Uncertain Conic Quadratic and Semidefinite Optimization

♣ "Canonical" Conic problem:

$$\min_x \left\{ c^T x + d : \begin{array}{c} A_1 x + b_1 \in \mathbf{K}_1 \\ \cdots \\ A_I x + b_I \in \mathbf{K}_I \end{array} \right\} \qquad \text{(CP)}$$

- x: decision vector
- $\mathbf{K}_i$: simple cone: *nonnegative orthant* $\mathbb{R}^m_+$, or *Lorentz cone* $\mathbf{L}^m = \{y \in \mathbb{R}^m : y_m \geq \sqrt{y_1^2 + ... + y_{m-1}^2}\}$, or *Semidefinite cone* $\mathbf{S}^m_+$ *comprised of positive semidefinite symmetric* $m \times m$ *matrices*, with $m = m_i$.
- **Structure** of (CP): the collection of cones $\mathbf{K}_1, ..., \mathbf{K}_I$
- **Data** of (CP): $c, d, A_1, b_1, ..., A_I, b_I$

$$\min_x \left\{ c^T x + d : \begin{array}{c} A_1 x + b_1 \in \mathbf{K}_1 \\ \cdots \\ A_I x + b_I \in \mathbf{K}_I \end{array} \right\} \tag{CP}$$

♠ **Uncertain canonical conic problem $\mathcal{P}$:** *a collection of canonical conic programs* ("instances") *with common structure and with the data running through a given uncertainty set $\mathcal{U}$.*

♠ *We always assume that the data is affinely parameterized by a perturbation vector ζ running through perturbation set $\mathcal{Z}$:*

$$\mathcal{U} = \Big\{ (c[\zeta], d[\zeta], A_1[\zeta], ..., b_I[\zeta]) := (c^0, d^0, A_1^0, ..., b_I^0) + \sum_{\ell=1}^{L} \zeta_\ell (c^\ell, d^\ell, A_1^\ell, ..., b_I^\ell) : \zeta \in \mathcal{Z} \Big\}$$

♠ **Robust Counterpart** *of uncertain canonical conic problem $\mathcal{P}$: the problem*

$$\min_{t,x} \left\{ t : \left. \begin{array}{rcl} t - c[\zeta]^T x - d[\zeta] & \in & \mathbb{R}_+ \\ A_1[\zeta]x + b_1[\zeta] & \in & \mathbf{K}_1 \\ & \cdots & \\ A_I[\zeta]x + b_I[\zeta] & \in & \mathbf{K}_I \end{array} \right\} \forall \zeta \in \mathcal{Z} \right\} \tag{RC}$$

$$\min_x \left\{ c^T x + d : \begin{array}{c} A_1 x + b_1 \in \mathbf{K}_1 \\ \cdots \\ A_I x + b_I \in \mathbf{K}_I \end{array} \right\}_{(c,d,A_1,\dots,b_I)\in\mathcal{U}}$$

$$\begin{array}{rcl} \mathcal{U} & = & \Big\{ (c[\zeta], d[\zeta], A_1[\zeta], \dots, b_I[\zeta]) := (c^0, d^0, A_1^0, \dots, b_I^0) \\ & & \quad + \sum_{\ell=1}^{L} \zeta_\ell (c^\ell, d^\ell, A_1^\ell, \dots, b_I^\ell) : \zeta \in \mathcal{Z} \Big\} \end{array} \tag{UCP}$$

♠ **Same as in the LO case**, we w.l.o.g. can assume the objective to be certain (and then skip d), in which case the RC becomes

$$\min_x \left\{ c^T x : \left\{ \begin{array}{c} A_1[\zeta]x + b_1[\zeta] \ \in \ \mathbf{K}_1 \\ \dots \\ A_I[\zeta]x + b_I[\zeta] \ \in \ \mathbf{K}_I \end{array} \right\} \forall \zeta \in \mathcal{Z} \right\} \tag{RC}$$

Note: Same as in LO, the RC remains intact when extending $\mathcal{Z}$ to its closed convex hull

$\Rightarrow$ *From now on, we always assume $\mathcal{Z}$ to be convex and closed.*

♠ **Same as in the LO case**, the RC of (UCP) with certain objective is a constraint-wise construction, and we can focus on the RC

$$A[\zeta]x + b[\zeta] \in \mathbf{K} \tag{RC}$$

of a *single* uncertain conic constraint.

$$\begin{array}{c} A[\zeta]x + b[\zeta] \in \mathbf{K}\ \forall \zeta \in \mathcal{Z} \\ [A[\zeta], b[\zeta]] = [A^0, b^0] + \sum_{\ell=1}^{L} \zeta_\ell [A^\ell, b^\ell] \end{array} \tag{RC}$$

♣ **Questions of primary importance:**

• When the semi-infinite conic constraint (RC) is computationally tractable?

• What to do when (RC) is intractable?

♠ **Fact:** Tractability of (RC) depends on "tradeoff" between the geometries of the perturbation set $\mathcal{Z}$ and the cone $\mathbf{K}$: the simpler is $\mathcal{Z}$, the more complicated can be $\mathbf{K}$.

• When $\mathbf{K}$ is as simple as possible – just a nonnegative orthant (uncertain LO), $\mathcal{Z}$ can be a whatever computationally tractable convex set, e.g., one given by well-structured conic representation.

• When $\mathcal{Z}$ is as simple as possible:

$$\mathcal{Z} = \mathrm{Conv}\left\{\zeta^1, ..., \zeta^N\right\}$$

(scenario uncertainty), (RC) is equivalent to

$$A[\zeta^i]x + b[\zeta^i] \in \mathbf{K},\ \ i = 1, ..., N,$$

that is, (RC) is tractable whenever the cone $\mathbf{K}$ is so (as is the case for nonnegative orthants, Lorentz and Semidefinite cones we are interested in).

$$\begin{array}{c} A[\zeta]x + b[\zeta] \in \mathbf{K}\ \forall \zeta \in \mathcal{Z} \\ [A[\zeta], b[\zeta]] = [A^0, b^0] + \sum_{\ell=1}^{L} \zeta_\ell [A^\ell, b^\ell] \end{array} \qquad \text{(RC)}$$

• In the "in-between" situations, tractability of (RC) is a "rare commodity:"

• When $\mathbf{K}$ is the Lorentz cone, (RC) is tractable when $\mathcal{Z}$ is an ellipsoid, and is intractable when $\mathcal{Z}$ is a box.

Indeed, checking feasibility of $x = 0$ for the semi-infinite Least Squares inequality

$$\begin{array}{c} \|Ax + b\|_2 \leq 1\ \forall b \in \mathcal{U} = \{B\zeta : \|\zeta\|_\infty \leq 1\} \\ \Leftrightarrow [Ax + B\zeta; 1] \in \mathbf{L}^{m+1}\ \forall \zeta, \|\zeta\|_\infty \leq 1 \end{array}$$

reduces to checking whether $\|B\zeta\|_2 \leq 1$ for all ζ with $\|\zeta\|_\infty \leq 1$, or, equivalently, to the maximization of the positive semidefinite quadratic form $\zeta^T[B^TB]\zeta$ over the unit box, which is NP-hard even when accuracy of 4% is sought.

• When $\mathbf{K}$ is a Semidefinite cone, (RC) is intractable when $\mathcal{Z}$ is either an ellipsoid or a box.

♠ As a result, *In Robust Conic Optimization the goals of primary importance are*

• *To discover special cases where* (RC) *is tractable*, and

• *To build tight safe tractable approximations to* (RC) *when* (RC) *"'as it is" is intractable.*

$$
\begin{array}{c}
A[\zeta]x + b[\zeta] \in \mathbf{K}\ \forall \zeta \in \mathcal{Z} \\
[A[\zeta], b[\zeta]] = [A^0, b^0] + \sum_{\ell=1}^{L} \zeta_\ell [A^\ell, b^\ell]
\end{array} \qquad \text{(RC)}
$$

Definition. A system $\mathcal{S}$ of efficiently computable convex constraints in variables x and, perhaps, additional variables u is called a *safe tractable approximation* of (RC) if the projection $X[\mathcal{S}]$ of the feasible set of the system on the plane of x-variables is contained in the feasible set of (RC), that is,

$$(x, u) \text{ is feasible for } \mathcal{S} \ \Rightarrow\ x \text{ is feasible for (RC)}$$

♡ Replacing the RC's of the conic constraints in an uncertain conic problem $\mathcal{P}$ with their safe tractable approximations, we end up with a *computationally tractable* problem such that (the x-components of) its feasible solutions are *feasible for the RC* of the uncertain problem.

♠ **Question:** How to quantify "tightness" of a safe approximation?

♠ **Answer:** Assume, as it usually is the case, that $0 \in \mathcal{Z}$ ($\zeta = 0$ corresponds to the nominal data).

⇒ We can embed (RC) in a parametric family of RC's

$$A[\zeta]x + b[\zeta] \in \mathbf{K}\ \forall \zeta \in \mathcal{Z}_\rho := \rho\mathcal{Z} \qquad \text{(RC}[\rho])$$

As the *uncertainty level* ρ grows, the perturbation set $\mathcal{Z}_\rho$ extends, and the feasible set X_ρ of (RC[ρ]) shrinks.

$$A[\zeta]x + b[\zeta] \in \mathbf{K} \ \forall \zeta \in \mathcal{Z}_\rho := \rho\mathcal{Z} \qquad (\mathrm{RC}[\rho])$$

Definition: A *safe tractable approximation* of (RC[ρ]) is a system $\mathcal{S}[\rho]$ of efficiently computable convex constraints, depending on $\rho \geq 0$ as on a parameter, in variables x and additional variables u such that the projection $\boldsymbol{\mathcal{X}}_\rho$ of the feasible set of $\mathcal{S}[\rho]$ on the plane of x-variables is, for every $\rho \geq 0$, contained in the feasible set X_ρ of (RC[ρ]).

Such an approximation is called *ϑ-tight*, if

$$\forall \rho \geq 0 : X_\rho \supset \boldsymbol{\mathcal{X}}_\rho \supset X_{\vartheta\rho}.$$

Equivalently: $\mathcal{S}[\cdot]$ is ϑ-tight safe approximation of (RC[$\cdot$]), if, for every $\rho \geq 0$,

• [safety] whenever x can be extended to a feasible solution to $\mathcal{S}[\rho]$, x is feasible for (RC[ρ]), and

• [tightness] whenever x cannot be extended to a feasible solution to $\mathcal{S}[\rho]$, x is not feasible for (RC[$\vartheta\rho$]).

• We call an approximation scheme *tight*, if its tightness factor is independent of the numerical values of the data specifying $\mathcal{Z}$.

Example: The semi-infinite Least Squares inequality with box uncertainty

$$\begin{array}{c}\forall\zeta, \|\zeta\|_\infty \leq \rho : \|A[\zeta]x + b[\zeta]\|_2 \leq \tau \\ \Updownarrow \\ [A[\zeta]x + b[\zeta]; \tau] \in \mathbf{L}^{m+1}\ \forall\zeta, \|\zeta\|_\infty \leq \rho \\ [A[\zeta], b[\zeta]] = [A^0, b^0] + \sum_{\ell=1}^L \zeta_\ell [A^\ell, b^\ell]\end{array} \qquad (\mathrm{RC}[\rho])$$

is, in general, computationally intractable. It, however, admits a *tight within the factor* $\pi/2$ safe tractable approximation

$$\begin{array}{l}\left[\begin{array}{c|c}\tau & [A^0x+b^0]^T \\ \hline A^0x+b^0 & \tau I\end{array}\right] - \rho \sum\limits_{\ell=1}^L Y_\ell \succeq 0 \\ Y_\ell \succeq \pm \left[\begin{array}{c|c} & [A^\ell x+b^\ell]^T \\ \hline A^\ell x+b^\ell & \end{array}\right] \succeq 0,\ 1 \leq \ell \leq L\end{array} \qquad \mathcal{S}[\rho]$$

(variables are x and symmetric matrices $Y_1, ..., Y_L$).

♡ As far as the x-components of feasible solutions are concerned, the above system can be replaced with a system with a much smaller number of variables, namely

$$\left[\begin{array}{c|c|cccc}\tau - \sum_{\ell=1}^L \lambda_\ell & [A^0x+b^0]^T & & & \\ \hline A^0x+b^0 & \tau I_m & A^1x+b^1 & \ldots & A^Lx+b^L \\ \hline & [A_1x+b_1]^T & \lambda_1 & & \\ & \vdots & & \ddots & \\ & [A^Lx+b^L]^T & & & \lambda_L\end{array}\right] \succeq 0$$

(variables are x and $\lambda_1, ..., \lambda_L$).

Note: Let $\mathcal{P}$ be an uncertain canonical conic problem with certain objective. Assume that the RC's of all conic constraints of $\mathcal{P}$ admit ϑ-tight safe tractable approximations. Then the optimal value $\mathrm{Opt}_{\mathrm{Appr}}(\rho)$ of the resulting safe tractable approximation of $\mathcal{P}$, treated as a function of the uncertainty level ρ, satisfies

$$\mathrm{Opt}_{\mathcal{P}}(\rho) \leq \mathrm{Opt}_{\mathrm{Appr}}(\rho) \leq \mathrm{Opt}_{\mathcal{P}}(\vartheta\rho).$$

Tractable Reformulations and Tight Tractable Approximations of Semi-Infinite Conic Quadratic Inequalities

♣ The fact that a vector $[Ax+b; c^T x+d] \in \mathbb{R}^{m+1}$ affinely depending on x belongs to the Lorentz cone $\mathbf{L}^{m+1}$ can be equivalently represented by *conic quadratic inequality* (c.q.i.)

$$\|Ax+b\|_2 \leq c^T x + d \tag{CQI}$$

When $[A, b; c^T, d]$ are affinely parameterized by a perturbation vector ζ:

$$\begin{array}{l} [A, b; c^T, d] = [A[\zeta], b[\zeta]; c^T[\zeta], d[\zeta]] \\ = [A^0, b^0; [c^0]^T, d^0] + \sum_{\ell=1}^{L} \zeta_\ell [A^\ell, b^\ell; [c^\ell]^T, d^\ell] \end{array}$$

and we want the inclusion $[Ax+b; c^T x+d] \in \mathbf{L}^{m+1}$ (or, equivalently, the c.q.i. (CQI)) to hold true for all $\zeta \in \mathcal{Z}$, we end up with *semi-infinite* c.q.i.

$$\|A[\zeta]x + b[\zeta]\|_2 \leq c^T[\zeta]x + d[\zeta] \ \forall \zeta \in \mathcal{Z} \tag{RC}$$

Note: Convex quadratic constraint

$$x^T A^T A x + 2b^T x + c \leq 0,$$

the data being A, b, c, is equivalent to the c.q.i.

$$\|[2Ax; 1 + 2b^T x + c]\|_2 \leq 1 - 2b^T x - c$$

$\Rightarrow$ *What follows covers, in particular, the RC's of uncertainty-affected convex quadratic constraints.*

$$\|A[\zeta]x+b[\zeta]\|_2\leq c^T[\zeta]x+d[\zeta]\ \forall\zeta\in\mathcal{Z} \qquad \text{(RC)}$$

♠ Aside of the trivial case of scenario-generated perturbation set $\mathcal{Z}$, essentially the only known generic case when (RC) is computationally tractable/admits tight computationally tractable approximation *independently of any assumptions on how the affine perturbations enter the problem* is the case of *an ellipsoid $\mathcal{Z}$.*

All other tractability results known to us deal with the case of *semi-infinite Least Squares inequality*

$$\|A[\zeta]x+b[\zeta]\|_2\leq\tau\ \forall\zeta\in\mathcal{Z}$$

in variables x,τ, or, which is the same, with the case when the right hand side in (RC) is not affected by the uncertainty, since in this case (RC) is equivalent to

$$\|A[\zeta]x+b[\zeta]\|_2\leq\tau\ \forall\zeta\in\mathcal{Z}\ \&\ \tau\leq c^Tx+d$$

and tractable reformulation/building tight safe tractable approximation of the latter system reduces to the same problem for the "troublemaking" semi-infinite Least Squares inequality.

$$\|A[\zeta]x + b[\zeta]\|_2 \le c^T[\zeta]x + d[\zeta]\ \forall \zeta \in \mathcal{Z} \qquad \text{(RC)}$$

♡ **Note:** (RC) with *side-wise* uncertainty, where the perturbation set $\mathcal{Z}$ is the product $\mathcal{Z}^{\text{left}} \times \mathcal{Z}^{\text{right}}$ of sets of perturbations affecting the left- and the right hand side data, reduces to the system of semi-infinite constraints

$$\begin{array}{ll} (a) & \|A[\zeta^{\text{left}}]x + b[\zeta^{\text{left}}]\|_2 \le \tau\ \forall \zeta^{\text{left}} \in \mathcal{Z}^{\text{left}}, \\ (b) & c^T[\zeta^{\text{right}}]x + d[\zeta^{\text{right}}] \ge \tau\ \forall \zeta^{\text{right}} \in \mathcal{Z}^{\text{right}} \end{array}$$

in variables x, τ.

• Whenever the right hand side perturbation set is tractable, the semi-infinite constraint (b) is computationally tractable

⇒ *All we need to process the RC efficiently is to build a tractable reformulation/tight tractable approximation of the semi-infinite Least Squares inequality* (a).

$$\|A[\zeta]x + b[\zeta]\|_2 \leq c^T[\zeta]x + d[\zeta] \ \forall \zeta \in \mathcal{Z} \qquad \text{(RC)}$$

Tractable Case I: $\mathcal{Z}$ is an Ellipsoid

♣ W.l.o.g. we can assume that the ellipsoid $\mathcal{Z}$ is just the unit ball centered at the origin:

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_2 \leq 1\}.$$

♠ A well-structured representation of (the feasible set of) (RC) was recently found by R. Hildenbrandt
(Highly nontrivial construction with highly nontrivial justification.)

♠ We restrict ourselves with an easy demonstration that $\mathcal{Z}$ *being the ball, the feasible set X of (RC)* (which clearly is a closed convex set) *admits an efficient Separation oracle – a routine which, given on input x, reports whether $x \in X$, and if it is not the case, returns a separator - a linear form e such that*

$$e^T x > \max_{x' \in X} e^T x'.$$

Note: Given an efficient Separation oracle for X, we can use, say, the Ellipsoid method to solve efficiently convex problems of the form

$$\min_{x \in X} \{f(x) : g_i(x) \leq 0, \, i = 1, ..., m\}$$

provided that f and g_i are efficiently computable.

$$X = \left\{ x : \|A[\zeta]x + b[\zeta]\|_2 \leq c^T[\zeta]x + d[\zeta] \ \forall \zeta, \|\zeta\|_2 \leq 1 \right\} \tag{RCBall}$$

♠ Since $A[\zeta]$,..., $d[\zeta]$ are affine in ζ, (RCBall) reads

$$X = \{x : \|\alpha[x]\zeta + \beta[x]\|_2 \leq \gamma^T[x]\zeta + \delta[x] \ \ \forall \zeta, \|\zeta\|_2 \leq 1\}$$

with $\alpha[x]$,...,$\delta[x]$ affine in x.

♣ Clearly, $x \in X$ *iff*

A. $\|\zeta\|_2 \leq 1 \to \gamma^T[x]\zeta + \delta[x] \geq 0 \Leftrightarrow \|\gamma[x]\|_2 \leq \delta[x]$

and

B. *The quadratic form of* ζ: $\zeta^T P[x]\zeta + 2p^T[x]\zeta + q[x]$ $\equiv [\gamma^T[x]\zeta + \delta[x]]^2 - \|\alpha[x]\zeta + \beta[x]\|_2^2$ *is* ≥ 0 *on the domain* $\zeta^T\zeta \leq 1$.

♣ **Miracle # 1: $\mathcal{S}$-Lemma:** *Consider two quadratic forms*

$$f(z) = z^T Az + 2a^T z + \alpha, \ g(z) = z^T Bz + 2b^T z + \beta$$

and let the set $\{z : g(z) > 0\}$ *be nonempty. Then the implication*

$$g(z) \geq 0 \Rightarrow f(z) \geq 0$$

holds true iff $\exists \lambda \geq 0 : f(z) \geq \lambda g(z) \ \forall z$, *that is, iff*

$$\exists \lambda \geq 0 : \left[\begin{array}{c|c} A - \lambda B & a - \lambda b \\ \hline [a - \lambda b]^T & \alpha - \lambda\beta \end{array} \right] \succeq 0.$$

$$X = \{x : \|\alpha[x]\zeta + \beta[x]\|_2 \le \gamma^T[x]\zeta + \delta[x]\ \ \forall \zeta, \|\zeta\|_2 \le 1\}$$

♠ $x \in X$ iff the following two properties take place:
A. $\|\gamma[x]\|_2 \le \delta[x]$
B. $g(z) = 1 - z^T z \ge 0 \Rightarrow f(z) := z^T P[x] z + 2p^T[x]z + q[x]$
$\Leftrightarrow \exists \lambda > 0 : \left[\begin{array}{c|c} P[x] + \lambda I & p[x] \\ \hline p^T[x] & q[x] - \lambda \end{array}\right] \succeq 0$

♡ Given x, we can easily verify **A** and **B**; if both the properties hold true, we report that $x \in X$.

• Now let *either* **A**, *or* **B**, *or both do not take place.* On a closest inspection, here we can find efficiently $\bar\zeta$, $\|\bar\zeta\|_2 \le 1$, such that the vector

$$\bar y = [\alpha[x]\bar\zeta + \beta[x]; \gamma^T[x]\bar\zeta + \delta[x]]$$

does *not* belong to $\mathbf{L}^{m+1}$, and thus can be easily separated from $\mathbf{L}^{m+1}$, That is, we can efficiently build $\eta \in \mathbb{R}^{L+1}$ such that

$$\eta^T \bar y > s := \sup_{y \in \mathbf{L}^{m+1}} \eta^T y.$$

$\Rightarrow$ The affine form $e[\xi] = \eta^T[\alpha[\xi]\bar\zeta + \beta[\xi]; \gamma^T[\xi]\bar\zeta + \delta[\xi]]$ separates x and X, and we can return this form as a required separator.

Tractable Case II: Semi-Infinite Least Squares Inequality with Unstructured Norm-Bounded Uncertainty

$$\|A[\zeta]x + b[\zeta]\|_2 \leq \tau \ \forall \zeta \in \mathcal{Z} \tag{RC}$$

Definition: We say that the uncertainty in (RC) is *unstructured norm-bounded*, if

• $\mathcal{Z}$ is the set of $p \times q$ matrices of matrix norm $\|\cdot\|_{2,2}$ not exceeding 1

• $A[\zeta]x + b[\zeta] \equiv A^0 x + b^0 + L^T[x]\zeta R[x]$ with matrices $L[x]$, $R[x]$ of appropriate sizes affinely depending on x and such that *either $L[x]$, or $R[x]$ is constant.*

♠ **Example: $\mathcal{Z}$ is an ellipsoid.** W.l.o.g. we can assume that $\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_2 \leq 1\}$, that is, $\mathcal{Z}$ is comprised of $L \times 1$ matrices ζ of norm ≤ 1. We have

$$A[\zeta]x + b[\zeta] \equiv \alpha[x]\zeta + \beta[x]$$

with affine in x $\alpha[x]$ and $\beta[x]$, and we can set

$$A^0 x + b^0 = \beta[x],\ L^T[x] = \alpha[x],\ R[x] = 1.$$

$$\begin{array}{c} \|A[\zeta]x + b[\zeta]\|_2 \le \tau \ \forall \zeta \in \mathbb{R}^{p\times q} : \|\zeta\|_{2,2} \le 1 \\ A[\zeta]x + b[\zeta] \equiv A^0 x + b^0 + L^T[x]\zeta R[x] \end{array} \qquad \text{(RC)}$$

Theorem: *When $R[x]$ is independent of x,* (RC) *can be represented equivalently by the Linear Matrix Inequality*

$$\left[\begin{array}{c|c|c} \tau - \lambda R^T R & [A^0 x + b^0]^T & \\ \hline A^0 x + b^0 & \tau I_m & L^T[x] \\ \hline & L[x] & \lambda I_p \end{array}\right] \succeq 0,$$

in variables x, λ.
When $L[x]$ is independent of x, (RC) *can be represented equivalently by the LMI*

$$\left[\begin{array}{c|c|c} \tau & [A^0 x + b^0]^T & R^T[x] \\ \hline A^0 x + b^0 & \tau I_m - \lambda L^T L & \\ \hline R[x] & & \lambda I_q \end{array}\right] \succeq 0$$

in variables x, λ.
Proof: to be given later.

Example [Robust Linear Estimation] *Given noisy observation*

$$w = [A + B\Delta C]v + \xi \qquad [\xi \sim \mathcal{N}(0,\Sigma)]$$

of unknown signal v known to belong to a given ellipsoid $\{v : v^T Qv \le R^2\}$, estimate the value at v of a given linear form $\langle f, \cdot\rangle$.

Here: • A, B, C are given matrices, • Δ is unknown perturbation known to be norm-bounded: $\|\Delta\|_{2,2} \le \rho$ with a given ρ.

We restrict ourselves with linear in w estimates $x^T w$ (x is the weight vector to be found) *and want to minimize the worst-case expected squared recovery error.* Thus, our problem is

$$\min_x \left\{ \max_{\substack{v:v^T Qv \le R^2 \\ \Delta:\|\Delta\|_{2,2}\le\rho}} \sqrt{\mathbf{E}\left\{\left(f^T v - x^T[[A + B\Delta C]v + \xi]\right)^2\right\}} \right\}$$

We have

$$\begin{array}{l}
\mathbf{E}\left\{\left(f^T v - x^T[[A + B\Delta C]v + \xi]\right)^2\right\} = \left[v^T[f - [A^T + C^T\Delta^T B^T]x]\right]^2 + x^T\Sigma x \\
\Rightarrow \max\limits_{v:v^T AQv \le R^2} \mathbf{E}\left\{\left(f^T v - x^T[[A + B\Delta C]v + \xi]\right)^2\right\} \\
\qquad\qquad = R^2[f - [A^T + C^T\Delta^T B^T]x]^T Q^{-1}[f - [A^T + C^T\Delta^T B^T]x] + x^T\Sigma x.
\end{array}$$

$\Rightarrow$ *the problem of interest is*

$$\min_{x,\tau,s,r} \left\{ r : \begin{array}{c} \|\Sigma^{1/2}x\|_2 \le s, \sqrt{R^2\tau^2 + s^2} \le r \\ \|Q^{-1/2}[f - [A^T + C^T\Delta^T B^T]x]\|_2 \le \tau \\ \forall \Delta, \|\Delta\|_{2,2} \le \rho \end{array} \right\}.$$

$$\min_{x,\tau,s,r}\left\{ r:\ \begin{array}{r} \|\Sigma^{1/2}x\|_2 \leq s, \sqrt{R^2\tau^2+s^2} \leq r \\ \|Q^{-1/2}[f-[A^T+C^T\Delta^T B^T]x]\|_2 \leq \tau \\ \forall \Delta, \|\Delta\|_{2,2} \leq \rho \end{array} \right\}.$$

♠ The only "troublemaker" is the semi-infinite Least Squares inequality

$$\begin{array}{l} \|Q^{-1/2}[f-[A^T+C^T\Delta^T B^T]x]\|_2 \\ \equiv \| \overbrace{Q^{-1/2}[f-A^T x]}^{A^0 x+b^0} + \overbrace{[-Q^{-1/2}C^T\Delta^T B^T x]}^{L^T\zeta R[x],\zeta=\Delta^T/\rho} \|_2 \leq \tau \\ \hfill \forall \zeta : \|\zeta\|_{2,2} \leq 1 \end{array}$$

Passing to its tractable reformulation, the problem of interest becomes an explicit canonical conic program

$$\min_{x,\tau,r,s,\lambda}\left\{ r:\ \begin{array}{l} \|\Sigma^{1/2}x\|_2 \leq s, \sqrt{R^2\tau^2+s^2} \leq r \\ \left[\begin{array}{c|c|c} \tau & [Q^{-1/2}[f-A^T x]]^T & \rho x^T B \\ \hline Q^{-1/2}[f-A^T x] & \tau I-\lambda Q^{-1/2}C^T C Q^{-1/2} & \\ \hline \rho B^T x & & \lambda I \end{array}\right] \succeq 0 \end{array} \right\}$$

Tight Approximation of Semi-Infinite Least Squares Inequality with Structured Norm-Bounded Uncertainty

$$\|A[\zeta]x + b[\zeta]\|_2 \leq \tau \ \forall \zeta \in \rho\mathcal{Z} \tag{RC}$$

Definition: We say that the uncertainty in (RC) is *structured norm-bounded*, if

• $\mathcal{Z}$ is the set of collections of L $p_\ell \times q_\ell$ matrices ζ^ℓ of matrix norm $\|\cdot\|_{2,2}$ not exceeding 1

• $A[\zeta]x + b[\zeta] \equiv A^0x + b^0 + \sum_{\ell=1}^{L} L_\ell^T[x]\zeta^\ell R_\ell[x]$ with matrices $L_\ell[x]$, $R_\ell[x]$ of appropriate sizes affinely depending on x and such that *for every ℓ, either $L_\ell[x]$, or $R_\ell[x]$ is constant*.

♠ **Example: $\mathcal{Z}$ is a box.** W.l.o.g. we can assume that $\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq 1\}$, that is, $\mathcal{Z}$ is comprised of L 1×1 matrices $\zeta^\ell = \zeta_\ell$ of norm ≤ 1. We have

$$A[\zeta]x + b[\zeta] \equiv \sum_{\ell=1}^{L} \zeta_\ell \alpha_\ell[x] + \beta[x]$$

with affine in x $\alpha_\ell[x]$ and $\beta[x]$, and we can set

$$A^0x + b^0 = \beta[x],\ L_\ell^T[x] = \alpha_\ell[x],\ R_\ell[x] = 1.$$

$$
\begin{array}{l}
\|A[\zeta]x + b[\zeta]\|_2 \\
\equiv \|A^0 x + b^0 + \sum_{\ell=1}^{L} L_\ell^T[x]\zeta^\ell R_\ell[x]\|_2 \leq \tau \ \forall \zeta \in \rho\mathcal{Z}, \quad \text{(RC)} \\
\mathcal{Z} = \{\zeta = (\zeta^1, ..., \zeta^L) : \zeta^\ell \in \mathbb{R}^{p_\ell \times q_\ell}, \|\zeta^\ell\|_{2,2} \leq 1\}.
\end{array}
$$

Theorem: *Semi-infinite Least Squares inequality with structured norm-bounded uncertainty admits safe tractable approximation given by an explicit system of LMIs and tight within the factor* $\pi/2$. *This approximation is precise when* $L = 1$ (i.e., in the case of unstructured norm-bounded perturbation).

Explicit representation of the approximation and its justification will be presented later.

Example: Antenna Design with Least Squares fit. When speaking on Robust LO, we have considered the Antenna Design problem with uniform fit. Now consider similar problem with Least Squares fit:

$$\min_{x,\tau}\left\{\tau : \|D_* - \sum_{\ell=1}^{10} x_\ell D_\ell\|_{2,w} \le \tau\right\}$$
$$\left[\begin{array}{c} x_\ell \mapsto (1+\zeta_\ell)x_\ell,\ \zeta_\ell \in [-\rho,\rho] \\ \Updownarrow \\ D_\ell \mapsto (1+\zeta_\ell)D_\ell,\ \zeta_\ell \in [-\rho,\rho] \end{array}\right]$$

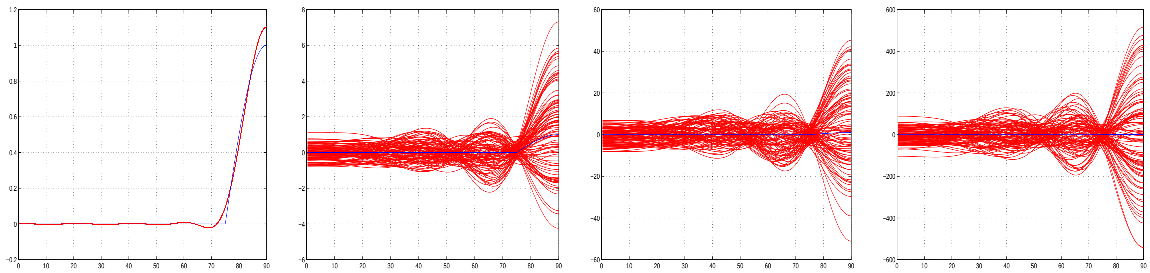
• D_*, $D_1, ..., D_{10}$: restrictions onto the grid $\delta_i = i\pi/480$, $1 \le i \le 240$ of altitude angles of the target diagram and the diagrams of 10 antenna elements.

• $\|D\|_{2,w}^2 = \frac{1}{240}\sum_{i=1}^{240}\cos(\theta_i)D^2(\theta_i)$

Origin of weights: Physically, diagrams in question are functions on the upper hemisphere S depending solely of the altitude angle, and our weighted 2-norm mimics the standard norm of $L_2(S)$.

$$\min_{\tau,x}\left\{\tau : \|D_* - \sum_{\ell=1}^{10} x_\ell(1+\zeta_\ell)D_\ell\|_{2,w} \leq \tau\right\}$$
$$-\rho \leq \zeta_\ell \leq \rho,\ 1 \leq \ell \leq 10$$

♠ **Nominal optimal design**, same as in the case of uniform fit, is a complete disaster when implementation errors are present:



$\rho = 0$ $\quad$ $\rho = 0.0001$ $\quad$ $\rho = 0.001$ $\quad$ $\rho = 0.01$

"Dream and reality," nominal optimal design: samples of 100 actual diagrams (red) for different uncertainty levels. Blue: the target diagram

	Dream	Reality								
	$\rho = 0$	$\rho = 0.0001$			$\rho = 0.001$			$\rho = 0.01$		
	value	min	mean	max	min	mean	max	min	mean	max
$\|\cdot\|_{2,w}$ distance to target	0.011	0.077	0.424	0.957	1.177	4.687	9.711	8.709	45.15	109.5
energy concentration	99.4%	0.23%	20.1%	77.7%	0.70	19.5%	61.5%	0.53%	18.9%	61.5%

Quality of nominal antenna design. Data over 100 samples of actuation errors per each value of ρ.

$$\min_{\tau,x}\left\{\tau : \|D_* - \sum_{\ell=1}^{10} x_\ell(1+\zeta_\ell)D_\ell\|_{2,w} \le \tau \forall \zeta, \|\zeta\|_\infty \le \rho\right\}_{\text{RC}}$$

♠ We are in the case of box (and thus – structured norm-bounded) uncertainty, whence (RC) admits a tight, within the factor $\pi/2$, tractable approximation. The approximation reads

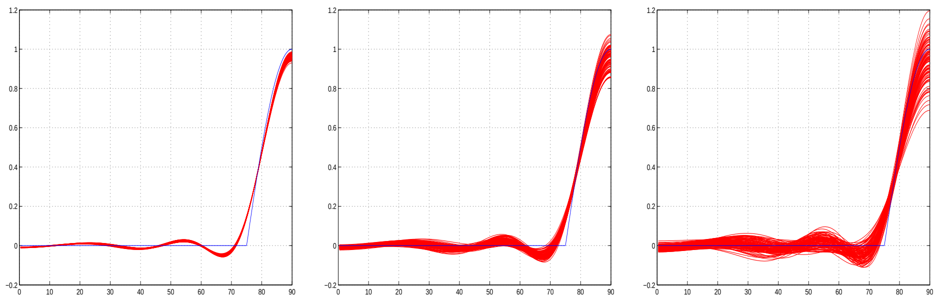
$$\begin{array}{ll}\min\limits_{\tau,x,\gamma} & \tau \\ \text{s.t.} & \end{array}$$

$$\left[\begin{array}{c|c|c} \tau - \sum_{\nu=1}^{L}\gamma_\nu & [WDx-b]^T & \\ \hline WDx-b & \tau I & \rho WD\mathrm{Diag}\{x\} \\ \hline & \rho[WD\mathrm{Diag}\{x\}]^T & \mathrm{Diag}\{\gamma_1,...,\gamma_{10}\} \end{array}\right] \succeq 0$$

where:

- $D = [D_{i\ell} = D_\ell(\theta_i)]_{\substack{1\le\ell\le10,\\1\le i\le240}}$,
- $W = \mathrm{Diag}\{\cos(\theta_1),...,\cos(\theta_{240})\}/\sqrt{240}$
- $b = WD_*$

Setting $\rho = 0.01$, we end up with *robust design* which withstands implementation errors incomparably better than the nominal one:



$\rho = 0.01$ $\qquad$ $\rho = 0.05$ $\qquad$ $\rho = 0.1$

"Dream and reality," robust optimal design: samples of 100 of actual diagrams (red) for different uncertainty levels. Blue: the target diagram.

	Reality								
	$\rho = 0.01$			$\rho = 0.05$			$\rho = 0.1$		
	min	mean	max	min	mean	max	min	mean	max
$\|\cdot\|_{2,w}$ distance to target	0.021	0.021	0.021	0.021	0.023	0.030	0.021	0.030	0.048
energy concentration	96.5%	96.7%	96.9%	93.0%	95.8%	96.8%	80.6%	92.9%	96.7%

Quality of robust antenna design. Data over 100 samples of actuation errors per each value of ρ.

For comparison: For nominal design, with $\rho = 0.001$, the average $\|\cdot\|_{2,w}$-distance of the actual diagram to target is as large as 4.69, and the expected energy concentration is as low as 19.5%.

♠ How conservative is our safe tractable approximation?

• The robust design was obtained from *safe tractable approximation* of the true RC rather than from the RC itself $\Rightarrow$ the guaranteed value $\mathrm{ApprOpt}(0.01) = 0.0212$ of the objective can be larger than the true optimal value RobOpt(0.01) of the RC at the uncertainty level 0.01. How large is the loss in optimality?

• Our approximation is tight within the factor $\pi/2$. This means only that

$$\mathrm{RobOpt}(0.01) \le \mathrm{ApprOpt}(0.01) \le \mathrm{RobOpt}(0.01 \cdot \pi/2)$$

and does not allow to make any conclusion on how far is ApprOpt(0.01) from RobOpt(0.01).

• The nominal optimal value $\mathrm{NomOpt} = 0.011$ is a lower bound on $\mathrm{RobOpt}(0.01) \Rightarrow \mathrm{ApprOpt}(0.01) = 0.0212$ is within 90% of the true robust optimal value.

• Fortunately, our perturbation set is a box in $\mathbb{R}^{10}$ and thus can be treated as a set given by a large, but not prohibitively so, number $2^{10} = 1024$ of scenarios. This allows to compute the true robust optimal value *exactly*, *and it turns out to be by just 0.2% worse than its upper bound* ApprOpt(0.01).

Tight Tractable Approximation of Semi-Infinite Least Squares Inequality with $\bigcap$-Ellipsoidal Perturbation Set

$$\|A[\zeta]x - b[\zeta]\|_2 \equiv \|\alpha[x]\zeta + \beta[x]\|_2 \leq \tau \ \forall \zeta \in \rho\mathcal{Z} \qquad \text{RC}$$

♣ Consider the case of $\bigcap$-*ellipsoidal perturbations*

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \zeta^T Q_j \zeta \leq 1, \ 1 \leq j \leq J\}$$
$$[Q_j \succeq 0, \sum_j Q_j \succ 0]$$

Geometrically: $\mathcal{Z}$ is bounded and is the intersection of J ellipsoids/elliptic cylinders centered at the origin.

Examples: • Unit ball; • A polytope $\mathcal{Z}$, $0 \in \mathrm{int}\mathcal{Z}$, symmetric w.r.t. the origin (e.g., a box centered at the origin). Indeed, such a polytope can be represented as $\mathcal{Z} = \{\zeta : (q_j^T\zeta)^2 \leq 1, \ 1 \leq j \leq J\}$.

♠ **Deriving the approximation.** We ask when

$$\{\zeta^T Q_j \zeta \leq \rho^2, \ 1 \leq j \leq J\} \Rightarrow \|\alpha[x]\zeta + \beta[x]\|_2^2 \leq \tau^2,$$

or, equivalently, when

$$\begin{array}{c}\{\zeta^T Q_j \zeta \leq \rho^2, \ 1 \leq j \leq J, t^2 \leq 1\} \\ \Rightarrow \|\alpha[x]\zeta + t\beta[x]\|_2^2 \leq \tau^2\end{array} \qquad (!)$$

An evident *sufficient* condition for (!) is:

$$\exists \{\lambda_j \geq 0\}_{j=0}^J : \left\{\begin{array}{l}\lambda_0 + \rho^2\lambda_1 + \ldots + \rho^2\lambda_J \leq \tau^2 \\ \lambda_0 t^2 + \lambda_1 \zeta^T Q_1 \zeta + \ldots + \lambda_J \zeta^T Q_J \zeta \\ \qquad\qquad \geq \|\alpha[x]\zeta + t\beta[x]\|_2^2 \ \forall (t,\zeta).\end{array}\right.$$

$$\begin{array}{c} \|A[\zeta]x - b[\zeta]\|_2 \equiv \|\alpha[x]\zeta + \beta[x]\|_2 \leq \tau \\ \forall \zeta : \zeta^T Q_j \zeta \leq \rho^2, 1 \leq j \leq J \end{array} \qquad \text{RC}$$

♡Assuming $\tau > 0$ and setting $\mu_j = \lambda_j/\tau$, the above sufficient condition for the validity of (RC) reads

$$\begin{array}{l} \exists \mu_j \geq 0,\ 0 \leq j \leq J : \\ (a) \qquad \mu_0 + \rho^2 \sum_{j=1}^J \mu_j \leq \tau \\ (b) \quad \left[\begin{array}{c|c} \mu_0 & \\ \hline & \sum_{j=1}^J \mu_j Q_j \end{array}\right] - \frac{1}{\tau}[\beta[x], \alpha[x]]^T [\beta[x], \alpha[x]] \succeq 0 \end{array}$$

Now let us use

Miracle # 2: Schur Complement Lemma: *A symmetric block matrix* $P = \left[\begin{array}{cc} A & B \\ B^T & C \end{array}\right]$ *with* $C \succ 0$ *is* $\succeq 0$ *iff* $A - BC^{-1}B^T \succeq 0$.

Proof: $P \succeq 0$ iff $[u;v]^T P [u;v] \geq 0$ for all u, v, that is, iff $\min\limits_v [u;v]^T P[u;v] \geq 0$ for all u. Since $C \succ 0$, the latter minimum is $u^T[A - BC^{-1}B^T]u$. $\square$

♡ Thus, a sufficient condition for $(x, \tau > 0)$ to satisfy (RC) is

$$\exists \{\mu_j \geq 0\}_{j=0}^J : \begin{array}{c} \left[\begin{array}{c|c|c} \mu_0 & & \beta^T[x] \\ \hline & \sum_{j=1}^J \mu_j Q_j & \alpha^T[x] \\ \hline \beta[x] & \alpha[x] & \tau I \end{array}\right] \succeq 0 \\ \mu_0 \leq \tau - \rho^2 \sum_{j=1}^J \mu_j \end{array}$$

$$
\begin{array}{r}
\|A[\zeta]x - b[\zeta]\|_2 \equiv \|\alpha[x]\zeta + \beta[x]\|_2 \leq \tau \\
\forall \zeta : \zeta^T Q_j \zeta \leq \rho^2, 1 \leq j \leq J
\end{array}
\qquad \mathrm{RC}
$$

♡ We have proved the first part of the following

Theorem. (i) *The explicit system of convex constraints*

$$
\left[\begin{array}{c|c|c}
\tau - \rho^2 \sum_{j=1}^J \mu_j & & \beta^T[x] \\
\hline
 & \sum_{j=1}^J \mu_j Q_j & \alpha^T[x] \\
\hline
\beta[x] & \alpha[x] & \tau I
\end{array}\right] \succeq 0 \qquad (+)
$$

in variables $x, \tau, \mu_1, ..., \mu_J$ *is a safe tractable approximation of* (RC) — *whenever* (x, τ) *can be extended to a feasible solution of* $(+)$, *this pair is feasible for* (RC).

(ii) *The approximation is precise when* $J = 1$, *and is tight within the factor* $O(1)\sqrt{\ln(J)}$ *otherwise.*

Tractable Reformulations/Tight Safe Tractable Approximations of Semi-Infinite Linear Matrix Inequalities

♣ Here we are interested in tractable reformulations/tight safe tractable approximations of a semi-infinite Linear Matrix Inequality

$$\mathcal{A}[\zeta, x] \succeq 0 \ \forall \zeta \in \rho\mathcal{Z} \tag{RC}$$

where $\mathcal{A}[\zeta, x]$ is a symmetric matrix which is bi-affine in x and in ζ:

$$\mathcal{A}[\zeta, x] \equiv A_0[\zeta] + \sum_{j=0}^{n} x_j A_j[\zeta] \equiv \alpha_0[x] + \sum_{i=1}^{d} \zeta_i \alpha_i[x],$$

where the symmetric matrices $A_j[\zeta]$ and $\alpha_i[x]$ are affine in their arguments.

♠ As a matter of fact, aside of the "universally tractable" (and trivial) case of scenario-generated $\mathcal{Z}$, in the LMI case no "universally good" – leading to tractable or "nearly so" (i.e., admitting tight safe tractable approximations) – geometries of $\mathcal{Z}$ are known.

♠ All known to us generic tractability results impose structural restrictions on how the uncertainty enters the body of the LMI, specifically, they assume *norm-bounded* , *structured or not*. *model of perturbations*

$$\mathcal{A}[\zeta, x] \succeq 0 \;\forall \zeta \in \rho\mathcal{Z} \tag{RC}$$

Definition: We say that (RC) is with *norm-bounded perturbations*, if

A. $\mathcal{Z}$ is comprised of collections $\zeta = \{\zeta^1, ..., \zeta^L\}$ of matrices $\zeta^\ell \in \mathbb{R}^{p_\ell \times q_\ell}$ of the spectral norm not exceeding 1.

In addition, prescribed part of the matrices ζ^ℓ — those with indices from a given set $\mathcal{I}_s$ — are marked as "scalar perturbation blocks" and should be *scalar* – proportional to the unit matrix (for those ℓ, of course, $p_\ell = q_\ell$). The remaining "full perturbation blocks" ζ^ℓ in ζ can be arbitrary $p_\ell \times q_\ell$ matrices of norm ≤ 1. Thus,

$$\mathcal{Z} = \left\{ \zeta = \{\zeta^1, ..., \zeta^L\} : \begin{array}{c} \zeta^\ell \in \mathbb{R}^{p_\ell \times q_\ell} \,\forall \ell \\ \|\zeta^\ell\|_{2,2} \leq 1 \,\forall \ell \\ \zeta^\ell \in \mathbb{R} \cdot I_{p_\ell}, \ell \in \mathcal{I}_s \end{array} \right\}$$

B. The body $\mathcal{A}[\zeta, x]$ of (RC) is

$$\mathcal{A}[\zeta, x] = \mathcal{A}^0[x] + \sum_{\ell=1}^{L} \left[L_\ell^T[x] \zeta^\ell R_\ell[x] + R_\ell^T[x] [\zeta^\ell]^T L_\ell[x] \right],$$

where $L_\ell[x]$, $R_\ell[x]$ are affine in x and for every ℓ at least one of these matrices is constant.

Note: W.l.o.g. we assume that $R_\ell[x] \equiv R_\ell$ for all ℓ.

• Norm-bounded uncertainty is called *structured*, if $L > 1$, and *unstructured* otherwise.

♣ **Relation to norm-bounded uncertainty in c.q.i.**

♠ The Lorentz cone $\mathbf{L}^m$ can be represented as the intersection of the semidefinite cone $\mathbf{S}^m_+$ and a linear subspace. Specifically, by the Schur Complement Lemma

$$[u;t] \in \mathbf{L}^m \Leftrightarrow \mathrm{Arrow}(u,t) := \left[\begin{array}{c|c} t & u^T \\ \hline u & tI_{m-1} \end{array}\right] \succeq 0$$

♠ Consequently, a semi-infinite c.q.i. can be reformulated equivalently as a semi-infinite LMI. In particular, a semi-infinite Least Squares inequality is equivalent to an "arrow" semi-infinite LMI with uncertainty-affected off-diagonal entries of the first row and column:

$$\begin{array}{c} \|A[\zeta]x + b[\zeta]\|_2 \leq \tau \,\forall \zeta \in \rho\mathcal{Z} \\ \Leftrightarrow \mathcal{A}[\zeta,x] := \left[\begin{array}{c|c} \tau & [A[\zeta]x+b]^T \\ \hline A[\zeta]x+b & \tau I \end{array}\right] \succeq 0 \;\forall \zeta \in \rho\mathcal{Z} \end{array}$$

With this correspondence, *norm-bounded uncertainty in the semi-infinite Least Squares inequality:*

$$A[\zeta]x + b[\zeta] = A^0x + b^0 + \sum_{\ell=1}^{L} L_\ell^T[x]\zeta^\ell R_\ell[x]$$

induces norm-bounded uncertainty with no scalar perturbation blocks in the semi-infinite LMI:

$$\begin{array}{rcl} \mathcal{A}[\zeta,x] &=& \mathrm{Arrow}(A^0x+b,\tau) \\ && + \sum_{\ell=1}^{L} \left[\mathcal{L}_\ell^T[x]\zeta^\ell \mathcal{R}_\ell[x] + \mathcal{R}_\ell^T[x][\zeta^\ell]^T \mathcal{L}_\ell[x]\right]. \end{array}$$

♠ As a consequence, *every result on tractability/tight safe tractable approximation of semi-infinite LMI with unstructured or structured norm-bounded uncertainty induces similar results on semi-infinite Least Squares inequalities*, and this is how the results of the latter type we have mentioned in the "c.q.i.-part" were obtained.

Derivation of Tight Safe Tractable Approximation of Semi-Infinite LMI with Norm-Bounded Uncertainty

$$
\begin{array}{l}
A_0[x] + \sum_{\ell=1}^{L} [L_\ell^T[x]\zeta^\ell R_\ell + R_\ell^T[\zeta^\ell]^T L_\ell[x]] \succeq 0 \\
\forall \{\zeta^\ell \in \mathbb{R}^{p_\ell \times q_\ell}\} : \|\zeta^\ell\|_{2,2} \le \rho \,\forall \ell,\ \zeta^\ell \in \mathbb{R}\cdot I_{p_\ell}\,\forall \ell \in \mathcal{I}_{\rm s}
\end{array}
\qquad \text{(RC)}
$$

The idea is pretty simple: an evident *sufficient* condition for x to be feasible for (RC) is existence of matrices Y_ℓ such that

$$
\begin{array}{ll}
(a) & Y_\ell \succeq [L_\ell^T[x]\zeta^\ell R_\ell + R_\ell^T[\zeta^\ell]^T L_\ell[x]] \\
 & \qquad \forall \zeta^\ell = \lambda_\ell I_{p_\ell} : \|\zeta_\ell\|_{2,2} \le 1,\ \ell \in \mathcal{I}_{\rm s} \\
(b) & Y_\ell \succeq [L_\ell^T[x]\zeta^\ell R_\ell + R_\ell^T[\zeta^\ell]^T L_\ell[x]] \\
 & \qquad \forall \zeta^\ell : \|\zeta_\ell\|_{2,2} \le 1,\ \ell \not\in \mathcal{I}_{\rm s} \\
(c) & A_0[x] - \rho \sum_{\ell=1}^{L} Y_\ell \succeq 0
\end{array}
\qquad (!)
$$

• "Semi-infinite" LMIs (a) are equivalent to finite system of LMIs

$$
Y_\ell \succeq \pm [L_\ell^T[x] R_\ell + R_\ell^T L_\ell[x]],\ \ell \in \mathcal{I}_{\rm s} \qquad (A)
$$

• From $\mathcal{S}$-Lemma one can derive without much thought that semi-infinite LMIs (b) cam ne represented equivalently by finite system of LMIs

$$
\left[\begin{array}{c|c} Y_\ell - \lambda_\ell R_\ell^T R_\ell & L_\ell^T[x] \\ \hline L_\ell[x] & \lambda I_{p_\ell} \end{array}\right] \succeq 0,\ \ \ell \not\in \mathcal{I}_{\rm s} \qquad (B)
$$

$$
\begin{array}{l}
A_0[x] + \sum_{\ell=1}^{L} [L_\ell^T[x]\zeta^\ell R_\ell + R_\ell^T[\zeta^\ell]^T L_\ell[x]] \succeq 0 \\
\forall \{\zeta^\ell \in \mathbb{R}^{p_\ell \times q_\ell}\} : \|\zeta^\ell\|_{2,2} \le \rho \, \forall \ell, \; \zeta^\ell \in \mathbb{R} \cdot I_{p_\ell} \, \forall \ell \in \mathcal{I}_s
\end{array}
\qquad \text{(RC)}
$$

We have arrived at the first part of the following

Theorem: (i) *The explicit system of LMIs*

$$
\begin{array}{ll}
(a) & Y_\ell \succeq \pm [L_\ell^T[x] R_\ell + R_\ell^T L_\ell[x]], \; \forall \ell \in \mathcal{I}_s \\
(b) & \left[\begin{array}{c|c} Y_\ell - \lambda_\ell R_\ell^T R_\ell & L_\ell^T[x] \\ \hline L_\ell[x] & \lambda I_{p_\ell} \end{array} \right] \succeq 0, \; \ell \not\in \mathcal{I}_s \\
(c) & A_0[x] - \rho \sum_{\ell=1}^{L} Y\ell \succeq 0
\end{array}
$$

in variables x, $\lambda_\ell \in \mathbb{R}$, $\ell \not\in \mathcal{I}_s$, Y_ℓ, $1 \le \ell \le L$, is a safe tractable approximation of (RC).

(ii) *This approximation is exact when $L = 1$* (unstructured norm-bounded perturbations) *and is tight within factor $\pi/2$ when $L > 1$, provided that there are no nontrivial (with $p_\ell > 1$) scalar perturbation blocks.*

When there are nontrivial scalar perturbation blocks, the tightness factor of the approximation is a universal function $\vartheta(\mu)$ of $\mu = 2 \max_{\ell \in \mathcal{I}_s} p_\ell$ such that

$$
\vartheta(\mu) \le \sqrt{\pi \mu / 2}.
$$

♣ Example: Robust Truss Topology Design

• A *truss* is a mechanical construction, like electric mast, railroad bridge, or Eiffel Tower, comprised of thin elastic *bars* linked to each other at *nodes*.

• Under external load, the truss deforms until the internal forces caused by the deformation compensate the external forces. At the resulting static equilibrium, the truss capacitates certain energy, called *compliance*. Compliance is a natural measure of the rigidity of the truss w.r.t. a load: the less the compliance, the more rigid is the truss.

• In a Truss Topology Design problem, one is given

A. A *list of tentative nodes* – an m-point grid in $\mathbb{R}^d$ ($d = 2/d = 3$), along with *boundary conditions* which declare some nodes partially or completely fixed by supports, and thus define for every $i \leq m$ a linear subspace $V_i \subset \mathbb{R}^d$ of *virtual displacements* of node i. A *virtual displacement of the nodal set* is a collection of allowed displacements of the nodes, and these virtual displacements form a linear space $\mathcal{V} = \bigoplus_{i=1}^{m} V_i$;

B. A collection of n *tentative bars* – pairs of nodes which can be linked by a bar;

C. A finite set $\mathcal{F}$ of *loading scenarios* – vectors $f \in \mathcal{V}$ comprised of external forces acting at the nodes and representing the load in question.

The goal in a TTD problem is *to assign the with nonnegative volumes $t = [t_1; ...; t_n] \in \mathcal{T}$ in a way which mini-*

mizes the worst, over loading scenarios, compliance of the construction w.r.t. a scenario. Here $\mathcal{T} \subset \mathbb{R}^n_+$ is a given polytope of *admissible designs*, most typically – just the simplex $\{t \geq 0, \sum_i t_i \leq w\}$, where w is an a priori upper bound on the weight of the construction.

♠ Mathematically, the fact that the compliance of a truss t w.r.t. a load $f \in \mathcal{V}$ is $\leq \tau$ is expressed as

$$\left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0,$$

where

$$A(t) = \sum_{i=1}^{n} t_i b_i b_i^T$$

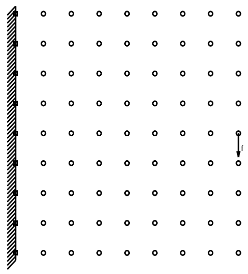
is the *stiffness matrix* of the truss; $b_i \in \mathcal{V}$ are readily given by the geometry of the nodal set.

Thus, the TTD problem is the semidefinite program

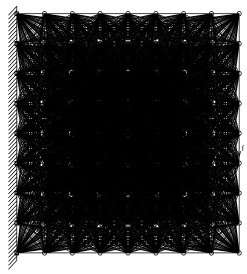
$$\min_{\tau,t} \left\{ \tau : \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0,\ f \in \mathcal{F} \right\}.$$

♠ In TTD, one starts with a dense nodal grid and allows for all pair connections of nodes with bars. At optimality, most of the bars get zero volume, thus revealing the *optimal topology* of the construction, along with its *optimal sizing*.

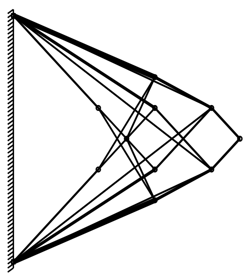
♠ The set of loading scenarios $\mathcal{F}$ usually is comprised of small (1-2-3) number of "loads of interest." In reality, the truss will be subject to small "unforeseen occasional loads" which can crush it:



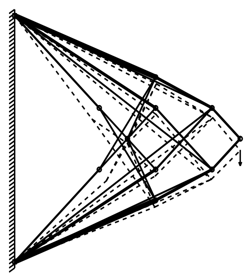
(a): 9×9 nodal grid with most left nodes fixed and the load of interest. $M =$ 144 degrees of freedom.



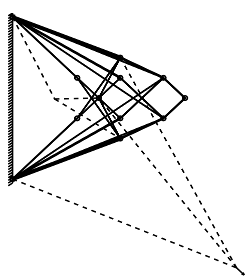
(b): 2,039 tentative bars



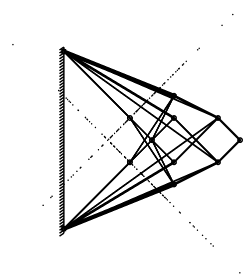
(c): Single-load optimal design, 12 nodes, 24 bars. Compliance w.r.t. load of interest 1.00.



(d): Deformation of nominal design under the load of interest f.



(e): Deformation of nominal design under "occasional" load 10^8 times less than f.



(f): "Dotted lines": positions of nodes in deformed nominal design, sample of 100 loads $\sim \mathcal{N}(0, 10^{-16} I_{20})$

♠ To avoid potential instability of the designed truss w.r.t. small occasional loads, it makes sense to control its compliance w.r.t. both loads of actual interest and *all "occasional" loads of norm* $\leq \rho$.

• **Question:** *Where should the occasional loads be applied?* Usually, most of the nodes from the original grid are not used in the resulting construction; why to bother about forces acting at *nonexisting* nodes?

Possible answer: When "robustifying" the nominal design, we can choose, as the nodal set, the nodes actually used in this design and to allow the occasional loads to act at all these nodes.

• **Question:** *How to choose* ρ*?*

Possible answer: Let τ_* be the nominal optimal value in the TTD problem, and let $\tau^+ > \tau_*$ be the compliance "we are ready to tolerate." When robustifying the nominal design, we can look for the design which ensures, *for as large* ρ *as possible*, that the compliance w.r.t. the loads of interest *and all occasional loads* g *with* $\|g\|_2 \leq \rho$ is $\leq \tau^+$.

♠ The resulting Robust TTD problem reads

$$\rho_* = \max_{\rho,t}\left\{\rho : \begin{array}{l}\left[\begin{array}{c|c}2\tau^+ & f^T\\ \hline f & A(t)\end{array}\right] \succeq 0\ \forall f \in \mathcal{F}\\ \left[\begin{array}{c|c}2\tau^+ & \zeta^T\\ \hline \zeta & A(t)\end{array}\right] \succeq 0\ \forall \zeta, \|\zeta\|_2 \leq \rho\\ t \in \mathcal{T}\end{array}\right\}. \qquad (*)$$

Note: Let $M = \dim \zeta$ be the number of degrees of freedom in the (reduced) nodal set. The body of the semi-infinite LMI in $(*)$ is

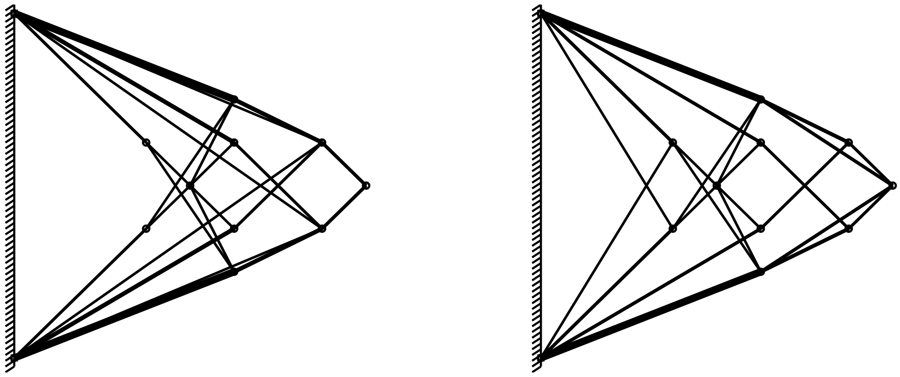
$$\left[\begin{array}{c|c}2\tau^+ & f^T\\ \hline f & A(t)\end{array}\right] = \left[\begin{array}{c|c}2\tau^+ & \\ \hline & A(t)\end{array}\right] + \overbrace{[0_{M\times 1}, I_M]^T}^{L^T}\zeta\overbrace{[1, 0_{1\times M}]}^{R} + R^T\zeta^T L,$$

with $\zeta \in \mathbb{R}^{m\times 1}$, so that $\|\zeta\|_{2,2} = \|\zeta\|_2$

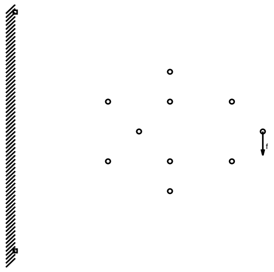
$\Rightarrow$ *We are in the case of unstructured norm-bounded uncertainty and thus $(*)$ admit a tractable reformulation*, namely,

$$\max_{\rho,t,s}\left\{\rho : \begin{array}{l}\left[\begin{array}{c|c}2\tau^+ & f^T\\ \hline f & A(t)\end{array}\right] \succeq 0\ \forall f \in \mathcal{F}\\ A(t) \succeq sI_M, \left[\begin{array}{c|c}2\tau^+ & \rho\\ \hline \rho & s\end{array}\right] \succeq 0\\ t \in \mathcal{T}\end{array}\right\}. \qquad (!)$$

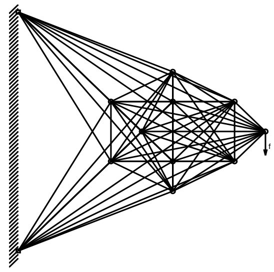
♠ Applying the outlined approach in the console design, we set $\tau^+ = 1.02\tau_* = 1.025$ and end up with $\rho_* = 0.362$. The resulting design, being only marginally inferior to the nominally optimal one as far as the load of interest is concerned, is incomparably more rigid w.r.t. occasional loads.



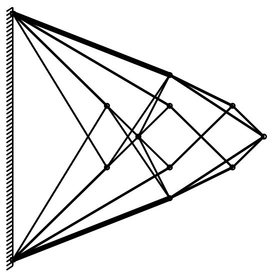
nominal design robust design



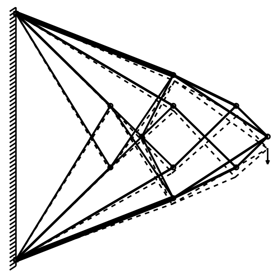
(a): reduced 12-node set with most left nodes fixed and the load of interest. $M = 20$ degrees of freedom.



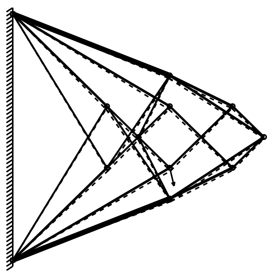
(b): 54 tentative bars



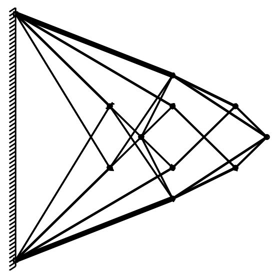
(c): Robust optimal design, 12 nodes, 24 bars. Compliance w.r.t. load of interest 1.025.



(d): Deformation of robust design under the load of interest.



(e): Deformation of robust design under "occasional" load 10 times less than the load of interest.



(f): "Bold dots": positions of nodes in deformated robust design over 100 loads $\sim \mathcal{N}(0, 10^{-2} I_{20})$

Robust design of a console

Applications in Robust Control

♣ Semi-infinite LMIs arise on numerous occasions in *Robust Control*.

Example: Lyapunov Stability Analysis. Consider an *uncertain time-varying linear dynamic system*

$$\dot{x}(t) = A_t x(t),$$

where A_t for all t is known to belong to a given convex compact set $\mathcal{U}$. How to certify that the system is *stable*, that is, all trajectories of (all realizations of) the system go to 0 as $t \to \infty$?

♠ The standard *sufficient* stability condition is existence of *Lyapunov Stability Certificate:* a matrix X such that

$$X \succ 0 \;\&\; A^T X + XA \prec 0 \;\forall A \in \mathcal{U}.$$

Indeed, given and LSC X, by compactness of $\mathcal{U}$

$$\exists \alpha > 0 : A^T X + XA \preceq -\alpha X \;\forall A \in \mathcal{A}$$

whence for every trajectory

$$\frac{d}{dt}[x^T(t)Xx(t)] = x^T(t)[A_t^T X + XA_t]x(t) \leq -\alpha[x^T(t)Xx(t)]$$
$$\Downarrow$$
$$x^T(t)Xx(t) \leq \exp\{-\alpha t\}x^T(0)Xx(0) \Rightarrow x(t) \to 0,\; t \to \infty.$$

♠ **Note:** Existence of an LSC is equivalent to the feasibility of the semi-infinite *Lyapunov LMI*

$$X \succeq I,\;\; A^T X + XA \preceq -I \;\forall A \in \mathcal{U}$$

$$X \succeq I,\ A^T X + XA \preceq -I\ \forall A \in \mathcal{U}$$

♠ In many Control applications, $\mathcal{U}$ is given by norm-bounded perturbations:

$$\mathcal{U} = \mathcal{U}_\rho = A^{\rm n} + \rho\mathcal{Z},$$

$$\mathcal{Z} = \left\{ A = \sum_{\ell=1}^L P_\ell \zeta^\ell Q_\ell : \begin{array}{c} \zeta^\ell \in \mathbb{R}^{p_\ell \times q_\ell} \\ \|\zeta^\ell\|_{2,2} \leq 1\ \forall \ell \\ \zeta^\ell \in \mathbb{R}I,\ \ell \in \mathcal{I}_{\rm s} \end{array} \right\}$$

Example 1: Interval uncertainty. In this case

$$\mathcal{U}_\rho = \{A : |A_{ij} - A^{\rm n}_{ij}| \leq \rho d_{ij}\},$$

that is,

$$\mathcal{Z} = \{Z = \sum_{i,j} d_{ij} e_i \zeta^{ij} e_j^T : |\zeta^{ij}| \leq 1\,\forall i,j\}$$

(full 1×1 perturbation blocks).

Example 2: Closed loop dynamical system. Consider a time invariant Linear Dynamical system "closed" by a linear feedback control:

$$\begin{array}{rcl}\dot{x}(t) & = & Px(t) + Qu(t) + T\xi(t) \text{ [state equations]}\\ y(t) & = & Rx(t) + U\xi(t) \text{ [observed outputs]}\\ u(t) & = & Sy(t) \text{ [feedback]}\\ \Rightarrow & & \\ \dot{x}(t) & = & [P + QSR]x(t) + [T + QSU]\xi(t) \text{ [closed loop]}\end{array}$$

When one (or more) of the matrices P, Q, R, S drifts with time around its nominal value, the system becomes uncertain. Assuming norm-bounded uncertainty in P, Q, R, S:

$$\|\zeta^P := P_t - P\|_{2,2} \leq \rho d_P, \ldots, \|\zeta^S := S_t - S\|_{2,2} \leq \rho\delta_S$$

we can approximate the range $\mathcal{U}_\rho$ of the matrix

$$A_t = P_t + Q_t S_t R_t$$

of the closed loop system by the norm-bounded perturbation set:

$$\begin{array}{c}\mathcal{U}_\rho \approx [P + QSR] + \rho\mathcal{Z},\\ \mathcal{Z} \;=\; \{\zeta^P + \zeta^Q SR + Q\zeta^S R + QS\zeta^R : \\ \|\zeta^P\|_{2,2} \leq 1, \ldots, \|\zeta^S\|_{2,2} \leq 1\}\end{array}$$

(this approximation is exact when only one of the matrices P, Q, R, S is subject to drift).

$$\mathcal{A}[A,X] := -I - A^TX - XA \succeq 0 \;\forall A \in \mathcal{U}_\rho \qquad \text{(L)}$$

$$\mathcal{U}_\rho = A^{\rm n} + \rho\mathcal{Z},$$

$$\mathcal{Z} = \left\{ \sum_{\ell=1}^L P_\ell \zeta^\ell Q_\ell : \begin{array}{c} \zeta^\ell \in \mathbb{R}^{p_\ell \times q_\ell} \\ \|\zeta^\ell\|_{2,2} \le 1 \;\forall \ell \\ \zeta^\ell \in \mathbb{R}I,\; \ell \in \mathcal{I}_{\rm s} \end{array} \right\} \qquad \text{(NB)}$$

Observation: *Norm-bounded uncertainty* (NB) *induces norm-bounded uncertainty in the Lyapunov LMI* (L), *the number, sizes and types (full/scalar) of the perturbation blocks being preserved.*

Corollary: *When* $L = 1$ (unstructured norm-bounded uncertainty, case A), (L) *admits a tractable reformulation, otherwise:*

— *when there are no nontrivial ($p_\ell > 1$) scalar perturbations blocks* (case B), (L) *admits tight within the factor $\pi/2$ safe tractable approximation,*

— *otherwise* (case C) (L) *admits a safe tractable approximation tight within factor* $\vartheta(\mu) \le \sqrt{\pi\mu/2}$, $\mu = 2\max\limits_{\ell \in \mathcal{I}_{\rm s}} p_\ell$.

♡ In particular, *The Lyapunov Stability Radius of* $(A^{\rm n}, \mathcal{Z})$ *– the largest ρ for which* (L) *has a positive definite solution X – admits an efficiently computable lower bound which is*

— *exact in case A,*

— *tight within the factor $\pi/2$ in case B,*

— *tight within the factor $\vartheta(\mu)$ in case C.*