ON TRACTABLE APPROXIMATIONS OF UNCERTAIN LINEAR MATRIX INEQUALITIES AFFECTED BY INTERVAL UNCERTAINTY*

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Abstract. We present efficiently verifiable sufficient conditions for the validity of specific NP-hard semi-infinite systems of Linear Matrix Inequalities (LMI's) arising from LMI's with uncertain data and demonstrate that these conditions are "tight" up to an absolute constant factor. In particular, we prove that given an $n \times n$ interval matrix $\mathcal{U}_\rho = \{A \mid |A_{ij} - A^*_{ij}| \le \rho C_{ij}\}$, one can build a computable lower bound, accurate within the factor $\frac{\pi}{2}$, on the supremum of those ρ for which all instances of $\mathcal{U}_\rho$ share a common quadratic Lyapunov function. We then obtain a similar result for the problem of Quadratic Lyapunov Stability Synthesis. Finally, we apply our techniques to the problem of maximizing a homogeneous polynomial of degree 3 over the unit cube.

Key words. Robust semidefinite optimization, data uncertainty, Lyapunov stability synthesis, relaxations of combinatorial problems

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1. Introduction.

In this paper, we focus on the following "Matrix Cube" problem:

> MatrCube: *Given an affine mapping* $u \to \mathcal{B}(u) = B^0 + \sum_{\ell=1}^{L} u_\ell B^\ell$ *from* $\boldsymbol{R}^L$ *to the space* $\boldsymbol{S}^m$ *of* $m \times m$ *real symmetric matrices and* $\rho > 0$, *check whether the image*
>
> $$\mathcal{C}[\rho] = \{A \mid \exists (u, \|u\|_\infty \le \rho) : A = \mathcal{B}(u)\}$$
>
> *of the box* $\{\|u\|_\infty \le \rho\}$ *under this mapping is contained in the cone* $\boldsymbol{S}^m_+$ *of positive semidefinite matrices.*

Problem MatrCube is closely related to what is called *uncertain semidefinite programming with interval uncertainty*. Specifically, consider a Linear Matrix Inequality (LMI)

$$A_0 + \sum_{j=1}^{n} x_j A_j \succeq 0; \tag{1}$$

here $x \in \boldsymbol{R}^n$ is the vector of variables, $A_0, ..., A_n \in \boldsymbol{S}^m$, and $A \succeq B$ means that $A - B \in \boldsymbol{S}^m_+$. Assume that the *data* $[A_0, ..., A_n]$ of the LMI "are uncertain" – we only know that the data belongs to a given *uncertainty set* $\mathcal{U}$. Our aim is to find *robust* solutions of the resulting "uncertain LMI", i.e., solutions x of the semi-infinite system of LMI's

$$A_0 + \sum_{j=1}^{n} x_j A_j \succeq 0 \quad \forall [A_0, ..., A_n] \in \mathcal{U}. \tag{2}$$

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We say that the uncertainty is *interval*, if $\mathcal{U}$ is the image of a box under an affine mapping:

$$\mathcal{U} = \mathcal{U}_\rho = \left\{ [A_0, ..., A_n] = [A_0^0, ..., A_n^0] + \sum_{\ell=1}^{L} u_\ell [A_0^\ell, ..., A_n^\ell] \mid \|u\|_\infty \le \rho \right\}. \tag{3}$$

As an example, consider the following *Lyapunov Stability Analysis* problem.

(LSA): *Given an uncertain linear time-varying system*

$$\frac{d}{dt} x(t) = A(t)x(t), \quad A(t) \in \mathcal{A} \ \forall t, \tag{4}$$

where $\mathcal{A}$ is a given compact set of matrices, check whether the system admits a quadratic Lyapunov function, i.e., whether there exists a positive definite matrix X such that

$$\frac{d}{dt}(x^T(t) X^{-1} x(t)) < 0$$

for all nonzero trajectories $x(t)$ of (4).

Recall that the existence of a quadratic Lyapunov function is a standard *sufficient* condition for the stability of the system (i.e., for the fact that $x(t) \to 0, t \to \infty$, for every trajectory of the system, whatever is a (measurable) choice of $A(\cdot)$ taking values in $\mathcal{A}$). It is easily seen that the existence of a quadratic Lyapunov function for (4) is equivalent to the solvability of the semi-infinite system of LMI's

$$\begin{array}{ll} (a) & X \succeq I, \\ (b) & AX + XA^T \preceq -I \quad \forall A \in \mathcal{A}, \end{array} \tag{5}$$

and every solution X of the latter system defines a quadratic Lyapunov function for (4). Note that (5) is of the form of (2), so that to find a quadratic Lyapunov stability certificate for uncertain linear dynamic system (4) is exactly the same as to solve a semi-infinite system of LMI's (2) associated with an appropriately chosen uncertainty set $\mathcal{U}$. Note also that the latter set is an interval uncertainty, provided that $\mathcal{A}$ is so, e.g., provided that $\mathcal{A}$ is an interval matrix:

$$\mathcal{A} = \mathcal{A}_\rho = \{A : |A_{ij} - A^*_{ij}| \le \rho D_{ij} \ \ \forall i, j\} \tag{6}$$

(A^* is the "nominal" matrix, $D = [D_{ij} \ge 0]_{i,j}$ is a "perturbation scale", and $\rho > 0$ is a "perturbation level").

The Lyapunov Analysis example, same as other examples which can be found in [1, 2, 4, 6], demonstrate the importance of robust solutions to semidefinite problems affected by data uncertainty, in particular, an interval one. Theoretically speaking, the major difficulty with this concept is that (2) is a semi-infinite system of LMI's and as such it can be computationally intractable. However, the set $\mathcal{X}$ of solutions to (2) clearly is closed and convex; it follows that, essentially (for details, see [8]), "computational tractability" of (2) (i.e., ability to find efficiently a point in $\mathcal{X}$ or to maximize efficiently a linear function over $\mathcal{X}$) is equivalent to a possibility to solve efficiently the associated *Analysis* problem:

Anal$[x]$: *Given a candidate solution x, check whether it satisfies* (2).

The role of the Matrix Cube problem in the context of uncertain semidefinite programming comes from the evident fact that *in the case of interval uncertainty* (3), *problem* Anal$[x]$ *is equivalent to problem* MatrCube *with the data*

$$B^\ell = A_0^\ell + \sum_{j=1}^{n} x_j A_j^\ell, \ \ell = 0, 1, ..., L.$$

Unfortunately, the Matrix Cube problem in general is NP-hard. This is so even in the case when all "edges" $B^1, ..., B^L$ of the "matrix box" $\mathcal{C}[\rho]$ are of rank ≤ 2 (see [9] or Section 4 below). Consequently, one is enforced to look for *verifiable sufficient conditions* for the inclusion $\mathcal{C}[\rho] \subset \boldsymbol{S}^m_+$. The simplest condition of this type is evident:

(S) *Assume that there exist matrices $X_1, ..., X_L$ satisfying the system of LMI's*

$$
\begin{array}{llcl}
(a) & X_\ell & \succeq & \pm\rho B^\ell, \ \ell = 1, ..., L, \\
(b) & \sum\limits_{\ell=1}^{L} X_\ell & \preceq & B^0.
\end{array} \tag{7}
$$

Then $\mathcal{C}[\rho] \subset \boldsymbol{S}^m_+$.

In the context of semi-infinite system of LMI's (2) with interval uncertainty (3), (S) results in the following system of LMI's in variables $x, \{X_\ell\}$:

$$
\begin{array}{rcl}
X_\ell & \succeq & \pm\rho \left[A_0^\ell + \sum\limits_{j=1}^{n} x_j A_j^\ell \right], \ \ell = 1, ..., L, \\
\sum\limits_{\ell=1}^{L} X_\ell & \preceq & A_0^0 + \sum\limits_{j=1}^{n} x_j A_j^0.
\end{array} \tag{8}
$$

This system is a "computationally tractable conservative approximation" of (2) in the sense that whenever x can be extended to a feasible solution of (8), x is feasible for (2) (by (S)).

The main result of this paper is as follows:

(N) *The simple sufficient condition* (S) *for the inclusion $\mathcal{C}[\rho] \subset \boldsymbol{S}^m_+$ is not too conservative, provided that the edges $B^1, ..., B^L$ of the matrix box $\mathcal{C}[\rho]$ are of small ranks. Specifically, if* (S) *is not satisfied, then $\vartheta(\mu)$-enlargement $\mathcal{C}[\vartheta(\mu)\rho]$ of the box $\mathcal{C}[\rho]$ is not contained in $\boldsymbol{S}^m_+$. Here*

$$
\mu = \max_{\ell=1,...,L} \text{rank}(B^\ell),
$$

and $\vartheta(\mu)$ is certain universal function such that

$$
\vartheta(1) = 1; \ \vartheta(2) = \tfrac{\pi}{2} = 1.57...; \ \vartheta(3) = 1.73...; \vartheta(4) = 2; \\
\vartheta(\mu) \leq \tfrac{\pi\sqrt{\mu}}{2} \ \forall \mu.
$$

Note that in typical semi-infinite systems of LMI's arising in Control, perturbation of a single data entry results in small rank perturbations of the LMI's; whenever this is the case, (S) and (N) allow to build "tight" (up to a moderate absolute constant factor), computationally tractable conservative approximation of the semi-infinite system in question, provided that the uncertainty is an interval one. For example, in the Lyapunov Stability Analysis system (5), by perturbing a single entry in A, we perturb the left-hand side of the semi-infinite LMI (5.b) by a matrix of rank ≤ 2; as we shall see, this observation combined with (N) allows to build efficiently a lower bound, tight up to the factor $\frac{\pi}{2}$, for the "Lyapunov Stability radius" of an interval matrix (i.e., for the supremum of those $\rho > 0$ for which all instances of the interval matrix (6) share a common quadratic Lyapunov function).

The rest of this paper is organized as follows. In Section 2, we prove our main result (N). Section 3 is devoted to Control applications of this result, specifically, those in Lyapunov Stability Analysis and Synthesis. In Section 4, we establish links between the Matrix Cube problem and the problem of maximizing a positive definite quadratic form over the unit cube; in particular, we demonstrate that (N) allows to re-derive the "$\frac{\pi}{2}$ Theorem" of Nesterov [12] stating that the standard semidefinite bound on the maximum of a positive definite quadratic form over the unit cube is tight within the factor $\frac{\pi}{2}$. In concluding Section 5, we apply our techniques to the problem of maximizing a homogeneous polynomial of degree 3 over the unit cube.

In what follows, we frequently use the *semidefinite duality*; for reader's convenience, we list here the relevant results (for proofs, see, e.g., [11]). Consider a semidefinite problem

$$\min_x \left\{ c^T x : \sum_{j=1}^n x_j A_j - A_0 \succeq 0 \right\}; \tag{Pr}$$

here $x \in \boldsymbol{R}^n$, $A_0, ..., A_n \in \boldsymbol{S}^m$. It is assumed that no nontrivial linear combination of the matrices $A_1, ..., A_n$ is zero.

The *semidefinite dual* of (Pr) is the problem

$$\max_X \{\mathrm{Tr}(A_0 X) : \mathrm{Tr}(A_j X) = c_j,\, j = 1, ..., n, X \succeq 0\}. \tag{Dl}$$

The duality is symmetric: (Dl) can be straightforwardly rewritten in the form of (Pr), and the semidefinite dual of this reformulation is (equivalent to) (Pr). The *Semidefinite Duality Theorem* says that if (Pr) is bounded below and strictly feasible (i.e., $\sum_j \bar{x}_j A_j - A_0 \succ 0$, for certain $\bar{x}$, where $A \succ B$ means that $A - B$ is positive definite), then (Dl) is solvable and has the same optimal value as (Pr).

2. The Matrix Cube problem. The formal statement of our main result (N) is given by the following.

THEOREM 2.1. *Consider problem* MatrCube *along with system of LMI's* (7) *in matrix variables $X_1, ..., X_L$, and let*

$$\mu = \max_{1 \le \ell \le L} \mathrm{rank}(B^\ell)$$

(note $1 \le \ell$ in the max*!). Then*

(i) *If system* (7) *is solvable, then the matrix box $\mathcal{C}[\rho]$ is contained in the positive semidefinite cone $\boldsymbol{S}^m_+$;*

(ii.a) *If system* (7) *is unsolvable, then $\vartheta(\mu)$–enlargement $\mathcal{C}[\vartheta(\mu)\rho]$ of the matrix box $\mathcal{C}[\rho]$ is not contained in the positive semidefinite cone, where the function $\vartheta(\cdot)$ is given by*

$$\frac{1}{\vartheta(k)} = \min_\alpha \left\{ \int_{\boldsymbol{R}^k} \left| \sum_{i=1}^k \alpha_i u_i^2 \right| (2\pi)^{-k/2} \exp\left\{ -\frac{u^T u}{2} \right\} du \;\middle|\; \sum_{i=1}^k |\alpha_i| = 1 \right\}. \tag{9}$$

(ii.b) *The function $\vartheta(\cdot)$ satisfies the relations*

$$\vartheta(k) \le \frac{\pi\sqrt{k}}{2} \quad \forall k; \quad \vartheta(2) = \frac{\pi}{2}. \tag{10}$$

Proof. (i) is evident: if $\{X_\ell\}_{\ell=1}^L$ solve (7), then $u_\ell B^\ell \succeq -X_\ell$ for all ℓ and all $u_\ell \in [-\rho,\rho]$ by (7.a), so that

$$\|u\|_\infty \le \rho \Rightarrow B^0 + \sum_{\ell=1}^{\ell} u_\ell B^\ell \succeq B^0 - \sum_{\ell=1}^{L} X_\ell \succeq 0$$

(we have used (7.b)), and so $\mathcal{C}[\rho] \subset \boldsymbol{S}^m_+$.

(ii.a): Assume that (7) is unsolvable, and let us prove that in this case $\mathcal{C}[\vartheta(\mu)\rho] \not\subset \boldsymbol{S}^m_+$.

1^0. Since (7) is unsolvable, the optimal value in the semidefinite program

$$\min_{t,\{X_\ell\}} \left\{ t \mid tI + B^0 \succeq \sum_{\ell=1}^{L} X_\ell,\ X_\ell \succeq \pm\rho B^\ell,\ \ell = 1,...,L \right\} \tag{P}$$

is positive. Since (P) is strictly feasible, it follows from the Semidefinite Duality Theorem that the semidefinite dual of (P), i.e., the program

$$\max_{U,\{Y_\ell,Z_\ell\}} \left\{ \rho \sum_{\ell=1}^{L} \mathrm{Tr}([Y_\ell - Z_\ell]B^\ell) - \mathrm{Tr}(UB^0) \,\middle|\, \begin{array}{c} \mathrm{Tr}(U)=1,\, U\succeq 0, \\ Y_\ell + Z_\ell = U,\, \ell=1,...,L, \\ Y_\ell, Z_\ell \succeq 0,\ \ell = 1,...,L. \end{array} \right\} \tag{D}$$

is solvable with a positive optimal value.

2^0. To proceed, we need the following simple result:

LEMMA 2.2. *Let $U \succeq 0$ and B be a symmetric matrix of the same size as U. Then*

$$\max_{Y,Z\succeq 0: Y+Z=U} \mathrm{Tr}([Y-Z]B) = \max_{V=V^T,\|V\|\le 1} \mathrm{Tr}(VU^{1/2}BU^{1/2}) = \|\lambda(U^{1/2}BU^{1/2})\|_1, \tag{11}$$

where $\lambda(Z)$ is the vector of eigenvalues of a symmetric matrix Z (counted with their multiplicities) and $\|Z\| = \|\lambda(Z)\|_\infty$ is the operator norm of Z.

Proof. We clearly have

$$\begin{array}{rcl} \max\limits_{Y,Z\succeq 0:Y+Z=U} \mathrm{Tr}([Y-Z]B) & = & \max\limits_{P,Q\succeq 0:P+Q=I} \mathrm{Tr}([U^{1/2}PU^{1/2} - U^{1/2}QU^{1/2}]B) \\ & = & \max\limits_{P,Q\succeq 0:P+Q=I} \mathrm{Tr}([P-Q][U^{1/2}BU^{1/2}]) \\ & = & \max\limits_{V=V^T:\|V\|\le 1} \mathrm{Tr}(V[U^{1/2}BU^{1/2}]), \end{array}$$

as stated in the first equality in (11). To get the second equality, it suffices to consider the case when the matrix $U^{1/2}BU^{1/2}$ is diagonal, where the equality becomes evident. □

In view of Lemma 2.2, the fact that (D) is solvable with positive optimal value means that there exists $U \succeq 0$ such that

$$\rho \sum_{\ell=1}^{L} \|\lambda(U^{1/2}B^\ell U^{1/2})\|_1 > \mathrm{Tr}(U^{1/2}B^0U^{1/2}). \tag{12}$$

We are about to provide a probabilistic interpretation of (12), and this interpretation will lead us to (ii.a).

3^0. Let us write $\xi \sim \mathcal{N}(0, I_k)$ to express that ξ is a random Gaussian k-dimensional vector with zero mean and unit covariance matrix, and let

$$p_k(u) = (2\pi)^{-k/2} \exp\{-u^T u/2\}$$

be the corresponding Gaussian density. We need the following fact:

LEMMA 2.3. *Whenever k is an integer, B is a symmetric $m \times m$ matrix with* $\mathrm{rank}(B) \le k$ *and* $\xi \sim \mathcal{N}(0, I_m)$*, one has*

$$\boldsymbol{E}\left\{|\xi^T B \xi|\right\} \ge \frac{\|\lambda(B)\|_1}{\vartheta(k)}.$$

Proof. It suffices to consider the case when B is diagonal, and in this case the relation in question immediately follows from the definition of $\vartheta(\cdot)$. □

4^0. Let $\xi \sim \mathcal{N}(0, I_m)$. We have

$$\begin{array}{rcll}
\boldsymbol{E}\left\{\vartheta(\mu)\rho \sum\limits_{\ell=1}^{L} |\xi^T U^{1/2} B^\ell U^{1/2} \xi|\right\} & = & \rho \sum\limits_{\ell=1}^{L} \vartheta(\mu) \boldsymbol{E}\left\{|\xi^T U^{1/2} B^\ell U^{1/2} \xi|\right\} & \\
& \ge & \rho \sum\limits_{\ell=1}^{L} \|\lambda(U^{1/2} B^\ell U^{1/2})\|_1 & \\
\multicolumn{4}{r}{\text{[by Lemma 2.3 and in view of } \mathrm{rank}(U^{1/2} B^\ell U^{1/2}) \le \mathrm{rank}(B^\ell) \le \mu]} \\
& > & \mathrm{Tr}(U^{1/2} B^0 U^{1/2}) & \text{[by (12)]} \\
& = & \boldsymbol{E}\left\{\xi^T U^{1/2} B^0 U^{1/2} \xi\right\} & \text{[evident]}
\end{array},$$

so that there exists $r \in \boldsymbol{R}^m$ such that

$$\sum_{\ell=1}^{L} \vartheta(\mu)\rho |r^T U^{1/2} B^\ell U^{1/2} r| > r^T U^{1/2} B^0 U^{1/2} r.$$

Consequently, there exists a collection $\{\epsilon_\ell = \pm 1, \ \ell = 1, ..., L\}$ such that

$$r^T \left[\sum_{\ell=1}^{L} \vartheta(\mu)\rho\epsilon_\ell U^{1/2} B^\ell U^{1/2}\right] r > r^T U^{1/2} B^0 U^{1/2} r,$$

i.e., the matrix $B^0 - \sum\limits_{\ell=1}^{L} \vartheta(\mu)\rho\epsilon_\ell B^\ell$ is not positive semidefinite. Thus, $\mathcal{C}[\vartheta(\mu)\rho] \not\subset \boldsymbol{S}^m_+$, as claimed in (ii.a).

(ii.b): Let $\alpha \in \boldsymbol{R}^k$, $\|\alpha\|_1 = 1$, let $\beta = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} \in \boldsymbol{R}^{2k}$, and let $\xi \sim \mathcal{N}(0, I_{2k})$. Setting

$$J = \int \left|\sum_{i=1}^{k} u_i^2 \alpha_i\right| p_k(u) du,$$

we have

$$\boldsymbol{E}\left\{\left|\sum_{i=1}^{2k} \xi_i^2 \beta_i\right|\right\} \le \boldsymbol{E}\left\{\left|\sum_{i=1}^{k} \xi_i^2 \alpha_i\right| + \left|\sum_{i=k+1}^{2k} \xi_i^2 \alpha_{i-k}\right|\right\} = 2J. \tag{13}$$

On the other hand, setting $\eta_i = (\xi_i - \xi_{i+k})/\sqrt{2}$, $\zeta_i = (\xi_i + \xi_{i+k})/\sqrt{2}$, we get

$$\left|\sum_{i=1}^{2k} \xi_i^2 \beta_i\right| = \left|\sum_{i=1}^{k} 2\alpha_i \eta_i \zeta_i\right| = 2\left|\widehat{\eta}^T \zeta\right|, \quad \widehat{\eta} = \begin{bmatrix} \alpha_1\eta_1 \\ \vdots \\ \alpha_k \eta_k \end{bmatrix}, \ \zeta = \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_k \end{bmatrix}. \tag{14}$$

Note that $\zeta \sim \mathcal{N}(0, I_k)$ and $\widehat{\eta}$, ζ are independent. Setting $\widetilde{\eta} = \begin{bmatrix} |\alpha_1\eta_1| \\ \vdots \\ |\alpha_k\eta_k| \end{bmatrix}$, we have

$$\begin{array}{rcl} \boldsymbol{E}\left\{|\widehat{\eta}^T\zeta|\right\} & = & \boldsymbol{E}\left\{\|\widehat{\eta}\|_2\right\} \int |t| p_1(t)dt \\ & & [\text{since } \widehat{\eta},\ \zeta \text{ are independent and } \zeta \sim \mathcal{N}(0, I_k)] \\ & = & \boldsymbol{E}\left\{\|\widehat{\eta}\|_2\right\} \frac{2}{\sqrt{2\pi}} = \frac{2}{\sqrt{2\pi}} \boldsymbol{E}\left\{\|\widetilde{\eta}\|_2\right\} \\ & \geq & \frac{2}{\sqrt{2\pi}} \|\boldsymbol{E}\left\{\widetilde{\eta}\right\}\|_2 = \frac{2}{\sqrt{2\pi}} \sqrt{\sum_{i=1}^{k} \alpha_i^2 \left(\frac{2}{\sqrt{2\pi}}\right)^2} \geq \frac{2}{\pi\sqrt{k}}. \end{array} \tag{15}$$

Combining (13), (14) and (15), we get $2J \geq \frac{4}{\pi\sqrt{k}}$, i.e., $\frac{1}{J} \leq \frac{\pi\sqrt{k}}{2}$, which yields the first relation in (10).

The second relation in (10) is given by the following computation:

$$\begin{array}{l} \frac{1}{\vartheta(2)} = \min\limits_{\substack{\alpha \in \boldsymbol{R}^2, \\ \|\alpha\|_1 = 1}} \left\{ \int \left|\alpha_1 u_1^2 + \alpha_2 u_2^2\right| p_2(u) du \right\} = \min\limits_{\theta \in [0,1]} \int \left|\theta u_1^2 - (1-\theta) u_2^2\right| p_2(u) du \\ = \frac{1}{2} \int \left|u_1^2 - u_2^2\right| p_2(u) du \\ [\text{since the function to be minimized is convex in } \theta \text{ and symmetric w.r.t. } \theta = 1/2] \\ = \left[\int |t| p_1(t) dt\right]^2 = \frac{2}{\pi}. \end{array}$$

□

Let us reformulate Theorem 2.1 in a more convenient form as follows:

Corollary 2.4. *Consider a semi-infinite system of LMI's* (2) *with interval data* (3)*:*

$$\begin{array}{c} A_0 + \sum\limits_{j=1}^{n} x_j A_j \succeq 0 \quad \forall [A_0, A_1, ..., A_n] \in \mathcal{U}_\rho, \\ \mathcal{U}_\rho = \left\{ [A_0, A_1, ..., A_n] = [A_0^0, A_1^0, ..., A_n^0] + \sum\limits_{\ell=1}^{L} u_\ell [A_0^\ell, A_1^\ell, ..., A_n^\ell] \middle| \|u\|_\infty \leq \rho \right\}, \end{array} \tag{Sys[ρ]}$$

and let

$$\begin{array}{rcl} B^\ell[x] & = & A_0^\ell + \sum\limits_{j=1}^{n} x_j A_j^\ell,\ \ell = 0, 1, ..., L, \\ \mu & = & \max\limits_{\substack{x, \\ 1 \leq \ell \leq L}} \operatorname{rank}(B^\ell[x]) \end{array}$$

(note $1 \leq \ell$ in the max*!).*

The system of LMI's in variables $x, \{X_\ell\}$:

$$\begin{array}{rcl} X_\ell & \succeq & \pm\rho B^\ell[x],\ \ell = 1, ..., L, \\ \sum\limits_{\ell=1}^{L} X_\ell & \preceq & B^0[x] \end{array} \qquad (\mathrm{Appr}[\rho])$$

is a $\vartheta(\mu)$-tight approximation of (Sys[ρ]), *i.e.,*

(i) *If x can be extended to a feasible solution of* (Appr[ρ]), *then x is feasible for* (Sys[ρ]);

(ii) *If x cannot be extended to a feasible solution of* (Appr[ρ]), *then x is not feasible for* (Sys[$\vartheta(\mu)\rho$]).

2.1. Simplification of (7). From the computational viewpoint, a shortcoming of the sufficient condition (7) for the inclusion $\mathcal{C}[\rho] \subset \boldsymbol{S}^m_+$ is that the sizes of the LMI system (7), although polynomial in the sizes of MatrCube, are "large": the system has $2L+1$ "big" ($m \times m$) LMI's and has $\frac{Lm(m+1)}{2}$ scalar decision variables. Our local goal is to demonstrate that in the case when $\mu \equiv \max\limits_{1\le\ell\le L} \mathrm{rank}(B^\ell)$ is small, (7) can be reduced to a much smaller system of LMI's.

Proposition 2.5. (i) *Let $S \in \boldsymbol{S}^m$ be a matrix of rank $k > 0$, so that*

$$S = P^T R P$$

with invertible $k \times k$ symmetric matrix R and $k \times m$ matrix P of rank k.

(i.1) *A matrix $X \in \boldsymbol{S}^m$ satisfies the relation $X \succeq \pm S$ if and only if there exist $k \times k$ symmetric matrices Y, Z satisfying the relations*

$$\begin{array}{lrcl} (a) & X & \succeq & \frac{1}{2}P^T(Y+Z)P, \\ (b) & \begin{bmatrix} Y & R \\ R & Z \end{bmatrix} & \succeq & 0. \end{array} \qquad (16)$$

(i.2) *In particular, $X \succeq \pm S$ if and only if there exists $U \succeq \pm R$ such that $X \succeq P^T U P$.*

(ii) *Consequently, the solvability of* (7) *is equivalent to the solvability of the system of LMI's*

$$\begin{array}{lrcl} (a) & \begin{bmatrix} Y_\ell & \rho R_\ell \\ \rho R_\ell & Z_\ell \end{bmatrix} & \succeq & 0,\ \ell = 1, ..., L, \\ (b) & \sum\limits_{\ell=1}^{L} P_\ell^T (Y_\ell + Z_\ell) P_\ell & \preceq & 2B^0 \end{array} \qquad (17)$$

in matrix variables $Y_\ell, Z_\ell \in \boldsymbol{S}^{k_\ell}$, $\ell = 1, ..., L$. Here $k_\ell = \mathrm{rank}(B^\ell)$ (without loss of generality, we can assume that $k_\ell > 0$), and $P_\ell, R_\ell = R_\ell^T$ are $k_\ell \times m$ and $k_\ell \times k_\ell$ matrices of rank k_ℓ such that $B^\ell = P_\ell^T R_\ell P_\ell$, $\ell = 1, ..., L$.

Proof. (i.1), "if" part: Assume that X, Y, Z satisfy (16); we should prove that then $X \succeq \pm S$. To this end it suffices to verify that if Y, Z satisfy (16.b), then $\frac{1}{2}(Y+Z) \succeq \pm R$, which is immediate:

$$\begin{array}{l} (16.b) \Rightarrow \left\{ 0 \le \begin{bmatrix} \xi \\ \epsilon\xi \end{bmatrix}^T \begin{bmatrix} Y & R \\ R & Z \end{bmatrix} \begin{bmatrix} \xi \\ \epsilon\xi \end{bmatrix} \quad \forall(\xi \in \boldsymbol{R}^k, \epsilon = \pm 1) \right\} \\ \Leftrightarrow \left\{ 0 \le \xi^T(Y+Z)\xi + 2\epsilon\xi^T R\xi \quad \forall(\xi \in \boldsymbol{R}^k, \epsilon = \pm 1) \right\} \Rightarrow \frac{1}{2}(Y+Z) \succeq \pm R. \end{array}$$

(i.1), "only if" part: Let $X \succeq \pm S$. We should prove that there exist Y, Z satisfying (16). Assume, on the contrary, that the system of LMI's (16) in variables Y, Z is unsolvable, and consider the semidefinite program

$$(18) \qquad t^* = \min_{t,Y,Z} \left\{ t : \begin{array}{c} tI + 2X - P^T(Y+Z)P \succeq 0, \\ \left[\begin{array}{cc} Y & R \\ R & Z \end{array}\right] \succeq 0 \end{array} \right\}.$$

Since P is of rank k, the intersections of the levels sets of the objective with the (nonempty!) feasible set of the problem are bounded, whence the problem is solvable; unsolvability of (16) implies that the optimal value t^* in the problem is positive. Since (18) clearly is strictly feasible, it follows that the semidefinite dual of (18), which is the semidefinite program

$$(19) \qquad \min_{U,V,W,Q} \left\{ -2\operatorname{Tr}(UX) - 2\operatorname{Tr}(RQ^T) : \begin{array}{c} V = PUP^T \\ W = PUP^T \\ \left[\begin{array}{cc} V & Q \\ Q^T & W \end{array}\right] \succeq 0 \\ \operatorname{Tr}(U) = 1 \\ U, V, W \succeq 0 \end{array} \right\},$$

is solvable with the same positive optimal value t^*. In other words, there exists $U \succeq 0$ and Q such that

$$(20) \qquad \begin{array}{rrcl} (a) & \operatorname{Tr}(UX) & < & \operatorname{Tr}(RQ^T), \\ (b) & \left[\begin{array}{cc} PUP^T & Q \\ Q^T & PUP^T \end{array}\right] & \succeq & 0. \end{array}$$

From (20.b), by standard arguments, it follows that $Q = PU^{1/2}MU^{1/2}P^T$ for appropriately chosen M such that $M^TM \le I$. Consequently, (20.a) reads

$$\operatorname{Tr}(\underbrace{U^{1/2}XU^{1/2}}_{\bar{X}}) < \operatorname{Tr}(RP^TU^{1/2}M^TU^{1/2}P) = \operatorname{Tr}((\underbrace{U^{1/2}SU^{1/2}}_{\bar{S}})M^T).$$

Since $M^TM \le I$, the quantity $\operatorname{Tr}(\bar{S}M^T)$ is $\le \|\lambda(\bar{S})\|_1$, and we come to the relation $\operatorname{Tr}(\bar{X}) < \|\lambda(\bar{S})\|_1$. This is the desired contradiction, since from $X \succeq \pm S$ it follows that $\bar{X} \succeq \pm\bar{S}$, whence $\operatorname{Tr}(\bar{X}) \ge \|\lambda(\bar{S})\|_1$ (look what happens in the orthonormal basis where $\bar{S}$ becomes diagonal). (i) is proved.

(i.2): If $X \succeq P^TUP$ with $U \succeq \pm R$, then of course $X \succeq \pm P^TRP = \pm S$. Vice versa, if $X \succeq \pm S$, then by (i.2) there exist Y, Z satisfying (16). Setting $U = \frac{1}{2}(Y+Z)$, we have $X \succeq P^TUP$ by (16.a), and applying (i.1) to R rather than to S, we have $U \succeq \pm R$.

(ii) is an immediate consequence of (i). □

Note that when the ranks k_ℓ of the matrices B^ℓ, $\ell = 1, ..., L$, are much less than the size m of these matrices, system (17) is much better suited for numerical processing than (7). Indeed, the latter system has $2L+1$ "big" ($m \times m$) LMI's and totally $\frac{Lm(m+1)}{2}$ scalar decision variables, while the former system has a single "big" LMI, L "small" ones (of the sizes at most $2\mu \times 2\mu$, $\mu = \max\limits_{1 \le \ell \le L} k_\ell$), and has no more than $L\mu(\mu+1)$ scalar decision variables. A shortcoming of the reformulated system, as compared to the original one, is that when the matrices B^ℓ depend affinely on certain vector of parameters x (as it is the case in the semi-infinite LMI (2) with interval uncertainty (3)), system (7) always is a system of LMI's in variables x, $\{X_\ell\}$ (cf. $(A[\rho])$), while (17) is a system of LMI's in x, Y_ℓ, Z_ℓ only under the additional (and restrictive) assumption that the matrices P_ℓ are independent of x. In Section 3.2 we shall see that in certain important applications this shortcoming can be avoided.

3. Application I: Quadratic Lyapunov Stability Analysis and Synthesis.

3.1. Lyapunov Stability Analysis/Synthesis. Consider a controlled time-varying linear dynamic system

$$\begin{array}{llcll} (a) & \frac{d}{dt}x(t) & = & A(t)x(t)+B(t)u(t) & \text{[open-loop system]}\\ (b) & u(t) & = & Kx(t) & \text{[feedback]}\\ & & & \Downarrow & \\ (c) & \frac{d}{dt}x(t) & = & [A(t)+B(t)K]x(t) & \text{[closed-loop system]} \end{array} \tag{21}$$

(x is n-dimensional, u is m-dimensional) which is uncertain in the sense that the dependency $t\mapsto [A(t),B(t)]$ is not known in advance; all we know is that

$$\forall t:\quad [A(t),B(t)]\in\mathcal{U}_\rho=\left\{[A,B]\mid |A_{ij}-B^*_{ij}|\le\rho C_{ij}, |B_{i\ell}-B^*_{i\ell}|\le\rho D_{i\ell}\quad\forall i,j,\ell\right\}. \tag{22}$$

Here A^*, B^* are given "nominal" data, C, D are given "scale matrices" with nonnegative entries, and $\rho\ge 0$ is the "perturbation level".

Consider the following pair of problems:

<u>*Lyapunov Stability Analysis:*</u> Given A^*, B^*, C, D and a feedback K, find the supremum $R^{\rm a}_*$ of those $\rho\ge 0$ for which all instances $A+BK$, $[A,B]\in\mathcal{U}_\rho$, of the closed-loop system matrix (21.c) share a common quadratic Lyapunov function:

$$\begin{array}{rl} R^{\rm a}_* = & \sup\limits_{\rho,X}\left\{\rho: X\succeq I, [A+BK]X+X[A+BK]^T\preceq -I,\ \forall [A,B]\in\mathcal{U}_\rho\right\}\\ = & \sup\limits_{\rho,X}\left\{\rho:\ \begin{array}{l} X\succeq I\\ \forall (u_{ij}, |u_{ij}|\le\rho, u^{i\ell}, |u^{i\ell}|\le\rho):\\ \sum\limits_{i,j}u_{ij}\underbrace{C_{ij}[E^{ij}X+XE^{ij}]}_{A_{ij}[X]}+\sum\limits_{i,\ell}u^{i\ell}\underbrace{D_{i\ell}[F^{i\ell}KX+XK^T(F^{i\ell})^T]}_{A^{i\ell}[X]}\\ \preceq\underbrace{[-I-(A^*+B^*K)X-X(A^*+B^*K)^T]}_{A[X]} \end{array}\right\}, \end{array} \tag{LA}$$

where E^{ij} are the basic $n\times n$ matrices (1 in cell ij, zeros in other cells), and $F^{i\ell}$ are the basic $n\times m$ matrices;

<u>*Lyapunov Stability Synthesis:*</u> Given A^*, B^*, C, D, find the supremum $R^{\rm s}_*$ of those $\rho\ge 0$ for which there exists a feedback K such that all instances $A+BK$, $[A,B]\in\mathcal{U}_\rho$, of the closed-loop system matrix (21.c) share a common quadratic Lyapunov function:

$$\begin{array}{rl} R^{\rm s}_* = & \sup\limits_{\rho,X,K}\left\{\rho: X\succeq I, [A+BK]X+X[A+BK]^T\preceq -I,\ \forall [A,B]\in\mathcal{U}_\rho\right\}\\ = & \sup\limits_{\rho,X,Z}\left\{\rho: X\succeq I, AX+XA^T+BZ+Z^TB^T\preceq -I,\ \forall [A,B]\in\mathcal{U}_\rho\right\}\\ & \hfill [Z=KX]\\ = & \sup\limits_{\rho,X,Z}\left\{\rho:\ \begin{array}{l} X\succeq I\\ \forall (u_{ij}, |u_{ij}|\le\rho, u^{i\ell}, |u^{i\ell}|\le\rho):\\ \sum\limits_{i,j}u_{ij}\underbrace{C_{ij}[E^{ij}X+XE^{ij}]}_{B_{ij}[X]}+\sum\limits_{i,\ell}u^{i\ell}\underbrace{D_{i\ell}[F^{i\ell}Z+Z^T(F^{i\ell})^T]}_{B^{i\ell}[Z]}\\ \preceq\underbrace{[-I-A^*X-X(A^*)^T-B^*Z-Z(B^*)^T]}_{B[X,Z]} \end{array}\right\}, \end{array} \tag{LS}$$

where E^{ij} are the basic $n \times n$, and $F^{i\ell}$ are the basic $n \times m$ matrices.

As we see, both problems (LA) and (LS) deal with *solvability of semi-infinite systems of LMI's.* Consider the approximations of these systems as follows:

$$\rho_*^{\rm a} = \max_{\rho, X, \{X^{ij}, Y^{i\ell}\}} \left\{ \rho : \begin{array}{c} A_{ij}[X] \le X^{ij}, -A_{ij}[X] \le X^{ij}, \forall i,j; \\ A^{i\ell}[X] \le Y^{i\ell}, -A^{i\ell}[X] \le Y^{i\ell}, \forall i,\ell; \\ \rho \left[\sum\limits_{i,j} X^{ij} + \sum\limits_{i,\ell} Y^{i\ell} \right] \preceq A[X]. \end{array} \right\} \quad \text{(ALA)}$$

$$\rho_*^{\rm s} = \max_{\rho, X, Z, \{X^{ij}, Y^{i\ell}\}} \left\{ \rho : \begin{array}{c} B_{ij}[X] \le X^{ij}, -B_{ij}[X] \le X^{ij}, \forall i,j; \\ B^{i\ell}[Z] \le Y^{i\ell}, -B^{i\ell}[Z] \le Y^{i\ell}, \forall i,\ell; \\ \rho \left[\sum\limits_{i,j} X^{ij} + \sum\limits_{i,\ell} Y^{i\ell} \right] \preceq B[X,Z]. \end{array} \right\} \quad \text{(ALS)}$$

Note that both (ALA) and (ALS) are Generalized Eigenvalue problems (see [3, 10]) and as such are "computationally tractable".

Taking into account that the ranks of the matrices $A_{ij}[X]$, $A^{i\ell}[X]$, $B_{ij}[X]$, $B^{i\ell}[Z]$ never exceed 2 and applying Corollary 2.4, we come to the result as follows:

THEOREM 3.1. (i) *Consider the Lyapunov Stability Analysis problem and assume that the matrix $A^* + B^*K$ of the nominal closed-loop system is stable (i.e., all its eigenvalues are in the open left half-plane). Then problem* (ALA) *is an approximation of* (LA) *(i.e., the ρ, X-component of a feasible solution of* (ALA) *is a feasible solution of* (LA)*), and the optimal value of* (ALA) *coincides with the one of* (LA) *within the factor $\frac{\pi}{2}$:*

$$\rho_*^{\rm a} \le R_*^{\rm a} \le \frac{\pi}{2} \rho_*^{\rm a}.$$

(ii) *Consider the Lyapunov Stability Synthesis problem and assume that the nominal system is stabilizable (i.e., there exists a feedback K^* such that the matrix $A^* + B^*K^*$ is stable). Then problem* (ALS) *is an approximation of* (LS) *(i.e., the ρ, X, Z-component of a feasible solution of* (ALS) *is a feasible solution of* (LS)*), and the optimal value in* (ALS) *coincides with the one of* (LS) *within the factor $\frac{\pi}{2}$:*

$$\rho_*^{\rm s} \le R_*^{\rm s} \le \frac{\pi}{2} \rho_*^{\rm s}.$$

3.2. Simplifications of (ALA) and (ALS). Although the dimensions of the approximating SDP-problems (ALA) and (ALS) are polynomial in the dimensions of the original system (21), they are nevertheless of huge design dimension (they have a matrix variable per every uncertain entry in the data of (21)). This fact may render the approximating problems too difficult for practical use. We are about to demonstrate that the design dimensions of (ALA) and (ALS) can be reduced dramatically.

Consider a "generic problem" of the same structure as (ALA), (ALS):

We are given $\rho > 0$ and $L+1$ symmetric $m \times m$ matrices $B^0[x]$, $B^1[x]$,...,$B^L[x]$ affinely depending on vector x of design variables, with $B^\ell[x]$, $\ell \ge 1$, of the form

$$B^\ell[x] = a_\ell b_\ell^T[x] + b_\ell[x] a_\ell^T, \quad (23)$$

where $a_\ell \neq 0$ and the vectors $b_\ell[x] \not\equiv 0$ are affine in x. We associate with these data the following semi-infinite system of LMI's in variables x, u:

$$B^0[x] + \sum_{\ell=1}^{L} u_\ell B^\ell[x] \succeq 0 \quad \forall (u : \|u\|_\infty \leq \rho) \tag{24}$$

along with its "tractable conservative approximation" – the system of LMI's in variables x and additional matrix variables $X_1, ..., X_\ell$ as follows:

$$\begin{array}{lrcl} (a) & X_\ell & \succeq & \pm\rho B^\ell[x],\ \ell = 1, ..., L, \\ (b) & \sum\limits_{\ell=1}^{L} X_\ell & \preceq & B^0[x]. \end{array} \tag{$\mathcal{P}[\rho]$}$$

The problem is to simplify $(\mathcal{P}[\rho])$, i.e., to pass from this system to a system of LMI's in variables x and, perhaps, additional variables λ in such a way that the new system, let it be called $(\mathcal{S}[\rho])$, is of smaller design dimension than $(\mathcal{P}[\rho])$ and is equivalent to $(\mathcal{P}[\rho])$ in the sense that an x can be extended to a feasible solution of $(\mathcal{S}[\rho])$ if and only if x can be extended to a feasible solution of $(\mathcal{P}[\rho])$.

Note that both (ALA) and (ALS) are of the form of $(\mathcal{P}[\rho])$.

The simplification of $(\mathcal{P}[\rho])$ to follow is similar to the construction presented in Proposition 2.5; it turns out that the specific form (23) of the dependence of B^ℓ on x allows to end up with an analogy of (17) which is a system of LMI's in x and additional variables. The key to our simplification is the following simple fact (which can be viewed as certain strengthening of Proposition 2.5.(i) for the case when $S = ab^T + ba^T$).

LEMMA 3.2. *Let $a, b \in \boldsymbol{R}^m$ be two nonzero vectors, and let X be an $m \times m$ symmetric matrix. Then $X \succeq \pm[ab^T + ba^T]$ if and only if there exists a positive real λ such that*

$$X \succeq \lambda aa^T + \frac{1}{\lambda} bb^T. \tag{25}$$

Proof. "if" part: It suffices to prove that if $\lambda > 0$, then $\lambda aa^T + \frac{1}{\lambda} \succeq \pm[ab^T + ba^T]$, which is immediate:

$$\forall \xi : \xi^T[\lambda aa^T + \frac{1}{\lambda} bb^T]\xi = \lambda(a^T\xi)^2 + \frac{1}{\lambda}(b^T\xi)^2 \geq 2|a^T\xi||b^T\xi| \geq |\xi^T[ab^T + ba^T]\xi|.$$

"only if" part: Assume that $a, b \neq 0$ and $X \succeq \pm[ab^T + ba^T]$; we should prove that there exists $\lambda > 0$ such that $X \succeq [\lambda aa^T + \frac{1}{\lambda} bb^T]$, or, which is clearly the same, that the system of LMI's

$$\begin{array}{rcl} X & \succeq & \lambda aa^T + \mu bb^T, \\ \begin{bmatrix} \mu & 1 \\ 1 & \lambda \end{bmatrix} & \succeq & 0 \end{array} \tag{26}$$

is solvable. Assume, on the contrary, that the system is unsolvable. Since $a, b \neq 0$, the semidefinite problem

$$\min_{t,\lambda,\mu} \left\{ t : \begin{array}{rcl} tI + X & \succeq & \lambda aa^T + \mu bb^T, \\ \begin{bmatrix} \mu & 1 \\ 1 & \lambda \end{bmatrix} & \succeq & 0 \end{array} \right\} \tag{27}$$

clearly is solvable; but then infeasibility of (27) means that the optimal value in problem(27) is positive. Since the problem clearly is strictly feasible, the problem

$$\max_{U,p,q,r}\left\{-\operatorname{Tr}(UX)-2r : \begin{array}{rcl} p & = & b^TUb \\ q & = & a^TUa \\ \operatorname{Tr}(U) & = & 1 \\ \begin{bmatrix} p & r \\ r & q \end{bmatrix} & \succeq & 0 \\ U & \succeq & 0 \end{array}\right\}, \tag{28}$$

which is the semidefinite dual of (27), is solvable with positive optimal value. Since at a feasible solution to this problem one clearly has $|r| \leq \sqrt{pq} = \sqrt{(a^TUa)(b^TUb)}$, the latter fact is equivalent to the existence of $U \succeq 0$ such that

$$2\sqrt{(a^TUa)(b^TUb)} > \operatorname{Tr}(UX). \tag{29}$$

Setting $\bar{a} = U^{1/2}a$, $\bar{b} = U^{1/2}b$, $\bar{X} = U^{1/2}XU^{1/2}$ and taking into account that $X \succeq \pm[ab^T + ba^T]$, we get

$$\begin{array}{lrcl} (a) & \bar{X} & \succeq & \pm Q, \quad Q = \bar{a}\bar{b}^T + \bar{b}\bar{a}^T, \\ (b) & \operatorname{Tr}(\bar{X}) & < & 2\|\bar{a}\|_2\|\bar{b}\|_2. \end{array} \tag{30}$$

This is the desired contradiction. Indeed, from (30.a) it follows that $\operatorname{Tr}(\bar{X}) \geq \|\lambda(Q)\|_1$ (pass to the orthonormal basis where Q is diagonal); on the other hand, an immediate computation demonstrates that $\|\lambda(Q)\|_1 = 2\|\bar{a}\|_2\|\bar{b}\|_2$, which is $> \operatorname{Tr}(\bar{X})$ by (30.b). □

Lemma 3.2 underlies the following

PROPOSITION 3.3. *The LMI system* ($\mathcal{P}[\rho]$) *is equivalent to the following system of LMI's in variables* x *and additional variables* $Y \in \boldsymbol{S}^m$, $\lambda \in \boldsymbol{R}^L$*:*

$$\begin{array}{ll} (a) & \begin{bmatrix} Y - \sum\limits_{\ell=1}^{L} \lambda_\ell a_\ell a_\ell^T & b_1[x] & b_2[x] & \dots & b_\ell[x] \\ b_1^T[x] & \lambda_1 & & & \\ b_2^T[x] & & \lambda_2 & & \\ \vdots & & & \ddots & \\ b_L^T[x] & & & & \lambda_L \end{bmatrix} \succeq 0, \\ (b) & \rho Y \preceq B^0[x]. \end{array} \tag{31}$$

Proof. We should prove that if x can be extended to a feasible solution of ($\mathcal{P}[\rho]$), then x can be extended to a feasible solution of (31), and vice versa.

1^0. Assume that $x, \{X_\ell\}$ is a feasible solution of ($\mathcal{P}[\rho]$), and let $J(x)$ be the set of those ℓ for which $b_\ell[x] = 0$. Let us extend x to a feasible solution of (31) as follows:

1. For $\ell \in J(x)$, we set $\lambda_\ell = 0$.
2. For $\ell \notin J(x)$, $a_\ell \neq 0$, $b_\ell[x] \neq 0$ and $\rho^{-1}X_\ell \succeq \pm[a_\ell b_\ell^T[x] + b_\ell[x]a_\ell^T]$. Applying Lemma 3.2, we can find $\lambda_\ell > 0$ such that $\rho^{-1}X_\ell \succeq \lambda_\ell a_\ell a_\ell^T + \lambda_\ell^{-1}b_\ell[x]b_\ell^T[x]$.
3. After we have defined $\lambda_\ell \geq 0$ for all $\ell = 1, ..., L$, we set

$$Y = \sum_{\ell \notin J(x)} \left[\lambda_\ell a_\ell a_\ell^T + \lambda_\ell^{-1} b_\ell[x] b_\ell^T[x]\right].$$

Let us prove that $x, Y, \{\lambda_\ell\}$ is feasible for (31). Indeed, $(31.a)$ is readily given by the definition of Y and the Schur Complement Lemma (note that a zero λ_ℓ on the diagonal of the left-hand side matrix in $(31.a)$ corresponds to a zero row and a zero column). Further, from the origin of λ_ℓ, $\ell \notin J(x)$, it follows that $Y \preceq \sum\limits_{\ell\notin J(x)} \rho^{-1}X_\ell \preceq \rho^{-1}\sum\limits_{\ell=1}^{L} X_\ell$, and since $x, \{X_\ell\}$ is feasible for $(\mathcal{P}[\rho])$, we conclude that $\rho Y \preceq B^0[x]$, i.e., $(31.b)$ is valid. Thus, $x, Y, \{\lambda_\ell\}$ is feasible for (31).

2^0. Now assume that $x, Y, \{\lambda_\ell\}$ is feasible for (31), and let us prove that x can be extended to a feasible solution of $(\mathcal{P}[\rho])$. Let, as above, $J(x)$ be the set of those ℓ for which $b_\ell[x] = 0$. Note that from $(31.a)$ it follows that $\lambda_\ell \geq 0$ for all ℓ and $\lambda_\ell > 0$ for $\ell \notin J(x)$. Let us set

$$X_\ell = \rho \begin{cases} 0, & \ell \in J(x) \\ \lambda_\ell a_\ell a_\ell^T + \lambda_\ell^{-1} b_\ell[x] b_\ell^T[x], & \ell \notin J(x) \end{cases}$$

Applying Lemma 3.2, we see that $(\mathcal{P}[\rho].a)$ holds true. Now, by Schur Complement Lemma from $(31.a)$ it follows that

$$\rho^{-1}\sum_\ell X_\ell = \sum_{\ell\notin J(x)} \left[\lambda_\ell a_\ell a_\ell^T + \lambda_\ell^{-1} b_\ell[x] b_\ell^T[x]\right] \preceq Y;$$

this observation combined with $(31.b)$ implies the validity of $(\mathcal{P}[\rho].\mathrm{b})$. Thus, $x, \{X_\ell\}$ is feasible for $(\mathcal{P}[\rho])$. □

We have reduced the system of LMI's $(\mathcal{P}[\rho])$ to (31). In the original system, there are $(\dim x + L\dim X)$ scalar design variables, while in the resulting system there are just $(\dim x + \dim X + L)$ design variables. To realize how large the reduction in the design dimension can be, consider the case when $(\mathcal{P}[\rho])$ is the problem (ALS). Here $x = X$ is a symmetric $n \times n$ matrix, and L is the total number of uncertain entries in the underlying uncertain interval matrix $[A, B]$. Here the original system $(\mathcal{P}[\rho])$ has $L + 1$ symmetric $n \times n$ matrix variables, i.e., totally $\frac{(L+1)n(n+1)}{2}$ scalar design variables, and $(2L + 1)$ "large" $(n \times n)$ LMI's. The reformulated system (31) has just two symmetric $n \times n$ matrix variables X, Y and $L \leq n^2 + nm$ scalar variables λ_ℓ, i.e., totally $L + n(n + 1) \leq 2n^2 + n(m + 1)$ scalar design variables. As about LMI's, (31) has one "large" $(n \times n)$ LMI (b) and one "very large" $((n + L) \times (n + L))$ LMI (a); note, however, that this LMI is of very simple "arrow" structure and is very sparse. Thus, (31) seems to be much better suited for numerical processing than $(\mathcal{P}[\rho])$.

3.3. Extensions. *An LMI region* is a set $\mathcal{H}$ in the complex plane $\boldsymbol{C}$ representable as

$$\mathcal{H} = \{z \in \boldsymbol{C} \mid f_{\mathcal{H}}(z) \equiv P + Qz + Q^T\bar{z} \prec 0\},$$

where $P = P^T$ and Q are real $k \times k$ matrices and $\bar{z}$ is the complex conjugate of z. The simplest examples of LMI regions are:

1. *Open left half-plane*: $f_{\mathcal{H}}(z) = z + \bar{z}$;
2. *Open disk* $\{z \mid |z + q| \leq r\}$, $q \in \boldsymbol{R}, r > 0$: $f_{\mathcal{H}}(z) = \begin{pmatrix} -r & \bar{z}+q \\ z+q & -r \end{pmatrix}$;
3. *The interior of the sector* $\{z \mid \pi - \theta \leq |\arg(z)| \leq \pi\}$ $(-\pi < \arg(z) \leq \pi$, $0 < \theta < \frac{\pi}{2})$:

$$f_{\mathcal{H}}(z) = \begin{pmatrix} (z+\bar{z})\sin\theta & -(z-\bar{z})\cos\theta \\ (z-\bar{z})\cos\theta & (z+\bar{z})\sin\theta \end{pmatrix};$$

4. *The stripe* $\{z \mid h_1 < \Re(z) < h_2\}$: $f_{\mathcal{H}}(z) = \begin{pmatrix} 2h_1 - (z+\bar{z}) & 0 \\ 0 & (z+\bar{z}) - 2h_2 \end{pmatrix}$.

It is known (see, e.g., [5]) that the spectrum $\Sigma(A)$ of a real $n \times n$ matrix A belongs to $\mathcal{H}$ if and only if there exists $X \in \boldsymbol{S}^m$, $X \succ 0$, such that the $k \times k$ block matrix $\mathcal{M}[X,A]$ with the $m \times m$ blocks

$$\mathcal{M}_{ij}[X,A] = P_{ij}X + Q_{ij}AX + Q_{ji}XA^T, \ i,j = 1,...,m$$

is negative definite. We can treat such an X as a certificate of the inclusion $\Sigma(A) \subset \mathcal{H}$, and by homogeneity reasons we can normalize this certificate to satisfy the relations $X \succeq I$, $\mathcal{M}[X,A] \preceq -I$. From now on, we speak about normalized certificates only.

The problem we are interested in now is as follows. Given an LMI region $\mathcal{H}$ and an "uncertain interval matrix"

$$\mathcal{U}_\rho = \Big\{[A,B] \in \boldsymbol{R}^{n\times n} \times \boldsymbol{R}^{n\times m} \mid |A_{ij} - A^*_{ij}| \le \rho C_{ij},\ |B_{i\ell} - B^*_{i\ell}| \le \rho D_{i\ell},\ 1 \le i,j \le n, 1 \le \ell \le m\Big\}$$

of the open-loop system (21), we ask what is the supremum R_* of those $\rho \ge 0$ for which there exists a linear feedback $K \in \boldsymbol{R}^{m\times n}$ such that all instances $A+BK$, $[A,B] \in \mathcal{U}_\rho$, of the uncertain matrix of the closed-loop system share a common certificate X of the inclusion $\Sigma(A+BK) \subset \mathcal{H}$. This important problem in Control is a natural extension of the Lyapunov Stability Synthesis problem. The problem can be treated in the same fashion as Lyapunov Analysis/Synthesis. Indeed, $X \succeq I$ certifies the inclusion $\Sigma(A+BK) \subset \mathcal{H}$ for all $[A,B] \in \mathcal{U}_\rho$ if and only if (X,K) solves the semi-infinite system of matrix inequalities

$$\mathcal{M}[X, A+BK] \preceq -I \quad \forall [A,B] \in \mathcal{U}_\rho.$$

Passing from the variables X,K to $X, Z = KX$, we convert this system to the semi-infinite system of LMI's

$$\begin{array}{l}\forall\,([A,B] \in \mathcal{U}_\rho): \\ \mathcal{N}(X,Z,A,B) \equiv \left[P_{ij}X + Q_{ij}AX + Q_{ij}BZ + Q_{ji}XA^T + Q_{ji}Z^TB^T\right]_{1\le i,j\le k} \preceq -I,\end{array} \tag{32}$$

where $[M_{ij}]_{1\le i,j\le k}$ denotes block matrix with blocks M_{ij}. We see that (X,Z) solves (32) if and only if (X,Z) solves the semi-infinite system of LMI's

$$\begin{array}{l}\forall\,(\{u_{ij}, |u_{ij}| \le 1\}, \{v_{i\ell}, |v_{i\ell}| \le 1\}): \\ \rho\left(\sum\limits_{(i,j):C_{ij}>0} u_{ij}\mathcal{N}^0(X,Z,C_{ij}E^{ij},0) + \sum\limits_{(i,\ell):D_{i\ell}>0} v_{i\ell}\mathcal{N}^0(X,Z,0,D_{i\ell}F^{i\ell})\right) \\ \qquad\qquad -I - \mathcal{N}(X,Z,A^*,B^*) \ \succeq\ 0, \\ \mathcal{N}^0(X,Z,A,B) = \left[Q_{ij}AX + Q_{ij}BZ + Q_{ji}XA^T + Q_{ji}Z^TB^T\right]_{1\le i,j\le k}\end{array} \tag{$\mathcal{I}[\rho]$}$$

where the basic matrices E^{ij}, $F^{i\ell}$ are the same as in (LA), (LS). As before, an evident sufficient condition for $X \succeq I$ and Z to solve $(\mathcal{I}[\rho])$ is the existence of matrices X^{ij}, $(i,j) \in \mathcal{C} = \{(i,j) \mid C_{ij} > 0\}$, $Z^{i\ell}$, $(i,\ell) \in \mathcal{D} = \{(i,\ell) \mid D_{i\ell} > 0\}$ such that

$(X, Z, X^{ij}, Z^{i\ell})$ solves the system of LMI's

$$\begin{array}{c} X^{ij} \succeq \mathcal{N}^0(X,Z,C_{ij}E^{ij},0),\ X^{ij} \succeq -\mathcal{N}^0(X,Z,C_{ij}E^{ij},0),\ (i,j)\in\mathcal{C}, \\ Z^{i\ell} \succeq \mathcal{N}^0(X,Z,0,D_{i\ell}F^{i\ell}),\ Z^{i\ell} \succeq -\mathcal{N}^0(X,Z,0,D_{i\ell}F^{i\ell}),\ (i,\ell)\in\mathcal{D}, \\ \rho\left(\sum\limits_{(i,j)\in\mathcal{C}} X^{ij} + \sum\limits_{(i,\ell)\in\mathcal{D}} Z^{i\ell}\right) \preceq -I - \mathcal{N}(X,Z,A^*,B^*), \\ X \succeq I. \end{array} \qquad (\mathcal{II}[\rho])$$

Invoking Corollary 2.4, we arrive at the following result:

THEOREM 3.4. *Let the system* $(\mathcal{I}[0])$ *(or, which is the same,* $(\mathcal{II}[0])$*) be solvable, and let*

$$\mu = \max\left[\max_{X,Z,i,j} \mathrm{rank}(\mathcal{N}^0(X,Z,C_{ij}E^{ij},0)), \max_{X,Z,i,\ell} \mathrm{rank}(\mathcal{N}^0(X,Z,0,D_{i\ell}F^{i\ell}))\right].$$

Then

(i) *If* $(\mathcal{II}[\rho])$ *is solvable, so is* $(\mathcal{I}[\rho])$*, and the* (X,Z)*-component of a solution of the former system solves the latter system;*

(ii) *If* $(\mathcal{II}[\rho])$ *is unsolvable, so is* $(\mathcal{I}[\vartheta(\mu)\rho])$*, where* $\vartheta(\cdot)$ *is the function given in* (9)*.*

In particular,

$$\frac{\sup\{\rho:\ (\mathcal{I}[\rho]) \text{ is solvable}\}}{\sup\{\rho:\ (\mathcal{II}[\rho]) \text{ is solvable}\}} \le \vartheta(\mu). \tag{33}$$

Note that the denominator in (33) is the optimal value in an explicit Generalized Eigenvalue problem and thus is efficiently computable. Note also that one always has $\mu \le 2k$, and that for our list of the 4 simple LMI regions: $\mu = 2$ in cases 1 and 2 ("half-plane" and "disk"), and $\mu = 4$ in cases 3 and 4 ("sector" and "stripe").

There are many other applications of Theorem 2.1 to semi-infinite systems of LMI's (2) arising in Control, provided that the uncertainty set $\mathcal{U}$ in (2) is an interval uncertainty. In a typical Control application, all matrices A_j in (2) share a common block-diagonal structure and are such that when perturbing a single data entry, every diagonal block in the matrix $A_0 + \sum\limits_j x_j A_j$ is perturbed by a small rank matrix, which is exactly the case considered in Theorem 2.1.

4. Application II: quadratic maximization over the unit cube. Here we demonstrate that the MatrCube problem in its simplest form, where all the edge matrices B^ℓ are very specific matrices of ranks ≤ 2, is equivalent to the problem

$$\omega_*(Q) = \max_x \left\{x^TQx : \|x\|_\infty \le 1\right\} \quad [Q \succ 0] \tag{34}$$

of maximizing a positive definite quadratic form over the unit cube. On one hand, this observation says that MatrCube (already in the case of "rank 2 edges") is NP-hard (since (34) is so). On the other hand, our observation allows to extract from Theorem 2.1 certain statement on the possibility to build efficiently a tight bound on the optimal value in (34). As it turns out, this bound is exactly the one given by the standard semidefinite relaxation of (34), and the corresponding "tightness" statement coming from Theorem 2.1 is nothing but the "$\frac{\pi}{2}$ Theorem" of Nesterov [12].

The link between the quadratic maximization over the unit cube and the Matrix Cube problem is given by the following simple observation:

PROPOSITION 4.1. *Assume that Q in* (34) *is positive definite. Then*

$$
(35)\qquad
\begin{array}{ll}
(a) & \omega \geq \omega_*(Q) \equiv \max\limits_{x:\|x\|_\infty \leq 1} x^T Q x \\
\text{(i)} & \Updownarrow \\
(b) & \omega \xi^T Q^{-1} \xi \geq \|\xi\|_1^2 \quad \forall \xi \\
\text{(ii)} & \Updownarrow \\
(c) & \omega Q^{-1} + \{A \in \boldsymbol{S}^m : |A_{ij}| \leq 1,\, 1 \leq i,j \leq m\} \subset \boldsymbol{S}^m_+
\end{array}
$$

Proof. Relation (a) means that the ellipsoid $\{x : x^T Q x \leq \omega\}$ contains the unit cube $\{x : \|x\|_\infty \leq 1\}$. Passing to polars, this is exactly the same as to say that the polar of the ellipsoid, which is the ellipsoid $\{\xi : \xi^T Q^{-1} \xi \leq \omega^{-1}\}$, is contained in the polar of the unit cube, which is the set $\{\xi : \|\xi\|_1 \leq 1\}$. But the latter inclusion is exactly what is stated in (b). We have proved the equivalence (i).

Now, (c) says exactly that

$$
(36)\qquad \omega \xi^T Q^{-1} \xi + \min_A \left\{\xi^T A \xi : A = A^T, |A_{ij}| \leq 1\right\} \geq 0 \quad \forall \xi.
$$

The minimum in the left hand side of this relation is equal to $-\|\xi\|_1^2$ (indeed, $\xi^T A \xi \geq -\|\xi\|_1^2$ whenever $|A_{ij}| \leq 1$ for all i,j, and $\xi^T A \xi = -\|\xi\|_1^2$ for $A_{ij} = -\mathrm{sign}(\xi_i)\mathrm{sign}(\xi_j)$, $i,j = 1,...,m$). Thus, (36) is equivalent to the relation $\omega \xi^T Q^{-1} \xi - \|\xi\|_1^2 \geq 0$ for all ξ, which is nothing but (b). We have proved the equivalence (ii). □

Now, let S^{ij} be the basic symmetric matrices (so that S^{ii} has a single nonzero entry, equal to 1, in the cell (i,i), and S^{ij}, $i \neq j$, has exactly two nonzero entries, both equal to 1, in the cells (i,j) and (j,i)). Relation (35.b) says exactly that the matrix box

$$
\mathcal{C}[\frac{1}{\omega}] = \left\{ Q^{-1} + \sum_{1 \leq i \leq j \leq m} u_{ij} S^{ij} : \|u\|_\infty \leq \frac{1}{\omega} \right\}
$$

is contained in the positive semidefinite cone. According to Theorem 2.1, a *sufficient* condition for this inclusion is the solvability of the system of LMI's as follows:

$$
(37)\qquad
\begin{array}{rcl}
X^{ij} & \succeq & \pm \rho S^{ij},\ 1 \leq i \leq j \leq m \\
\sum\limits_{1 \leq i \leq j \leq m} X^{ij} & \preceq & Q^{-1}
\end{array}
, \rho = \frac{1}{\omega}.
$$

Moreover, since the ranks of the edge matrices S^{ij} are ≤ 2, Theorem 2.1 says that the solvability of (37) is a "tight, within the factor $\frac{\pi}{2}$", sufficient condition for the validity of (35.b). Taking into account that the smallest value of ω for which (35.b) is valid is exactly $\omega_*(Q)$ (Proposition 4.1), we arrive at the following

PROPOSITION 4.2. *Let $Q \succ 0$. Consider the semidefinite program*

$$
(38)\qquad \rho(Q) = \max_{\rho, X^{ij}} \left\{ \rho : \begin{array}{rcl} X^{ij} & \succeq & \pm \rho S^{ij},\ 1 \leq i \leq j \leq m \\ \sum\limits_{1 \leq i \leq j \leq m} X^{ij} & \preceq & Q^{-1} \end{array} \right\}.
$$

The reciprocal of the optimal value in this problem is an upper bound on the optimal value $\omega_*(Q)$ in the problem of quadratic maximization (34)*, and this bound is tight within the factor $\frac{\pi}{2}$:*

$$
(39)\qquad \omega_*(Q) \leq \frac{1}{\rho(Q)} \leq \frac{\pi}{2} \omega_*(Q).
$$

Proposition 4.2 says that certain efficiently computable via Semidefinite Programming quantity (namely, $\frac{1}{\rho(Q)}$) is a tight, within the factor $\frac{\pi}{2}$, upper bound on the maximum $\omega_*(Q)$ of the positive definite quadratic form x^TQx over the unit cube. We are about to demonstrate that our bound is nothing but the standard semidefinite upper bound

$$\begin{array}{rcl}\omega^*(Q) & = & \max_X \{\mathrm{Tr}(QX) : X_{ii} \leq 1,\, i=1,...,m, X \succeq 0\} \\ & = & \min\limits_{\lambda} \left\{\sum\limits_{i=1}^m \lambda_i : \mathrm{Diag}\{\lambda\} \succeq Q\right\}\end{array} \tag{40}$$

on $\omega_*(Q)$.

Proposition 4.3. *For $Q \succ 0$, one has $\frac{1}{\rho(Q)} = \omega^*(Q)$.*

Proof. Let e_i be the standard basic orths in $\boldsymbol{R}^m$, so that $S^{ij} = \frac{1}{1+\delta_{ij}}[e_ie_j^T + e_je_i^T]$, where δ_{ij} are the Kronecker symbols. Applying Lemma 3.2, we see that

$$\begin{array}{l}\rho(Q) = \max\left\{\rho : \exists X^{ij} : \begin{array}{c} X^{ij} \succeq \pm\frac{\rho}{1+\delta_{ij}}[e_ie_j^T + e_je_i^T],\ 1 \leq i \leq j \leq m \\ \sum\limits_{1\leq i\leq j\leq m} X^{ij} \preceq Q^{-1}\end{array}\right\} \\ = \max\left\{\rho : \exists\{H_{ij} > 0\}_{1\leq i\leq j\leq m} : \sum\limits_{1\leq i\leq j\leq m} \frac{1}{1+\delta_{ij}}\left[H_{ij}e_ie_i^T + H_{ij}^{-1}e_je_j^T\right] \preceq \frac{1}{\rho}Q^{-1}\right\}.\end{array} \tag{41}$$

Let $\mathcal{H}$ be the set of all $m \times m$ matrices $H = [H_{ij}]$ with positive entries such that $H_{ij}H_{ji} \geq 1$ for all i, j. It is immediately seen that (41) can be rewritten as

$$\begin{array}{rcl}\rho^{-1}(Q) & = & \min\left\{\omega : \exists(H \in \mathcal{H}) : \Lambda(H) \preceq \omega Q^{-1}\right\} \\ & = & \min\left\{\omega : \exists(H \in \mathcal{H}) : Q \preceq \omega\Lambda^{-1}(H)\right\},\end{array} \tag{42}$$

where $\Lambda(H)$ is the diagonal matrix with the diagonal entries

$$\Lambda_{ii}(H) = \sum_{j=1}^m H_{ij},\ i=1,...,m.$$

Lemma 4.4. *The matrices which can be represented as $\Lambda^{-1}(H)$, $H \in \mathcal{H}$, are exactly the positive definite diagonal matrices with trace ≤ 1.*

Proof. A matrix $M = \mathrm{Diag}\{\mu_i\}$ with $\mu_i > 0$ and $s \equiv \sum\limits_i \mu_i \leq 1$ is $\Lambda^{-1}(H)$ for H given by $H_{ij} = \frac{\mu_j}{s\mu_i}$; note that $H \in \mathcal{H}$ due to $s \leq 1$. It remains to prove that if $H \in \mathcal{H}$, then $\mathrm{Tr}(\Lambda^{-1}(H)) \leq 1$. To this end observe that

(*) *For positive reals $\mu_1, ..., \mu_m$, one has*

$$\sum_i \mu_i \leq 1 \Leftrightarrow \forall\{a_i > 0\} : \sum_i \frac{a_i^2}{\mu_i} \geq (\sum_i a_i)^2.$$

Indeed, $\Rightarrow$ is given by the evident relation $\min\limits_{\mu_i>0:\sum\limits_i \mu_i\leq 1} \sum\limits_i \frac{a_i^2}{\mu_i} = (\sum\limits_i a_i)^2$ for all $a_i > 0$.

To verify $\Leftarrow$, set $a_i = \mu_i$ in the inequality $\sum\limits_i \frac{a_i^2}{\mu_i} \geq (\sum\limits_i a_i)^2$.

In view of (*), in order to prove that $H \in \mathcal{H}$ implies $\mathrm{Tr}(\Lambda^{-1}(H)) \le 1$, it suffices to verify that if $H \in \mathcal{H}$ and $a_i > 0$, then $\sum\limits_i a_i^2 \Lambda_{ii}(H) \ge (\sum\limits_i a_i)^2$, which is immediate:

$$\begin{array}{rcl} \sum\limits_{i=1}^m a_i^2 \Lambda_{ii}(H) & = & \sum\limits_{i,j=1}^m a_i^2 H_{ij} = \sum\limits_{i=1}^m a_i^2 H_{ii} + \sum\limits_{i<j} [a_i^2 H_{ij} + a_j^2 H_{ji}] \\ & \ge & \sum\limits_i a_i^2 + 2\sum\limits_{i<j} a_i a_j \\ & & \qquad\qquad [\text{since } H_{ij} > 0, H_{ij}H_{ji} \ge 1] \\ & = & \left(\sum\limits_i a_i\right)^2 . \end{array}$$

□

By Lemma 4.4, as H runs through $\mathcal{H}$, the matrix $\Lambda^{-1}(H)$ runs through the entire set of positive definite diagonal matrices with trace ≤ 1, so that the matrix $\omega\Lambda^{-1}(H)$ runs through the entire set of positive definite diagonal matrices with trace $\le \omega$. Consequently, (42) implies that

$$\rho^{-1}(Q) = \min \left\{ \sum_{i=1}^m \lambda_i : Q \preceq \mathrm{Diag}\{\lambda\} \right\},$$

so that $\rho^{-1}(Q) = \rho^*(Q)$ by (40). □

Note that the fact that the bound (40) on the optimal value $\omega_*(Q)$ of (34) is tight within the factor $\frac{\pi}{2}$ is known; it is the "$\frac{\pi}{2}$ Theorem" of Nesterov [12] established originally via a construction based on the famous MAXCUT-related "random hyperplane" technique of Goemans and Williamson [7]. Surprisingly, the alternative proof we have developed, although exploits randomization, seemingly uses nothing like the random hyperplane technique.

5. Maximizing homogeneous polynomial of degree 3 over the unit cube. Let $B[x^1, x^2, x^3]$ be a symmetric 3-linear form on $\boldsymbol{R}^m$, and let $P[x] = B[x, x, x]$ be the associated homogeneous polynomial, i.e.,

$$P[x] = \sum_{j=1}^m x_j (x^T B_j x) \quad \text{where} \quad B_j \in S^m .$$

Consider the problem of computing

$$\omega(P) = \max_x \{P[x] : \|x\|_\infty \le 1\}$$

along with the semidefinite program

$$\text{(43)}\quad \omega^*(P) = \min_{\lambda, X^1,\dots,X^m} \left\{ \sum_{j=1}^m \lambda_j : \sum_j X^j \preceq \mathrm{Diag}\{\lambda\}, X^j \succeq \pm B_j,\ j = 1, \dots, m \right\},$$

where B_j are the matrices of the symmetric bilinear forms $B[e_j, \cdot, \cdot]$; here e_j, $j = 1, \dots, m$, are the standard basic orths in $\boldsymbol{R}^m$. We intend to demonstrate that $\omega^*(P)$ is an upper bound on $\omega(P)$, and that the quality of this bound basically depends only on the "width"

$$d(P) = \max_{1 \le j \le m} \mathrm{rank}(B_j).$$

of P.

THEOREM 5.1. *One has:*

$$\omega(P) \le \omega^*(P) \le 4.652\vartheta(d(P))\ln(m+1)\omega(P) \le 7.31\sqrt{d(P)}\ln(m+1)\omega(P), \tag{44}$$

where $\vartheta(\cdot)$ *is given by* (9).

Proof. The proof is very much in the spirit of the Matrix Cube Theorem 2.1; it uses a *probabilistic* argument in order to validate the solvability/insolvability of a certain deterministic inequality system.

1^0. Let $\lambda, X_1, ..., X_m$ be a feasible solution of (43). We have

$$\begin{array}{rcl}\|x\|_\infty \le 1 \Rightarrow P[x] &=& \sum\limits_j x_j(x^TB_jx) \le \sum\limits_j |x_j|(x^TX_jx) \le \sum\limits_j x^TX_jx \\ &\le& x^T\mathrm{Diag}\{\lambda\}x \le \sum\limits_j \lambda_j,\end{array}$$

which gives the first inequality in (44).

2^0. Let us prove the second inequality in (44); w.l.o.g. we may assume that $P \not\equiv 0$, so that $\omega(P) > 0$. Problem (43) is strictly feasible and bounded below, so that its optimal value $\omega^*(P)$ is equal to that of its (solvable) semidefinite dual problem:

$$\omega^*(P) = \max_{U,Y_j,Z_j}\left\{\sum_j \mathrm{Tr}([Y_j - Z_j]B_j) : \begin{array}{l} U, Y_j, Z_j \succeq 0, \\ Y_j + Z_j = U, \\ U_{jj} = 1, j = 1, ..., m\end{array}\right\}. \tag{45}$$

Invoking Lemma 2.2, we see that there exists U such that

$$\begin{array}{lrcl}(a) & U \succeq & 0, & \\ (b) & U_{jj} &=& 1, j = 1, ..., m, \\ (c) & \sum\limits_j \|\lambda(U^{1/2}B_jU^{1/2})\|_1 &=& \omega^*(P).\end{array} \tag{46}$$

Now let $V = U^{1/2}$ and let $\xi \sim \mathcal{N}(0, I_m)$. By Lemma 2.3 and (46.$c$) we have

$$\vartheta(d(P))\boldsymbol{E}\left\{\sum_j |\xi^TVB_jV\xi|\right\} \ge \sum_j \|\lambda(VB_jV)\|_1 = \omega^*(P). \tag{47}$$

At the same time, by (46.b) the Euclidean norms of the rows of V are equal to 1, so that

$$\|V\xi\|_\infty = \max_{1\le j\le m}|\zeta_j|, \quad \zeta_j \sim \mathcal{N}(0,1),$$

whence, as it is well-known,

$$\boldsymbol{E}\left\{\|Y\xi\|_\infty^2\right\} \le 2\ln(m+1). \tag{48}$$

To make the paper self-contained, here is a derivation of (48). We have

$$
\begin{array}{rl}
t > 0 \Rightarrow & \\
\psi(t) & \equiv \mathrm{Prob}\{|\zeta_j| > t\} = 2\int_t^\infty \frac{1}{\sqrt{2\pi}} \exp\{-\frac{\tau^2}{2}\}d\tau \\
& \leq 2\int_t^\infty \frac{1}{\sqrt{2\pi}} \frac{\tau}{t} \exp\{-\frac{\tau^2}{2}\}d\tau = \sqrt{\frac{2}{\pi}} t^{-1} \exp\{-\frac{t^2}{2}\} \\
\Rightarrow & \\
\mathrm{Prob}\{\max\limits_{j\leq m} |\zeta_j| > t\} & \leq \min[1, m\psi(t)] \leq \min\left[1, m\sqrt{\frac{2}{\pi}} t^{-1} \exp\{-\frac{t^2}{2}\}\right] \\
\Rightarrow & \\
\boldsymbol{E}\left\{\max\limits_{j\leq m} |\zeta_j|^2\right\} & = 2\underbrace{\int\limits_{t>0} t\min[1, m\psi(t)]dt}_{J_m} \leq 2\int\limits_{t>0} \min[t, m\sqrt{\frac{2}{\pi}} \exp\{-\frac{t^2}{2}\}]dt \\
& \leq 2\int\limits_0^\tau t dt + 2\sqrt{\frac{2}{\pi}} m \int\limits_\tau^\infty \exp\{-\frac{t^2}{2}\}dt \\
& \leq \tau^2 + 2\sqrt{\frac{2}{\pi}} m\tau^{-1} \exp\{-\frac{\tau^2}{2}\}.
\end{array}
$$

The resulting bound

$$
\boldsymbol{E}\left\{\max_{j\leq m} |\zeta_j|^2\right\} \leq 2J_m \leq \tau^2 + 2\sqrt{\frac{2}{\pi}} m\tau^{-1} \exp\{-\frac{\tau^2}{2}\} \tag{49}
$$

is valid for all m and all $\tau > 0$. Assuming $m \geq 3$ and setting $\tau = \sqrt{2\ln(m/2)}$, one can easily conclude from (49) that $2J_m \leq 2\ln(m+1)$ for all $m \geq 25$. Numerical computation of J_m for $m \leq 25$ demonstrates that the latter inequality holds true for all m.

Combining (47), (48), we get

$$
\boldsymbol{E}\left\{\sum_j |\xi^T V B_j V \xi|\right\} \geq \frac{\omega^*(P)}{2\vartheta(d(P))\ln(m+1)} \boldsymbol{E}\left\{\|V\xi\|_\infty^2\right\} \tag{50}
$$

and the left-hand side in this inequality is positive. It follows that there exists $\eta \in \boldsymbol{R}^m$ and a vector $\epsilon \in \boldsymbol{R}^m$ with entries ± 1 such that

$$
\eta^T \left[\sum_j \epsilon_j B_j\right] \eta \geq \frac{\omega^*(P)}{2\vartheta(d(P))\ln(m+1)} \text{ and } \|\eta\|_\infty = 1,
$$

whence

$$
\max_{\epsilon,\eta} \{B[\epsilon, \eta, \eta] : \|\epsilon\|_\infty \leq 1, \|\eta\|_\infty \leq 1\} \geq \frac{\omega^*(P)}{2\vartheta(d(P))\ln(m+1)}. \tag{51}
$$

On the other hand,

$$
B[x + ty, x + ty, x + ty] = B[x, x, x] + 3tB[x, x, y] + 3t^2 B[x, y, y] + t^3 B[y, y, y],
$$

whence

$$
\begin{array}{l}
\forall t \neq 0 \forall x, y : \\
B[x, y, y] = \frac{B[x+ty,x+ty,x+ty]+B[x-ty,x-ty,x-ty]-2B[x,x,x]}{6t^2}
\end{array}
$$

It follows that

$$
\max_{\epsilon,\eta} \{B[\epsilon, \eta, \eta] : \|\epsilon\|_\infty \leq 1, \|\eta\|_\infty \leq 1\} \leq \frac{(1+t)^3 + 1}{3t^2} \omega(P) \quad \forall t > 0,
$$

which combines with (51) to yield the relation

$$\frac{\omega^*(P)}{2\vartheta(d(P))\ln(m+1)} \leq \min_{t>0} \frac{(1+t)^3+1}{3t^2}\omega(P) \leq 2.326\omega(P),$$

and the second inequality in (44) follows. The third inequality follows from $\vartheta(d) \leq \frac{\pi\sqrt{d}}{2}$, see (10). □

REMARK 5.1. There are two simple cases when $d(P)$ is small. The first is when $P[x]$ is "of small rank": $P[x] = \sum\limits_{\ell=1}^{L}(p_\ell^T x)^3$ with a small L (since clearly $d(P) \leq L$). The second case is when P is of a "band" structure, i.e., the quantity

$$\kappa = \max_{1\leq i\leq j\leq k\leq m}\{k-i : B[e_i,e_j,e_k] \neq 0\}$$

is small (since clearly $d(P) \leq 2\kappa+1$).

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