Lectures on Robust Convex Optimization

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Subject. The data of optimization problems of real world origin typically is *uncertain* - not known exactly when the problem is solved. With the traditional approach, "small" (fractions of percents) data uncertainty is merely ignored, and the problem is solved as if the nominal data — our *guesses* for the actual data — were identical to the actual data. However, experiments demonstrate that already pretty small perturbations of uncertain data can make the nominal (i.e., corresponding to the nominal data) optimal solution heavily infeasible and thus practically meaningless. For example, in 13 of 90 LP programs from the NETLIB library, 0.01% random perturbations of uncertain data lead to more than 50% violations of some of the constraints as evaluated at the nominal optimal solutions. Thus, in applications there is an actual need in a methodology which produces *robust*, "immunized against uncertainty" solutions. Essentially, the only traditional methodology of this type is offered by Stochastic Programming, where one assigns data perturbations a probability distribution and replaces the original constraints with their "chance versions", imposing on a candidate solution the requirement to satisfy the constraints with probability $\geq 1-\epsilon$, $\epsilon \ll 1$ being a given tolerance. In many cases, however, there is no natural way to assign the data perturbations with a probability distribution; besides this, Chance Constrained Stochastic Programming typically is computationally intractable – aside of a small number of special cases, chance constrained versions of simple – just linear – constraints are nonconvex and difficult to verify, which makes optimization under these constraints highly problematic.

Robust Optimization can be viewed as a complementary to Stochastic Programming approach to handling optimization problems with uncertain data. Here one uses "uncertain-but-bounded" model of data perturbations, allowing the uncertain data to run through a given *uncertainty set*, and imposes on a candidate solution the requirement to be *robust feasible* – to satisfy the constraints whatever be a realization of the data from this set. Assuming that the objective is certain (i.e., is not affected by data perturbations; in fact, this assumption does not restrict generality), one then looks for the *robust optimal* solution — a robust feasible solution with as small value of the objective as possible. With this approach, one associates with the original uncertain problem its *Robust Counterpart* – the problem of building the robust optimal solution. Originating from Soyster (1973) and completely ignored for over two decades after its birth, the Robust Optimization was "reborn" circa 1997 and during the last decade became one of the most rapidly developing areas in Optimization. The mini-course in question is aimed at overview of basic concepts and recent developments in this area.

The contents. Our course will be focused on the basic theory of Robust Optimization, specifically, on

- Motivation and detailed presentation of the Robust Optimization paradigm, including in-depth investigation of the outlined notion of the Robust Counterpart of an uncertain optimization problem and its recent extensions (Adjustable and Globalized Robust Counterparts);

- *Computational tractability* of Robust Counterparts. In order to be a working tool rather than wishful thinking, the RC (which by itself is a specific optimization problem) should be efficiently solvable. This, as a minimum, requires efficient solvability of every certain instance of the uncertain problem in question; to meet this requirement, we restrict ourselves in our course with uncertain *conic* optimization, specifically, with uncertain Linear, Conic Quadratic and Semidefinite Optimization problems. Note, however, that tractability of instances is necessary, but by far not sufficient for the RC to be tractable. Indeed, the RC of an uncertain conic problem is a *semi-infinite* program: every conic constraint of the original problem gives rise to *infinitely many* "commonly structured" conic constraints parameterized by the data running through the uncertainty set. It turns out that the tractability of the RC of an uncertain conic problem depends on interplay between the geometries of the underlying cones and uncertainty sets. It will be shown that "uncertain Linear Optimization is tractable" — the RC of an uncertain LO is tractable whenever the uncertainty set is so; this is the major good news about Robust Optimization and its major advantage as compared to Stochastic Programming. In contrast to this, the tractability of the RC of an uncertain Conic Quadratic or Semidefinite problem is a "rare commodity;" the related goal of the course is to overview a number of important cases where the RCs of uncertain Conic Quadratic/Semidefinite problems are tractable or admit "tight", in certain precise sense, tractable approximations.

- Links with Chance Constrained Linear/Conic Quadratic/Semidefinite Optimization. As it was already mentioned, chance constrained versions of randomly perturbed optimization problems, even as simple as Linear Programming ones, usually are computationally intractable. It turns out that Robust Optimization offers an attractive way to build "safe," in certain natural sense, tractable approximations of chance constrained LO/CQO/SDO problems. As a result, information on the stochastic properties of data perturbations, when available, allows to build meaningful (and often highly nontrivial) uncertainty sets.

Prerequisites. Participants are expected to possess basic mathematical culture and to know the most elementary facts from Linear Algebra, Convex Optimization and Probabilities; all more specific and more advanced facts we intend to use (Conic Duality, Semidefinite Relaxation, Concentration Inequalities,...) will be explained in the course.

Textbook. The course is more than covered by the book A. Ben-Tal, L. El Ghaoui, A. Nemirovski, *Robust Optimization*, Princeton University Press, 2009 (freely available at http://sites.google.com/site/robustoptimization
We hope to provide the participants, in an on-line fashion, with self-contained Lecture Notes.

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