(x) = 2^{-1/2} \lim_{\rho\to\infty} \frac{d}{d\rho}\widehat{F}_x(\rho)$.

♣ **Summary:** *the efficiently computable quantities* $\widehat{F}_x(\rho)$, $\widehat{\alpha}(x)$ *are tight, within the factor* $\sqrt{\pi/2}$, *upper bounds on* $F_\rho(x)$, $\alpha(x)$, *respectively*.

Since $\widehat{F}_x(\rho)$ **is convex and nondecreasing in** ρ **and** $\widehat{\alpha}(x) = 2^{-1/2}\lim\limits_{\rho\to\infty}\frac{d}{d\rho}\widehat{F}_x(\rho)$**, we have**

$$F_x(\rho) \leq \begin{cases} \frac{\bar{\rho}-\rho}{\bar{\rho}}\widehat{F}_x(\bar{\rho}) + \frac{\rho}{\bar{\rho}}\widehat{F}_x(0), & 0 \leq \rho < \bar{\rho} \\ \widehat{F}_x(\rho) + 2^{1/2}\widehat{\alpha}(x)[\rho - \bar{\rho}], & \rho \geq \bar{\rho} \end{cases}$$

and we can optimize, to some extent, the right hand side in order to ensure a desirable robust performance of our design in a wide range of values of ρ**.**

♠ **Illustration:** Setting $\bar{\rho} = 0$, we solve the problems

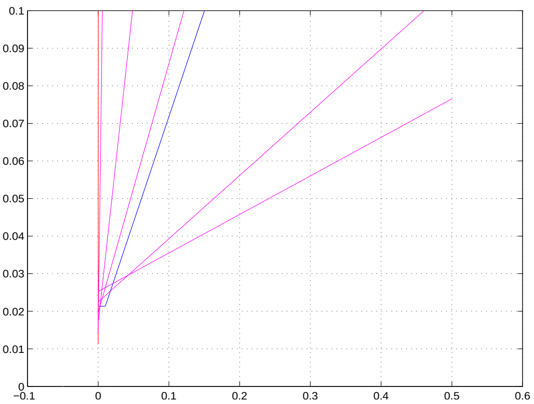
$$\begin{array}{rcl}\beta(\delta) & = & \min\limits_x \Big\{\widehat{\alpha}(x) : \widehat{F}_x(0) \equiv \|h - Hx\|_2 \\ & & \leq (1+\delta)\min\limits_u \|h - Hu\|_2\Big\}\end{array}$$

for several values of δ, thus getting global upper bounds

$$\widehat{F}_{x_\delta}(0) + 2^{1/2}\beta(\delta)\rho \qquad (*)$$

on the robust performances of the resulting designs. The end-user could then choose the design he finds the most appropriate.

δ	0	0.25	0.50	0.75	1.00	1.25
$\beta(\delta)$	9441.4	14.883	1.7165	0.6626	0.1684	0.1025



Red and Magenta: bounds $(*)$ for the values of δ from the table.
Blue: Bound on global performance of the RC design corresponding to $\rho = 0.01$ (global sensitivity ≤ 0.3962).