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ROBUST DISSIPATIVITY OF INTERVAL UNCERTAIN LINEAR SYSTEMS*

FLORIN DAN BARB†, AHARON BEN-TAL‡, AND ARKADI NEMIROVSKI‡

Abstract. In this paper we are concerned with the problem of robust dissipativity of linear systems with parameters affected by box uncertainty; our major goal is to evaluate the largest uncertainty level for which all perturbed instances share a common dissipativity certificate. While it is NP-hard to compute this quantity exactly, we demonstrate that under favorable circumstances one can build an $O(1)$-tight lower bound of this "intractable" quantity by solving an explicit semidefinite program of the size polynomial in the size of the system. We consider a number of applications, including the robust versions of the problems of extracting nearly optimal available storage, providing nearly optimal required supply, Lyapunov stability analysis, and linear-quadratic control.

Key words. interval matrices, dissipative linear systems, interval uncertain LMIs, positive-real systems, contractive systems, Riccati equation, semidefinite programming

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1. Introduction and motivation.

An important requirement of any modern control system is its robustness. In many system theory and control applications, the concept of robustness is related to the stability of the closed-loop system and its performance measured with respect to a certain objective function.

In this paper, we focus on robustness with respect to unknown-but-bounded (and possibly time-varying) perturbations of the entries in the matrix $\Sigma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of a continuous-time linear dynamical system

$$\begin{aligned} \dot{z} &= Az + Bu, \\ y &= Cz + Du. \end{aligned}$$

For the time being, we assume the simplest *interval* model of perturbations—every entry Σ_{ij} in Σ, independently of all other entries, can vary in the interval $\Sigma_{ij} \pm \rho d\Sigma_{ij}$, where Σ_{ij} are the nominal data, $d\Sigma_{ij}$ are given scale factors, and ρ is the uncertainty level. The set of matrices just defined will be referred to as *interval matrix* and will be denoted by $\mathcal{U}_\rho$.

The question we are addressing is as follows:

(?) *What is the supremum $\rho^\star$ of those uncertainty levels ρ under which all perturbations of level ρ preserve a particular property of the system, such as stability, passivity, contractiveness, etc.?*

Typically, it is computationally intractable to give a *precise* answer to such a question. For example, it is known to be NP-hard to check the stability of all instances of an interval matrix $\mathcal{U}_\rho$ [7]. In other words, we do not know how to check efficiently whether every one of the Lyapunov linear matrix inequalities (LMIs)

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†Delft University of Technology, Faculty of Information Technology and Systems, Mekelweg 4, 2628 CD Delft, The Netherlands (f.d.barb@its.tudelft.nl).
‡William Davidson Faculty of Industrial Engineering and Management, Technion—Israel Institute of Technology, Technion City, Haifa 32000, Israel (abental@ie.technion.ac.il, nemirovs@ie.technion.ac.il).

$$\text{find } X \text{ such that } X \succ 0 \text{ and } A^T X + XA \prec 0$$[1]

corresponding to the instances A of an interval square matrix is solvable.

The situation does not improve at all when we pass from the question "whether all instances of an interval matrix are stable" to the seemingly simpler question "whether all instances of an interval matrix admit a common quadratic Lyapunov stability certificate," or, which is the same, whether the aforementioned LMIs have a *common* solution. Although in the new form the question is to check the solvability of a *finite* system of LMIs

$$X \succ 0, \quad A^T X + XA \prec 0 \quad \forall (A \in \mathcal{V}),$$

where $\mathcal{V}$ is the (finite!) set of the extreme points of the original interval matrix, the number of LMIs in this system blows up exponentially with the size of the matrix (unless the number of uncertain entries in the matrix remains once and for ever fixed). It turns out that in general it is already NP-hard to check whether a given candidate solution X is feasible for the above system of LMIs.[2]

The difficulty arising when checking stability of (all instances of) an interval matrix is typical for other problems of the aforementioned type: the property of interest is equivalent to the solvability of certain LMI $\mathcal{L}_\Sigma(X) \succ 0$ with the data coming from the matrix Σ of the system in question. When Σ is subject to interval uncertainty, both of the following tasks become NP-hard:

(1.A) Checking whether every one of the LMIs

$$\mathcal{L}_\Sigma(X) \succ 0 \tag{1}$$

with $\Sigma \in \mathcal{U}_\rho$ is solvable (i.e., to verify that the desired property is possessed by all instances), and

(1.B) Checking whether the infinite system of LMIs

$$\mathcal{L}_\Sigma(X) \succ 0 \qquad \forall (\Sigma \in \mathcal{U}_\rho)$$

is solvable, i.e., whether all instances of our interval matrix share a common certificate for the property of interest (which normally is a *sufficient* condition for the property to be preserved also by dynamic perturbations).

Now, in light of the fact that it is NP-hard to answer questions (1.A), (1.B) *exactly*, a natural course of action is to *relax* the questions in order to make them tractable. We are not aware of any good relaxation of question (1.A). In contrast to this, recent progress in what is called robust semidefinite programming [1, 5, 6] (specifically, the matrix cube theorem [3]) leads to "tight" tractable relaxations of question (1.B). It turns out that *under favorable circumstances* (which do take place for a wide family of "properties of interest") *one can build efficiently a lower bound* $\widehat{\rho}$ *on the supremum* $\rho^\star$ *of those uncertainty levels* ρ *for which the answer to the question* (1.B) *is affirmative, and this lower bound is tight within an absolute constant factor* (the latter is in most of the cases $\frac{\pi}{2} = 1.57\ldots$). The goal of this paper is to justify the above claim.

[1]We write $A \succeq B$ ($A \succ B$) to express that A, B are symmetric matrices of the same size such that $A - B$ is positive semidefinite (respectively, positive definite).

[2]This "analysis" problem is not simpler than checking whether all instances of a given interval symmetric matrix are positive semidefinite; it is shown in [7] that the latter problem is NP-hard already in the case when all entries in the interval matrix, except for those from the first two rows and columns, are fixed.

A convenient general framework for our study is the dissipativity-based approach, as developed in the seminal papers of Willems [9, 10]. The notion of dissipativity is one of the most important concepts in systems and control theory, both from the theoretical point of view as well as from the practical perspective. In many mechanical and electrical engineering applications, dissipativity is related to the notion of energy. Here, a dissipative system is characterized by the following property: at any moment of time, the amount of energy which the system can supply to its environment cannot exceed the amount of energy that has been supplied to it. However, the dissipativity-based framework is not restricted to the energy-related issues; it allows us to investigate stability analysis and linear-quadratic control as well.

The rest of the paper is organized as follows. In section 2, we review basic notions and results from dissipativity theory. In section 3, we present the box model of uncertainty (which is slightly more general than the simple interval model) and pose and motivate three basic dissipativity-related versions of question (1.B): finding a common dissipativity certificate for all instances of a given uncertain system (a particular case of this problem is the Lyapunov stability analysis under box uncertainty); extracting available storage/providing required supply in the face of uncertainty (this covers, in particular, the optimal linear-quadratic control of uncertain systems). In the central section 4, we develop "tractable tight relaxations" of the problems posed in section 3. Finally, in section 5, we present several illustrating numerical examples.

On many occasions in this paper we use the term "efficient computability" of various quantities. An appropriate definition of this notion does exist,[3] but for our purposes here it suffices to agree that all "LMI-representable" quantities—those which can be represented as optimal values in semidefinite programs

$$\min_x \left\{ c^T x : A_0 + \sum_{i=1}^N x_i A_i \succeq 0 \right\}$$

or generalized eigenvalue problems

$$\min_{x,\omega} \left\{ \omega : \begin{array}{c} A(x) \equiv A_0 + \sum\limits_{i=1}^N x_i A_i \succeq 0 \\ B(x) \equiv B_0 + \sum\limits_{i=1}^N x_i B_i \preceq \omega A(x) \\ C(x) \equiv C_0 + \sum\limits_{i=1}^N x_i C_i \succeq 0 \end{array} \right\}$$

—are efficiently computable functions of the data $c, \{A_i \in \mathbf{S}^n\}_{i=0}^N$, respectively, $\{A_i, B_i, C_i \in \mathbf{S}^n\}_{i=0}^N$; where $\mathbf{S}^n$ is the space of real symmetric $n \times n$ matrices. From now on, missing blocks in block matrices are assumed to be zero.

2. Dissipative systems. In this section, we shall briefly review the dissipativity theory for linear systems with quadratic storage and supply functions as developed in [10]. The readers less familiar with the topic are referred to [8] for details.

Consider a continuous-time linear time-invariant dynamical system given by

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), \qquad z(0) = \zeta, \\ y(t) &= Cz(t) + Du(t), \end{aligned} \tag{2}$$

[3] For a definition which fits best of all the contents of the paper, see [2, Chapter 5].

where $\Sigma \equiv \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathbf{R}^{(n+p)\times(n+m)}$ is the matrix of system coefficients, $u(\cdot) \in \mathbf{R}^m$ is the *input* (which henceforth is assumed to be locally square integrable), $z(\cdot) \in \mathbf{R}^n$ is the *state*, and $y(\cdot) \in \mathbf{R}^p$ is the *output*. In what follows we refer to system (2) given by a matrix Σ as "system Σ."

Let us fix a quadratic *supply function*

$$\mathfrak{S} : \mathbf{R}^{p+m} \to \mathbf{R}, \qquad \mathfrak{S}(y,u) = \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q & L \\ L^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}; \tag{3}$$

here

$$\mathfrak{P} = \begin{bmatrix} Q & L \\ L^T & R \end{bmatrix}$$

is a symmetric *supply matrix* (Q is $p\times p$, R is $m\times m$). Given a trajectory $(z(\cdot), y(\cdot), u(\cdot))$ of (2) and two time instants $t_0 \le t_1$, we interpret the corresponding *supply*

$$\int_{t_0}^{t_1} \mathfrak{S}(y(t), u(t))dt$$

as the work carried on the system in the time interval $[t_0, t_1]$ along the trajectory in question, if the supply is nonnegative, and as minus the energy extracted from the system, if the supply is negative.

Note that along a trajectory of (2) the supply can be expressed in terms of the state and the input:

$$\begin{aligned} \mathcal{S}_\Sigma(z,u) &\equiv \mathfrak{S}(Cz + Du, u) \\ &= \begin{bmatrix} z \\ u \end{bmatrix}^T \underbrace{\begin{bmatrix} C^TQC & C^T(L+QD) \\ (L+QD)^TC & D^TQD + L^TD + D^TL + R \end{bmatrix}}_{\mathbf{S}_\Sigma} \begin{bmatrix} z \\ u \end{bmatrix}. \end{aligned} \tag{4}$$

DEFINITION 2.1. *System Σ is called dissipative with respect to supply $\mathfrak{S}$, if there exists a nonnegative storage function $V(z)$, $V(0) = 0$, such that*

$$V(z(0)) + \int_0^T \mathfrak{S}(y(t), u(t))dt \ge V(z(T)) \tag{5}$$

for all $T \ge 0$ and all trajectories $(z(\cdot), y(\cdot), u(\cdot))$ of the system.

The standard interpretation of a storage function is that $V(z)$ is the internal energy stored by system in state z; with this interpretation, (5) means that the work W on the system needed to move it from one state to another is at least the resulting change ΔV in the internal energy stored by the system; the excess $W - \Delta V \ge 0$ is thought of to be dissipated by the system.

The summary of facts on dissipativity we need in what follows is as follows. *Assume that system Σ is controllable*, and let $\mathfrak{S}$ be a quadratic supply:

D.1. $(\Sigma, \mathfrak{S})$ is dissipative if and only if $(\Sigma, \mathfrak{S})$ admits a quadratic storage function $V(z) = z^TZz$, where $Z \in \mathbf{S}^n_+$ (from now on, $\mathbf{S}^n_+$ is the cone of positive semidefinite matrices from $\mathbf{S}^n$).

D.2. A quadratic function $V(z) = z^TZz$ is a storage function for $(\Sigma, \mathfrak{S})$ if and only if $Z \in \mathbf{S}^n_+$ and

$$\mathfrak{S}(y(t), u(t)) - \frac{d}{dt}(z^T(t)Zz(t)) \ge 0$$

for all trajectories $(z(\cdot), y(\cdot), u(\cdot))$, or, which is the same, if and only if $Z \in \mathbf{S}^n$ solves the system of matrix inequalities (MIs)

$$Z \succeq 0, \tag{6a}$$

$$\mathbb{D}_\Sigma[Z] \equiv \mathbf{S}_\Sigma - \begin{bmatrix} A^T Z + ZA & ZB \\ B^T Z & \end{bmatrix} \succeq 0 \tag{6b}$$

(for notation, see (4)). Note that MI (6b) expresses a very transparent requirement that

$$\mathcal{S}_\Sigma(z(t), u(t)) \geq \frac{d}{dt}(z^T(t) Z z(t)) \tag{7}$$

for all t and all trajectories $(z(t), y(t), u(t))$ of Σ. In what follows, we call the solutions of (6) the *dissipativity certificates* for $(\Sigma, \mathfrak{S})$.

D.3. If $(\Sigma, \mathfrak{S})$ is dissipative, then we have the following:

(a) among the associated storage functions there exist the (pointwise) minimal one,

$$V_{av}(z) = \sup_{(z(\cdot),y(\cdot),u(\cdot))} \left\{ -\int_0^{t_1} \mathfrak{S}(y(t), u(t))dt : \begin{array}{l} (z(\cdot), y(\cdot), u(\cdot)) \text{ is a trajectory,} \\ z(0) = z \end{array} \right\}$$

("available storage"), and the (pointwise) maximal one,

$$V_{req}(z) = \inf_{(z(\cdot),y(\cdot),u(\cdot))} \left\{ \int_0^{t_1} \mathfrak{S}(y(t), u(t))dt : \begin{array}{l} (z(\cdot), y(\cdot), u(\cdot)) \text{ is a trajectory,} \\ z(0) = 0, z(t_1) = z \end{array} \right\}$$

("required supply"). Every storage function $V(\cdot)$ for $(\Sigma, \mathfrak{S})$ satisfies the relations $V_{av}(z) \leq V(z) \leq V_{req}(z)$ for all z, and every convex combination of $V_{av}(\cdot)$ and $V_{req}(\cdot)$ is a storage function for $(\Sigma, \mathfrak{S})$.

(b) Both the available storage and the required supply are quadratic functions of the state:

$$\begin{array}{rcl} V_{av}(z) & = & z^T Z_{av} z, \\ V_{req}(z) & = & z^T Z_{req} z, \end{array}$$

where the positive semidefinite matrices Z_{av}, Z_{req} are, respectively, the $\succeq$-minimal and the $\succeq$-maximal solutions of (6). The set of solutions to (6) is exactly the "matrix interval" $\{Z : Z_{av} \preceq Z \preceq Z_{req}\}$.

D.4. Assume that $(\Sigma, \mathfrak{S})$ is dissipative and that the matrix $D^T QD + L^T D + D^T L + R$ is positive definite. Then the state feedback

$$u = F_{av} z, \quad F_{av} = -(D^T QD + L^T D + D^T L + R)^{-1}(B^T Z_{av} - (L + QD)^T C)$$

stabilizes the system (i.e., the real parts of all eigenvalues of the matrix $A + BF_{av}$ of the closed-loop system are negative), and with this feedback, the energy extracted from the system, the initial state of the system being $\zeta \in \mathbb{R}^n$, is exactly the available storage $V_{av}(\zeta)$:

$$-\int_0^\infty \mathfrak{S}(y(t), u(t))dt = \zeta^T Z_{av} \zeta,$$

where $(y(t), u(t))$ are given by

$$\begin{aligned}
&\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = \zeta,\\
&u(t) = F_{av} z(t),\\
&y(t) = Cz(t) + Du(t).
\end{aligned}$$

Similarly, the state feedback

$$u = F_{req} z, \quad F_{req} = -(D^T QD + L^T D + D^T L + R)^{-1}(B^T Z_{req} - (L+QD)^T C)$$

stabilizes the "backward time" system (i.e., the real parts of all eigenvalues of the matrix $-(A + BF_{req})$ of the closed-loop system with backward time are negative), and with this feedback the supply required to move the system from the origin to a state ζ is exactly the required supply $V_{req}(\zeta)$:

$$\int_0^\infty \mathfrak{S}(y(t), u(t))dt = \zeta^T Z_{req} \zeta,$$

where $(y(t), u(t))$ are given by

$$\begin{aligned}
&\dot{z}(t) = -[Az(t) + Bu(t)], \qquad z(0) = \zeta,\\
&u(t) = F_{req} z(t),\\
&y(t) = Cz(t) + Du(t).
\end{aligned}$$

Let us list several important examples of supply functions.

EXAMPLE 1 (positive-real systems). *Here $m = p$, and the supply matrix is* $\mathfrak{P} = \left[\begin{smallmatrix} & I \\ I & \end{smallmatrix}\right]$, i.e.,

$$\mathfrak{S}(y, u) = 2y^T u.$$

Assuming that A is stable and (A, B, C) is minimal, the pair $(\Sigma, \mathfrak{S})$ is dissipative if and only if (2) *is passive, i.e., $\int_0^T y^T(t)v(t)dt \geq 0$ for all $T \geq 0$ and all trajectories $(z(t), y(t), v(t))$ with $z(0) = 0$. Under the same assumptions on A, B, C, the frequency domain characterization of passivity is that the transfer function*

$$H(s) = C(sI - A)^{-1}B + D$$

of the system is such that

$$\Re(s) \geq 0 \Rightarrow H(s) + H^*(s) \succeq 0,$$

where $H^*(s)$ is the Hermitian conjugate of $H(s)$ and $\Re(s)$ is the real part of $s \in \mathbb{C}$.

EXAMPLE 2 (nonexpansive systems [4]). *Here the supply matrix is* $\mathfrak{P} = \left[\begin{smallmatrix} I_p & \\ & I_m \end{smallmatrix}\right]$ *(I_k is the $k \times k$ unit matrix), i.e.,*

$$\mathfrak{S}(y, u) = u^T u - y^T y.$$

Assuming again that A is stable and (A, B, C) is minimal, dissipativity of $(\Sigma, \mathfrak{S})$ is equivalent to the fact that

$$\int_0^T y^T(t)y(t)dt \leq \int_0^T u^T(t)u(t)dt$$

for all $T \geq 0$ and trajectories $(z(t), y(t), u(t))$ of (2) *with $z(0) = 0$. Under the same assumption on A, B, C, the frequency domain characterization of nonexpansivity is that the transfer function $H(s)$ of the system is such that*

$$\Re(s) \geq 0 \Rightarrow H^*(s)H(s) \preceq I.$$

EXAMPLE 3 (linear-quadratic control [4]). *Here the supply matrix $\mathfrak{P}$ is positive semidefinite. Assuming that (A, B) is controllable, the pair $(\Sigma, \mathfrak{S})$ is always dissipative, with the available storage $V_{av}(z) \equiv 0$. The required supply $V_{req}(z)$ is the optimal value in the problem of optimal control where the goal is to minimize $\int_0^T \mathfrak{S}(y(t), u(t))dt$ when moving the system from the origin at time 0 to the state z at time T (to be chosen).*

3. Dissipativity under uncertainty. Now assume that the linear dynamic system in question is *uncertain*, so that all we know about the matrix Σ is that Σ belongs to a given *uncertainty set* $\mathcal{U}_\rho$ in the space of $(n+p) \times (m+p)$ real matrices. In this paper we focus on the case of *box uncertainty*:

$$\mathcal{U}_\rho = \left\{ \Sigma = \Sigma + \sum_{\ell=1}^{L} u_\ell d\Sigma_\ell : -\rho \leq u_\ell \leq \rho,\, \ell = 1, \ldots, L \right\}, \tag{8}$$

where

- $\Sigma = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ is the *nominal* system;
- $d\Sigma_\ell = \left[\begin{smallmatrix} dA_\ell & dB_\ell \\ dC_\ell & dD_\ell \end{smallmatrix}\right]$, $\ell = 1, \ldots, L$, are *basic perturbation matrices*;
- $\rho > 0$ is the *uncertainty level*.

In what follows, we refer to matrices $\Sigma \in \mathcal{U}_\rho$ as to *instances* of the uncertain system associated with the uncertainty set $\mathcal{U}_\rho$.

Let us fix a quadratic supply function (3); in what follows, when speaking about the dissipativity of a certain system, we mean the dissipativity with respect to this supply function. We assume from now on that the nominal pair $(\mathbf{A}, \mathbf{B})$ is controllable, and the nominal system Σ is dissipative, with the minimal and maximal dissipativity certificates $\mathbf{Z}_{av}$, $\mathbf{Z}_{req}$, respectively.

We intend to focus on three dissipativity-related problems for uncertain systems, specifically, the following problems:

1. Common dissipativity certificate. Find a common dissipativity certificate for all instances of the uncertain system.
2. Extracting available storage. Given $\epsilon \in (0, 1)$, find a feedback which stabilizes all instances of the uncertain system and allows us to extract from the initial state ζ of any instance energy at least $(1-\epsilon)\zeta^T \mathbf{Z}_{av} \zeta$.
3. Providing required supply. Given $\delta > 0$, find a feedback which stabilizes in backward time all instances of the uncertain system and allows to move every instance from the origin to a given state ζ with total supply at most $(1+\delta)\zeta^T \mathbf{Z}_{req} \zeta$.

Our next goal is to motivate and to model the outlined problems.

3.1. Common dissipativity certificate. The problem of finding a common dissipativity certificate for all instances of an uncertain system is as follows.

PROBLEM 1. *Given a supply $\mathfrak{S}$, a convex set $\mathcal{I}$ in the cone $\mathbf{S}^n_+$, and the data specifying $\mathcal{U}_\rho$, find the supremum of those $\rho \geq 0$ for which all instances from $\mathcal{U}_\rho$ admit a common dissipativity certificate in $\mathcal{I}$, or, which is the same in view of* D.2, *find the* *supremum of those* $\rho \geq 0$ *for which the system of constraints*

$$Z \in \mathcal{I}, \tag{9a}$$

$$\mathbb{D}_\Sigma[Z] \equiv \begin{bmatrix} C^T QC - A^T Z - ZA & C^T(L+QD) - ZB \\ (L+QD)^T C - B^T Z & D^T QD + L^T D + D^T L + R \end{bmatrix} \succeq 0 \tag{9b}$$

$$\forall \Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{U}_\rho$$

(see (6)*) in matrix variable* Z *is solvable.*

The motivation behind Problem 1 is quite transparent: there are cases when the dissipativity is a highly desirable property, and in these cases it is worthy of knowing what are the largest perturbations which for sure preserve this property. With this motivation, however, it remains unclear why we should be interested in a *common* dissipativity certificate for all $\Sigma \in \mathcal{U}_\rho$ rather than to ask what is the largest ρ for which every instance from $\mathcal{U}_\rho$ admits a dissipativity certificate (perhaps depending on the instance). The motivation behind seeking a common dissipativity certificate comes from the fact that such a certificate ensures dissipativity of the uncertain *time-varying* system

$$\begin{aligned} \dot{z}(t) &= A(t)z(t) + B(t)u(t), \\ y(t) &= C(t)z(t) + D(t)u(t), \end{aligned} \tag{10}$$

where the dependence of $\Sigma(t) \equiv \left[\begin{smallmatrix} A(t) & B(t) \\ C(t) & D(t) \end{smallmatrix}\right]$ on t is not known in advance; all we know is that $\Sigma(t)$ is a measurable function of t taking values in $\mathcal{U}_\rho$. The precise meaning of the claim "existence of a common dissipativity certificate for all instances $\Sigma \in \mathcal{U}_\rho$ implies dissipativity of the uncertain time-varying system (10)" is given by the following simple statement.

PROPOSITION 3.1. *Let* Z *be a common dissipativity certificate for all instances* $\Sigma \in \mathcal{U}_\rho$*, i.e., let* $Z \succeq 0$ *satisfy* (9b)*. Then for every* $T \geq 0$ *and every trajectory* $(z(t), y(t), u(t))$ *of the time-varying system* (10) *with* $\Sigma(t) \in \mathcal{U}_\rho$ *for all* t*, one has*

$$z^T(0)Zz(0) + \int_0^T \mathfrak{S}(y(t), u(t))dt \geq z^T(T)Zz(T). \tag{11}$$

Proof. It is immediately seen that (9b) implies that

$$\mathfrak{S}(y(t), u(t)) \geq \frac{d}{dt}(z^T(t)Zz(t))$$

for all t. Integrating this inequality, we arrive at (11). □

EXAMPLE 4 (Lyapunov stability analysis under box uncertainty). *Assume that we have designed a controller for a linear dynamical system, and let*

$$\dot{z} = Az$$

be the description of the closed-loop system (so that some components of z *represent states of the plant, while the remaining components of* z *represent states of the controller). After the design is completed, a natural question is how the performance of the system can be affected by perturbations in* A *(i.e., in the parameters of the plant and of the controller). Assuming a box model of perturbations*

$$A \in \mathcal{V}_\rho = \left\{ A = \mathbf{A} + \sum_{\ell=1}^{L} u_\ell dA_\ell : -\rho \leq u_\ell \leq \rho,\, \ell = 1, \ldots, L \right\},$$

an important component of the above question is, What is the supremum $\rho^\star$ of those uncertainty levels ρ for which all instances $A \in \mathcal{V}_\rho$ remain stable, moreover, such that

$$z^T(t)\mathbf{Z}z(t) \le \beta \exp\{-\alpha t\} z^T(0)\mathbf{Z}z(0) \qquad \forall t \ge 0 \tag{12}$$

for all trajectories $z(\cdot)$ of all perturbed instances? Here $\mathbf{Z} \succ 0$, $\beta > 1$, and $\alpha > 0$ are given in advance. A well-known sufficient condition for (12) *is the existence of an appropriate quadratic Lyapunov stability certificate, namely, a matrix Z satisfying the relations*

$$\beta^{-1}\mathbf{Z} \preceq Z \preceq \mathbf{Z}, \tag{13a}$$
$$A^T Z + ZA \preceq -\alpha\mathbf{Z} \qquad \forall A \in \mathcal{V}_\rho. \tag{13b}$$

Indeed, if Z satisfies (13) *and $z(t)$ is a trajectory of the time-varying system*

$$\dot{z}(t) = A(t)z(t) \qquad (A(t) \in \mathcal{V}_\rho \qquad \forall t),$$

then

$$\begin{aligned}\tfrac{d}{dt}(z^T(t)Zz(t)) &= z^T(t)[A^T(t)Z + ZA(t)]z(t)\\ &\le -\alpha z^T(t)\mathbf{Z}z(t) && \text{(cf. (13b))}\\ &\le -\alpha z^T(t)Zz(t) && \text{(cf. (13a))};\end{aligned}$$

hence

$$\begin{aligned}z^T(t)Zz(t) &\le \exp\{-\alpha t\}z^T(0)Zz(0)\\ &\le \exp\{-\alpha t\}z^T(0)\mathbf{Z}z(0) \quad \text{(cf. (13a))}\end{aligned}$$

and therefore

$$\begin{aligned}z^T(t)\mathbf{Z}z(t) &\le \beta z^T(t)Zz(t) && \text{(cf. (13a))}\\ &\le \beta\exp\{-\alpha t\}z^T(0)\mathbf{Z}z(0).\end{aligned}$$

On the other hand, it is immediately seen that relations (13) *say exactly that Z is a common dissipativity certificate, belonging to the matrix interval $\mathcal{I} = \{Z : \beta^{-1}\mathbf{Z} \preceq Z \preceq \mathbf{Z}\}$ for all instances $\Sigma \in \mathcal{U}_\rho$ of the system*

$$\begin{aligned}\dot{z} &= Az + 0_{n\times 1}\cdot u,\\ y &= z + 0_{n\times 1}\cdot u\end{aligned} \tag{14}$$

when the supply matrix is specified as

$$\mathfrak{P} = \begin{bmatrix} -\alpha\mathbf{Z} & \\ & I \end{bmatrix}; \tag{15}$$

here $\mathcal{U}_\rho$ is the box uncertainty given by

$$d\Sigma_\ell = \begin{bmatrix} dA_\ell & 0_{n\times 1}\\ 0_{n\times n} & 0_{n\times 1}\end{bmatrix}, \qquad \ell = 1, \dots, L.$$

We see that Problem 1 *can be used to find the largest uncertainty level ρ for which the validity of* (13) *can be guaranteed by a quadratic Lyapunov stability certificate.*

3.2. Extracting available storage. Assume that we are interested in retrieving the energy stored in the initial state ζ. If there were no perturbations, the maximal amount of energy we could retrieve would be the nominal available storage $\zeta^T \mathbf{Z}_{av} \zeta$, and the corresponding control could be chosen in the state feedback form (see D.4). With perturbations, we hardly could guarantee the same amount of retrieved energy; however, it is reasonable to look for a state feedback which stabilizes all instances of the uncertain system in question and allows us to retrieve, whatever is an instance and an initial state ζ, at least a given fraction $(1-\epsilon)\zeta^T \mathbf{Z}_{av} \zeta$ of the nominal available storage. To model this target mathematically, we start with the following simple observation.

PROPOSITION 3.2. *Assume that $0 \prec \mathbf{Z}_{av}$, and let $\epsilon \in [0,1)$, $\rho \geq 0$ be given. Assume that matrices $G, H \in \mathbf{S}^n_+$ and a state feedback $u = Fz$ are such that*

1.

$$\begin{bmatrix} C^T Q C & C^T(L+QD) \\ (L+QD)^T C & D^T Q D + L^T D + D^T L + R \end{bmatrix} - \begin{bmatrix} A^T G + GA & GB \\ B^T G & \end{bmatrix} \succeq 0 \quad \forall \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{U}_\rho, \tag{16}$$

i.e., G is a common dissipativity certificate for all instances of $\mathcal{U}_\rho$;

2.

$$\begin{bmatrix} I & F^T \end{bmatrix} \begin{bmatrix} C^T Q C & C^T(L+QD) \\ (L+QD)^T C & D^T Q D + L^T D + D^T L + R \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} \prec \left[(A+BF)^T H + H(A+BF)\right] \quad \forall \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{U}_\rho; \tag{17}$$

3.

$$(1-\epsilon)\mathbf{Z}_{av} \preceq H \prec G. \tag{18}$$

Then all instances of the uncertain time-varying closed-loop system

$$\begin{array}{rcl} \dot{z}(t) & = & A(t)z(t) + B(t)u(t), \\ y(t) & = & C(t)z(t) + D(t)u(t), \\ u(t) & = & Fz(t), \end{array} \qquad \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \in \mathcal{U}_\rho \qquad \forall t \tag{19}$$

share a common quadratic Lyapunov function $z^T(G-H)z$. Moreover, for every initial state $\zeta = z(0)$ of (19)*, one has*

$$-\int_0^\infty \mathfrak{S}(y(t), u(t))dt \geq \zeta^T H \zeta \geq (1-\epsilon)\zeta^T \mathbf{Z}_{av} \zeta, \tag{20}$$

i.e., the state feedback F allows us to extract at least $(1-\epsilon)$ times the nominal available storage $\zeta^T \mathbf{Z}_{av} \zeta$.

Proof. Consider a time-invariant instance $\Sigma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of (19), and let $(z(t), y(t), u(t))$ be a trajectory of this instance. By (16), the quadratic function $V(z) = z^T G z$ is a storage function for $(\Sigma, \mathfrak{S})$; hence for every $t_0 \leq t_1$

$$z^T(t_0) G z(t_0) + \int_{t_0}^{t_1} \mathfrak{S}(y(t), u(t))dt \geq z^T(t_1) G z(t_1). \tag{21}$$

On the other hand, (17) implies that

$$\mathfrak{S}(y(t),u(t)) \leq \frac{d}{dt}\left(z^T(t)Hz(t)\right) - \theta z^T(t)z(t)$$

for certain $\theta > 0$; hence

$$\int_{t_0}^{t_1} \mathfrak{S}(y(t),u(t))dt \leq \left[z^T(t_1)^T Hz(t_1) - z^T(t_0)^T Hz(t_0)\right] - \theta \int_{t_0}^{t_1} z^T(t)z(t)dt.$$

Substituting this inequality into (21), we see that for every trajectory of every time-invariant instance of (19) and every pair $t_0 \leq t_1$ of time instants one has

$$z^T(t_0)[G-H]z(t_0) - \theta \int_{t_0}^{t_1} z^T(t)z(t)dt \geq z^T(t_1)[G-H]z(t_1);$$

hence $\frac{d}{dt}(z^T(t)[G-H]z(t)) \leq -\theta z^T(t)z(t)$ for all $t \geq 0$ and all trajectories, so that

$$(A+BF)^T[G-H] + [G-H](A+BF) \preceq -\theta I.$$

Since this relation is valid for all $\Sigma \in \mathcal{U}_\rho$, and since $G-H \succ 0$ by (18), $G-H$ is indeed a quadratic Lyapunov stability certificate for (19).

Now consider a trajectory $(z(t),y(t),u(t))$ of (19). Same as above, we have

$$\mathfrak{S}(y(t),u(t)) \leq \frac{d}{dt}\left(z^T(t)Hz(t)\right).$$

Integrating both sides of this inequality from 0 to ∞ and taking into account that $(z(t),y(t),u(t))$ converges exponentially fast to 0 as $t \to \infty$ (we have seen that (19) admits quadratic Lyapunov stability certificate!), we get

$$\int_0^\infty \mathfrak{S}(y(t),u(t))dt \leq -z^T(0)Hz(0),$$

as required in the first inequality in (20); the second inequality in (20) is readily given by (18). □

In view of Proposition 3.2, we could pose the problem of extracting available storage as the problem of finding the supremum of those uncertainty levels ρ for which the semi-infinite system of MIs (16), (17), (18) in matrix variables G, H, F is solvable. This problem, however, is too difficult; it is completely unclear how to check efficiently the solvability of this *nonlinear* in F, H MI even in the nominal case $\rho = 0$. This is why we are forced to simplify our task by assuming that either F or H are given in advance. With this simplification, we arrive at the following pair of problems.

PROBLEM 2A. *Given a supply* $\mathfrak{S}$, *a feedback matrix* F, *parameter* $\epsilon \in (0,1)$, *and the data specifying* $\mathcal{U}_\rho$, *find the supremum of those* $\rho \geq 0$ *for which the system of MIs* (16)–(18) *in matrix variables* G, H *is solvable.*

With F specified as the ideal nominal feedback F_{av}, see D.4, Problem 2A becomes a quite natural question of finding the largest uncertainty level for which we can certify the fact that whatever is an initial state ζ of an instance of the uncertain system, the nominal feedback allows us to extract at least the fraction $(1-\epsilon)$ of the corresponding nominal available storage $\zeta^T \mathbf{Z}_{av}\zeta$.

PROBLEM 2B. *Given a supply* $\mathfrak{S}$, *parameter* $\epsilon \in (0,1)$, *an* $n \times n$ *positive definite matrix* $H \succeq (1-\epsilon)\mathbf{Z}_{av}$, *and the data specifying* $\mathcal{U}_\rho$, *find the supremum of those* $\rho \geq 0$ *for which the system of MIs* (16), (18) *in matrix variables* G *and* F *is solvable.*

A simple choice for the matrix H in Problem 2B is the solution of Problem 2A.

3.3. Providing required supply. The motivation behind this problem is completely similar to the one for the extracting storage problem; the only difference is that now we want to drive the system from the origin to a given state ζ and we are interested in achieving this target with the total supply not exceeding $(1+\delta)$ times the nominal required supply $\zeta^T \mathbf{Z}_{req}\zeta$. We have the following analogy of Proposition 3.2.

PROPOSITION 3.3. *Assume that $0 \prec \mathbf{Z}_{req}$, and let $\delta \in [0, 1)$, $\rho \geq 0$ be given. Assume that matrices $G, H \in \mathbf{S}^n_+$ and a state feedback $u = Fz$ are such that the conditions* (16), (17) *and the condition*

$$G \prec H \preceq (1+\delta)\mathbf{Z}_{req} \tag{22}$$

are satisfied.

Then all instances of the uncertain time-varying closed-loop system

$$\begin{array}{l} \dot{z}(t) = -[A(t)z(t) + B(t)u(t)], \\ y(t) = C(t)z(t) + D(t)u(t), \\ u(t) = Fz(t), \end{array} \qquad \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \in \mathcal{U}_\rho \qquad \forall t \tag{23}$$

(which is the backward time version of system (19)*) share a common quadratic Lyapunov function $z^T[H - G]z$. Moreover, for every initial state $\zeta = z(0)$ of* (23)*, one has*

$$\int_0^\infty \mathfrak{S}(y(t), u(t))dt \leq (1+\delta)\zeta^T \mathbf{Z}_{req}\zeta, \tag{24}$$

i.e., the state feedback F allows us to move system (19) *from the origin to a given state ζ with total supply at most $(1+\delta)$ times the nominal required supply $\zeta^T \mathbf{Z}_{req}\zeta$.*

The proof is similar to the one of Proposition 3.2.

In view of Proposition 3.3, a natural way to model the providing required supply problem would be to look for the largest ρ for which the semi-infinite system (16), (17), (22) in matrix variables F, G, H is solvable; however, "tractability reasons" similar to those in section 3.2 force us to simplify the setting and restrict ourselves to the following pair of problems.

PROBLEM 3A. *Given a supply $\mathfrak{S}$, a feedback matrix F, parameter $\delta > 0$, and the data specifying $\mathcal{U}_\rho$, find the supremum of those $\rho \geq 0$ for which the system of MIs* (16), (17), (22) *in matrix variables G, H is solvable.*

PROBLEM 3B. *Given a supply $\mathfrak{S}$, parameter $\delta > 0$, an $n \times n$ positive definite matrix $H \preceq (1+\delta)\mathbf{Z}_{req}$, and the data specifying $\mathcal{U}_\rho$, find the supremum of those $\rho \geq 0$ for which the system of MIs* (16), (17) *in matrix variables G, $0 \preceq G \prec H$, and F is solvable.*

In contrast to the situation of section 3.2, now there exists a particular "tractable case" where one can treat in the system of interest (which is now the system (16), (17), (22)) both F and H as design variables; this is the case of positive semidefinite supply matrix $\left[\begin{smallmatrix} Q & L \\ L^T & R \end{smallmatrix}\right]$ (as it happens in linear-quadratic control).[4] In this case it makes sense to specify the common dissipativity certificate G of the perturbed instances as the zero matrix; this choice ensures the validity of (16) and is "ideal" from the viewpoint of the constraint (22). Setting $G = 0$ and treating F, H as the design variables in the

[4]Note that this case makes no sense in the extracting storage problem, since there it would imply that "there is nothing to extract" – $\mathbf{Z}_{av} = 0$

system (16), (17), (22), we arrive at the following version of the problem of providing required supply.

PROBLEM 3C. *Given a supply $\mathfrak{S}$ such that the supply matrix $\mathfrak{P}$ is positive semidefinite, parameter $\delta > 0$, and the data specifying $\mathcal{U}_\rho$, find the supremum of those $\rho \geq 0$ for which the system comprised of semi-infinite MI* (17) *and the LMI*

$$0 \prec H \preceq (1+\delta)\mathbf{Z}_{req} \tag{25}$$

in matrix variables F, H is solvable.

4. Processing the problems. Every one of Problems 1, 2A, 2B, 3A, 3B, and 3C asks for finding the largest ρ such that a given system of MIs (depending on ρ as on a parameter) is solvable. The systems in question are *semi-infinite*—they involve infinitely many MIs with the data running through the uncertainty sets. It is well known that semi-infinite systems of MIs are, in general, NP-hard; it is easy to show that in general this is the case with the specific semi-infinite systems arising in Problems 1, 2A, 2B, 3A, 3B, and 3C. What we intend to do is to replace these NP-hard systems with their computationally tractable *conservative approximations*, the latter notion being defined as follows.

DEFINITION 4.1. *Let $\mathcal{S}$ be a system of constraints on a design vector x. We say that a system $\mathcal{A}$ of constraints on x and a vector of additional variables y is a conservative approximation of $\mathcal{S}$ if the x-component of every feasible solution (x, y) of the approximating system $\mathcal{A}$ is a feasible solution of the original system $\mathcal{S}$.*

Our plan for processing Problems 1, 2A, 2B, 3A, 3B, and 3C is as follows: we start with reviewing the basic results we intend to use when building computationally tractable approximations of the problems and then apply these results to the problems of interest.

4.1. The matrix cube theorem. Consider an uncertain LMI with affine box uncertainty

$$\mathcal{A}^0(x) + \sum_{\ell=1}^{L} u_\ell \mathcal{A}^\ell(x) \succeq 0 \qquad \forall (u : \|u\|_\infty \leq \rho), \tag{26}$$

where

- $x \in \mathbf{R}^d$ is the vector of decision variables;
- $\mathcal{A}^\ell(x)$, $\ell = 0, 1, \dots, L$, are symmetric $m \times m$ matrices affinely depending on x;
- $u_1, \dots, u_L$ are *perturbations*, and $\rho \geq 0$ is the *uncertainty level*.

It is known that in general, it is NP-hard to solve (26) or even to check whether a given candidate solution x is feasible. However, (26) admits a computationally tractable conservative approximation which is a system of LMIs in original variables x and additional symmetric matrix variables $X_1, \dots, X_L$. Let us write $X \succeq \pm Y$ as a shortcut for the system of two matrix inequalities $X \succeq Y$, $X \succeq -Y$. The aforementioned conservative approximation of (26) is as follows:

$$X_\ell \succeq \pm \mathcal{A}^\ell(x), \qquad \ell = 1, \dots, L; \tag{27a}$$

$$\rho \sum_{\ell=1}^{L} X_\ell \preceq \mathcal{A}^0(x). \tag{27b}$$

The fact that (27) is indeed a conservative approximation of (26) is evident: if x can be extended by appropriately chosen $X_1, \dots, X_L$ to a feasible solution of (27), then from (27a) it follows that $u_\ell \mathcal{A}^\ell(x) \succeq -\rho X_\ell$ for all u_ℓ such that $|u_\ell| \le \rho$; hence

$$\mathcal{A}^0(x) + \sum_{\ell=1}^{L} u_\ell \mathcal{A}^\ell(x) \succeq \mathcal{A}^0(x) - \rho \sum_{\ell=1}^{L} X_\ell \qquad \forall (u : \|u\|_\infty \le \rho);$$

the right-hand side matrix in the latter relation is $\succeq 0$ by (27b), so that x indeed satisfies (26).

It turns out that the "level of conservativeness" of the approximation (27) is not too big, provided that the matrices $\mathcal{A}^1(x), \dots, \mathcal{A}^L(x)$ are of small ranks.

PROPOSITION 4.1 (matrix cube theorem [3]). *Let* $\mu = \max_x \max_{\ell \ge 1} \mathrm{Rank}(\mathcal{A}^\ell(x))$. *(Note* $\ell \ge 1$ *in the* max*!). Then the relation between the feasible sets of* (26) *and* (27) *is as follows:*

1. *If* x *can be extended to a feasible solution of* (27)*, then* x *is feasible for* (26).
2. *If* x *cannot be extended to a feasible solution of* (27)*, then* x *is infeasible for* (26) *with* ρ *replaced by* $\vartheta(\mu)\rho$*, where* $\vartheta(\cdot)$ *is certain universal function such that* $\vartheta(\mu) \le \frac{\pi\sqrt{\mu}}{2}$ *for all* μ *and*

$$\vartheta(1) = 1, \quad \vartheta(2) = \frac{\pi}{2} = 1.57\dots, \quad \vartheta(3) = 1.73\dots, \quad \vartheta(4) = 2.$$

In particular, for every set $\mathcal{X} \subset \mathbf{R}^d$ *one has*

$$1 \le \frac{\sup\{\rho : (26) \text{ has a solution in } \mathcal{X}\}}{\sup\{\rho : (27) \text{ has a solution in } \mathcal{X}\}} \le \vartheta(\mu)$$

provided that the numerator in the fraction is positive.

Remark 1. Sometimes we shall be interested in a sufficient condition for the strict version

$$\mathcal{A}^0(x) + \sum_{\ell=1}^{L} u_\ell \mathcal{A}^\ell(x) \succ 0 \qquad \forall (u : \|u\|_\infty \le \rho)$$

of the semi-infinite LMI (26). Such a sufficient condition can be obtained from (27) by replacing the nonstrict LMI (27b) with its strict version. For the resulting pair of conditions, a statement completely similar to the matrix cube theorem takes place.

4.2. Approximating Problem 1.

Let

$$Q = Q_+ - Q_-$$

be the representation of Q as a difference of two positive semidefinite symmetric matrices with orthogonal image spaces, and let

$$S_+ = Q_+^{1/2}, \qquad S_- = Q_-^{1/2}.$$

From now on, we assume that the set $\mathcal{I}$ in Problem 1 is *LMI-representable*, i.e., it can be specified by LMI $\{Z : \mathcal{Z}[Z] \succeq 0\}$, where $\mathcal{Z}[\cdot]$ is an affine function taking values in the space of symmetric matrices. With this assumption, Problem 1 becomes the problem of finding the supremum $\rho_1^\star$ of those $\rho > 0$ for which the system of LMIs

$$\mathcal{Z}[Z] \succeq 0, \tag{28a}$$

$$\begin{bmatrix} -A^T Z - ZA & C^T L - ZB \\ L^T C - B^T Z & L^T D + D^T L + R \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} Q \begin{bmatrix} C & D \end{bmatrix} \succeq 0 \qquad \forall (A, B, C, D) \in \mathcal{U}_\rho \tag{28b}$$

in symmetric matrix variable Z has a solution. This system can be equivalently rewritten as

(29a)
$$\mathcal{Z}[Z] \succeq 0,$$
$$\left[\begin{array}{c|c} \delta C^T Q\mathbf{C} + \mathbf{C}^T Q\delta C + \mathbf{C}^T Q\mathbf{C} & \delta C^T Q\mathbf{D} + \mathbf{C}^T Q\delta D + \mathbf{C}^T Q\mathbf{D} \\ -A^T Z - ZA & +C^T L - ZB \\ \hline \mathbf{D}^T Q\delta C + \delta D^T Q\mathbf{C} + \mathbf{D}^T Q\mathbf{C} & \delta D^T Q\mathbf{D} + \mathbf{D}^T Q\delta D + \mathbf{D}^T Q\mathbf{D} \\ +L^T C - B^T Z & +L^T D + D^T L + R \end{array}\right]$$

(29b)
$$-\begin{bmatrix} \delta C^T S_- \\ \delta D^T S_- \end{bmatrix}\begin{bmatrix} S_-\delta C & S_-\delta D \end{bmatrix} + \begin{bmatrix} \delta C^T \\ \delta D^T \end{bmatrix} Q_+ \begin{bmatrix} \delta C & \delta D \end{bmatrix} \succeq 0$$

$$\forall \begin{bmatrix} A = \mathbf{A} + \delta A & B = \mathbf{B} + \delta B \\ C = \mathbf{C} + \delta C & D = \mathbf{D} + \delta D \end{bmatrix} \in \mathcal{U}_\rho.$$

Since $Q_+ \succeq 0$, the last term in the left-hand side of (29b) is positive semidefinite. Eliminating this term, we pass from (29) to a conservative approximation of this system. By the Schur complement lemma,[5] this approximation is equivalent to the system of LMIs

(30a)
$$\mathcal{Z}[Z] \succeq 0,$$

(30b)
$$\left[\begin{array}{c|c|c} \begin{array}{c}\delta C^T Q\mathbf{C} + \mathbf{C}^T Q\delta C + \mathbf{C}^T Q\mathbf{C} \\ -A^T Z - ZA\end{array} & \begin{array}{c}\delta C^T Q\mathbf{D} + \mathbf{C}^T Q\delta D + \mathbf{C}^T Q\mathbf{D} \\ +C^T L - ZB\end{array} & \delta C^T S_- \\ \hline \begin{array}{c}\mathbf{D}^T Q\delta C + \delta D^T Q\mathbf{C} + \mathbf{D}^T Q\mathbf{C} \\ +L^T C - B^T Z\end{array} & \begin{array}{c}\delta D^T Q\mathbf{D} + \mathbf{D}^T Q\delta D \\ +\mathbf{D}^T Q\mathbf{D} \\ +L^T D + D^T L + R\end{array} & \delta D^T S_- \\ \hline S_-\delta C & S_-\delta D & I_p \end{array}\right] \succeq 0$$

$$\forall \begin{bmatrix} A = \mathbf{A} + \delta A & B = \mathbf{B} + \delta B \\ C = \mathbf{C} + \delta C & D = \mathbf{D} + \delta D \end{bmatrix} \in \mathcal{U}_\rho.$$

Taking into account (8), we see that the latter semi-infinite system of LMIs is in the form of (26), and we can use the construction from section 4.1 to build a computationally tractable conservative approximation of this system (and thus of (28)). The approximation is the following system of LMIs in matrix variables $Z, \{X_\ell\}$:

(31)
$$\mathcal{Z}[Z] \succeq 0,$$

$$X_\ell \succeq \pm \overbrace{\left[\begin{array}{c|c|c} \begin{array}{c}dC_\ell^T Q\mathbf{C} + \mathbf{C}^T Q dC_\ell \\ -dA_\ell^T Z - Z dA_\ell\end{array} & \begin{array}{c}dC_\ell^T [L + Q\mathbf{D}] \\ +\mathbf{C}^T Q dD_\ell - Z dB_\ell\end{array} & dC_\ell^T S_- \\ \hline \begin{array}{c}[L + Q\mathbf{D}]^T dC_\ell \\ +dD_\ell^T Q\mathbf{C} - dB_\ell^T Z\end{array} & \begin{array}{c}dD_\ell^T [L + Q\mathbf{D}] \\ +[L + Q\mathbf{D}]^T dD_\ell\end{array} & dD_\ell^T S_- \\ \hline S_- dC_\ell & S_- dD_\ell & 0_{pp} \end{array}\right]}^{\mathcal{A}^\ell[Z]}, \quad \ell = 1, \ldots, L,$$

$$\rho \sum_{\ell=1}^{L} X_\ell \preceq \left[\begin{array}{c|c|c} \mathbf{C}^T Q\mathbf{C} - \mathbf{A}^T Z - Z\mathbf{A} & \mathbf{C}^T [L + Q\mathbf{D}] - Z\mathbf{B} & \\ \hline [L + Q\mathbf{D}]^T \mathbf{C} - \mathbf{B}^T Z & \begin{array}{c}L^T \mathbf{D} + \mathbf{D}^T L \\ +\mathbf{D}^T Q\mathbf{D} + R\end{array} & \\ \hline & & I_p \end{array}\right].$$

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[5] The Schur complement lemma (see, e.g., [2, Chapter 4]) states that a symmetric block matrix $\left[\begin{smallmatrix} P & L \\ L^T & Q \end{smallmatrix}\right]$ with $Q \succ 0$ is positive definite (positive semidefinite) if and only if the matrix $P - LQ^{-1}L^T$ is positive definite (positive semidefinite).

Note that the supremum $\widehat{\rho}_1$ of those $\rho \geq 0$ for which system (31) is solvable is efficiently computable—it is the optimal value in the problem

$$\max_{\rho,\{X_\ell\},Z} \{\rho : (\rho, \{X_\ell\}, Z) \text{ solves } (31)\}.$$

The latter is a generalized eigenvalue problem, so that its optimal value is efficiently computable. We intend to use the efficiently computable quantity $\widehat{\rho}$ as a bound for the "quantity of interest" $\rho_1^\star$. The properties of this bound are described in the following statement.

PROPOSITION 4.2. (i) *System* (31) *is a conservative approximation of* (28)*, so that the Z-component of a feasible solution to* (31) *is a feasible solution of* (28)*. In particular, $\widehat{\rho}_1$ is a lower bound for $\rho_1^\star$.*

(ii) *If either*

(a) $Q \preceq 0$ *(i.e., $Q_+ = 0$) (as it is the case, e.g., in Examples* 1, 2, 4*)*

or

(b) *D and C are certain (i.e., $dC_\ell = 0$, $dD_\ell = 0$ for all ℓ),*

then

$$1 \leq \frac{\rho_1^\star}{\widehat{\rho}_1} \leq \vartheta(\mu) \tag{32}$$

provided that $\rho_1^\star > 0$. Here $\vartheta(\mu)$ is the function from Proposition 4.1 *and*

$$\mu = \max_{\ell=1,\dots,L} \max_Z \operatorname{Rank}(\mathcal{A}^\ell[Z]);$$

see (31)*.*

Proof. The validity of the first claim is readily given by the origin of (31). To justify the second claim, note that in the case of $Q \preceq 0$, same as in the case when C, D are certain, system (30) is solvable if and only if (28) is solvable, so that $\rho_1^\star$ is the supremum of those $\rho \geq 0$ for which (30) is solvable; with this observation, (32) is readily given by Proposition 4.1. ▯

Unfortunately, we cannot bound from above fraction (32) in the case of uncertain C, D and $Q_+ \neq 0$, since here the derivation of the approximating system includes a step (passing from (28) to (30)) with an unknown "level of conservativeness."

4.3. Approximating Problems 2A and 3A. It suffices to process Problem 2A, since Problem 3A can be treated in a completely similar fashion. The semi-infinite LMI (16), similar to the semi-infinite LMI (28), admits the conservative approximation (cf. (30))

(33)
$$\left[\begin{array}{c|c|c} \begin{array}{c}\delta C^T Q\mathbf{C} + \mathbf{C}^T Q\delta C + \mathbf{C}^T Q\mathbf{C}\\ -A^T G - GA\end{array} & \begin{array}{c}\delta C^T Q\mathbf{D} + \mathbf{C}^T Q\delta D + \mathbf{C}^T Q\mathbf{D}\\ +C^T L - GB\end{array} & \delta C^T S_- \\ \hline \begin{array}{c}\mathbf{D}^T Q\delta C + \delta D^T Q\mathbf{C} + \mathbf{D}^T Q\mathbf{C}\\ +L^T C - B^T G\end{array} & \begin{array}{c}\delta D^T Q\mathbf{D} + \mathbf{D}^T Q\delta D\\ +\mathbf{D}^T Q\mathbf{D}\\ +L^T D + D^T L + R\end{array} & \delta D^T S_- \\ \hline S_-\delta C & S_-\delta D & I_p \end{array}\right] \succeq 0$$

$$\forall \begin{bmatrix} A = \mathbf{A} + \delta A & B = \mathbf{B} + \delta B \\ C = \mathbf{C} + \delta C & D = \mathbf{D} + \delta D \end{bmatrix} \in \mathcal{U}_\rho,$$

which is equivalent to (16) in the case of $Q_+ = 0$, as well as in the case of certain C, D. The semi-infinite LMI (17) can be rewritten as

(34)
$$\begin{aligned}
&\begin{bmatrix} I & F^T \end{bmatrix} \begin{bmatrix} \begin{array}{c} \mathbf{C}^T Q\mathbf{C} \\ +\delta C^T Q\mathbf{C} + \mathbf{C}^T Q\delta C \end{array} & \mathbf{C}^T(L+QD) + \delta C^T(L+Q\mathbf{D}) \\ (L+QD)^T\mathbf{C} + (L+Q\mathbf{D})^T\delta C & \begin{array}{c} \delta D^T(L+Q\mathbf{D}) + (L+Q\mathbf{D})^T\delta D \\ +\mathbf{D}^T Q\mathbf{D} + \mathbf{D}^T L + L^T\mathbf{D} + R \end{array} \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} \\
&+ (\delta C + \delta DF)^T S_+^2 (\delta C + \delta DF) \\
&- (\delta C + \delta DF)^T S_-^2 (\delta C + \delta DF) \prec \left[(A+BF)^T H + H(A+BF)\right] \\
&\qquad\qquad \forall \begin{bmatrix} A = \mathbf{A} + \delta A & B = \mathbf{B} + \delta B \\ C = \mathbf{C} + \delta C & D = \mathbf{D} + \delta D \end{bmatrix} \in \mathcal{U}_\rho.
\end{aligned}$$

The third term in the left-hand side of this MI is negative semidefinite; eliminating this term, we get a conservative approximation of (34), and this approximation, by the Schur complement lemma, is equivalent to the following semi-infinite LMI, where we set

$$\mathcal{F} = \left[\begin{array}{c|c} \begin{array}{c} I_n \\ F \end{array} & \\ \hline & I_p \end{array}\right]:$$

(35)
$$\begin{aligned}
&\mathcal{F}^T \left[\begin{array}{c|c|c} \begin{array}{c} A^T H + HA - \mathbf{C}^T Q\mathbf{C} \\ -\delta C^T Q\mathbf{C} - \mathbf{C}^T Q\delta C \end{array} & \begin{array}{c} HB - \mathbf{C}^T(L+QD) \\ -\delta C^T(L+Q\mathbf{D}) \end{array} & \delta C^T S_+ \\ \hline \begin{array}{c} B^T H - (L+QD)^T\mathbf{C} \\ -(L+Q\mathbf{D})^T\delta C \end{array} & \begin{array}{c} -\delta D^T(L+Q\mathbf{D}) - (L+Q\mathbf{D})^T\delta D \\ -\mathbf{D}^T L + L^T\mathbf{D} - \mathbf{D}^T Q\mathbf{D} - R \end{array} & \delta D^T S_+ \\ \hline S_+\delta C & S_+\delta D & I_p \end{array}\right] \mathcal{F} \succ 0 \\
&\qquad\qquad \forall \begin{bmatrix} A = \mathbf{A} + \delta A & B = \mathbf{B} + \delta B \\ C = \mathbf{C} + \delta C & D = \mathbf{D} + \delta D \end{bmatrix} \in \mathcal{U}_\rho
\end{aligned}$$

in matrix variable H. Thus, the system of semi-infinite LMIs (33), (35) in matrix variables G, H is a conservative approximation of (16), (17); in the cases when $Q = 0$ and/or C, D are certain, the former system in fact is equivalent to the latter one. The semi-infinite system (33), (35) is in the form of (26). Applying the construction from section 4.1, we end up with computationally tractable conservative approximation of the system (16), (17), (18). The approximation is the following system of LMIs in matrix variables $G, H, \{X_\ell, Y_\ell\}$:

(36a)
$$X_\ell \succeq \pm \overbrace{\left[\begin{array}{c|c|c} \begin{array}{c} dC_\ell^T Q\mathbf{C} + \mathbf{C}^T Q dC_\ell \\ -dA_\ell^T G - G dA_\ell \end{array} & \begin{array}{c} dC_\ell^T(L+Q\mathbf{D}) \\ +\mathbf{C}^T Q dD_\ell - G dB_\ell \end{array} & dC_\ell^T S_- \\ \hline \begin{array}{c} (L+Q\mathbf{D})^T dC_\ell \\ +dD_\ell^T Q\mathbf{C} - dB_\ell^T G \end{array} & \begin{array}{c} dD_\ell^T(L+Q\mathbf{D}) \\ +(L+Q\mathbf{D})^T dD_\ell \end{array} & dD_\ell^T S_- \\ \hline S_- dC_\ell & S_- dD_\ell & \end{array}\right]}^{B^\ell[G]}, \quad \ell = 1, \ldots, L,$$

(36b)
$$\rho \sum_{\ell=1}^{L} X_\ell \preceq \left[\begin{array}{c|c|c} \mathbf{C}^T Q\mathbf{C} - \mathbf{A}^T G - G\mathbf{A} & \mathbf{C}^T(L+Q\mathbf{D}) - G\mathbf{B} & \\ \hline (L+Q\mathbf{D})^T\mathbf{C} - \mathbf{B}^T G & \mathbf{D}^T Q\mathbf{D} + L^T\mathbf{D} + \mathbf{D}^T L + R & \\ \hline & & I_p \end{array}\right],$$

(36c)

$$Y_\ell \succeq \pm \mathcal{F}^T \overbrace{\left[\begin{array}{c|c|c} \begin{array}{c} dA_\ell^T H + H dA_\ell \\ -dC_\ell^T Q\mathbf{C} - \mathbf{C}^T Q dC_\ell \end{array} & \begin{array}{c} H dB_\ell \\ -dC_\ell^T (L + Q\mathbf{D}) - \mathbf{C}^T Q dD_\ell \end{array} & dC_\ell^T S_+ \\ \hline \begin{array}{c} dB_\ell^T H \\ -(L+Q\mathbf{D})^T dC_\ell - dD_\ell^T Q\mathbf{C} \end{array} & \begin{array}{c} -dD_\ell^T (L + Q\mathbf{D}) \\ -(L+Q\mathbf{D})^T dD_\ell \end{array} & dD_\ell^T S_+ \\ \hline S_+ dC_\ell & S_+ dD_\ell & \end{array}\right]}^{\mathcal{C}^\ell[H]} \mathcal{F},$$

(36d)

$$\rho \sum_{\ell=1}^{L} Y_\ell \prec \mathcal{F}^T \left[\begin{array}{c|c|c} \mathbf{A}^T H + H\mathbf{A} - \mathbf{C}^T Q\mathbf{C} & H\mathbf{B} - \mathbf{C}^T(L + Q\mathbf{D}) & \\ \hline \mathbf{B}^T H - (L+Q\mathbf{D})^T \mathbf{C} & -\mathbf{D}^T Q\mathbf{D} - L^T\mathbf{D} - \mathbf{D}^T L - R & \\ \hline & & I_p \end{array}\right] \mathcal{F},$$

(36e)

$$(1-\epsilon)\mathbf{Z}_{av} \preceq H \prec G.$$

The supremum $\widehat{\rho}_{2A}$ of those $\rho \geq 0$ for which system (36) is solvable is efficiently computable, and this efficiently computable quantity can be used as a bound for the optimal value $\rho^\star_{2\mathrm{A}}$ in Problem 2A. The properties of this bound are described in the following.

PROPOSITION 4.3. (i) *System* (36) *is a conservative approximation of* (16), (17), (18) *so that the* G, H*-components of a feasible solution to* (36) *are a feasible solution of* (16), (17), (18)*. In particular,* $\widehat{\rho}_{2A}$ *is a lower bound for* $\rho^\star_{2\mathrm{A}}$.

(ii) *If either*

(a) $Q = 0$ *(i.e.,* $Q_+ = Q_- = 0$*),*

or

(b) D *and* C *are certain (i.e.,* $dC_\ell = 0$, $dD_\ell = 0$ *for all* ℓ*),*

then

$$1 \leq \frac{\rho^\star_{2\mathrm{A}}}{\widehat{\rho}_{2A}} \leq \vartheta(\mu) \tag{37}$$

provided that $\rho^\star_{2\mathrm{A}} > 0$*. Here* $\vartheta(\mu)$ *is the function from Proposition* 4.1 *and*

$$\mu = \max\left[\max_{\ell \geq 1, G} \mathrm{Rank}(\mathcal{B}^\ell[G]), \max_{\ell\geq 1, H} \mathrm{Rank}(\mathcal{C}^\ell[H])\right];$$

see (36)*.*

Tractable conservative approximation of Problem 3A looks exactly as (36), up to the constraint (36e), which should be replaced with the constraint

$$0 \preceq G \prec H \preceq (1+\delta)\mathbf{Z}_{req}.$$

The properties of this approximation are completely similar to those established in Proposition 4.3.

4.4. Approximating Problems 2B and 3B. Our current goal is to build a tractable conservative approximation of the semi-infinite system of MIs associated with Problems 2B and 3B. Both problems have the same structure, so that it suffices to consider the system associated with Problem 2B, i.e., the system (16), (17), (22) in variables G, F (H now is fixed). We have already built a tractable conservative approximation of the semi-infinite MI (16); it is given by system of LMIs (36a), (36b) in matrix variables $G, \{X_\ell\}$. Let us focus on the semi-infinite MI (17) in variable F.

We can rewrite this inequality equivalently as

(38)

$$
\begin{array}{l}
(S_+\delta C + S_+\delta DF)^T(S_+\delta C + S_+\delta DF) - (S_-\delta C + S_-\delta DF)^T(S_-\delta C + S_-\delta DF) \\
+\left[\begin{array}{cc} I & F^T \end{array}\right]\left[\begin{array}{c|c} \begin{array}{c}\delta C^T Q\mathbf{C} + \mathbf{C}^T Q\delta C \\ +\mathbf{C}^T Q\mathbf{C}\end{array} & \begin{array}{c}\mathbf{C}^T(L+Q\mathbf{D}) + \delta C^T(L+Q\mathbf{D}) \\ +\mathbf{C}^T Q\delta D\end{array} \\ \hline \begin{array}{c}(L+Q\mathbf{D})^T\mathbf{C} + (L+Q\mathbf{D})^T\delta C \\ +\delta D^T Q\mathbf{C}\end{array} & \begin{array}{c}\mathbf{D}^T Q\mathbf{D} + L^T\mathbf{D} + \mathbf{D}^T L + R \\ \delta D^T Q\mathbf{D} + \mathbf{D}^T Q\delta D \\ +L^T\delta D + \delta D^T L\end{array}\end{array}\right]\left[\begin{array}{c} I \\ F \end{array}\right] \\
\hfill \prec ([A+BF]^T H + H[A+BF]) \\
\hfill \forall \left[\begin{array}{cc} A = \mathbf{A}+\delta A & B = \mathbf{B}+\delta B \\ C = \mathbf{C}+\delta C & D = \mathbf{D}+\delta D \end{array}\right] \in \mathcal{U}_\rho.
\end{array}
$$

The second term in the left-hand side of the latter MI always is negative semidefinite; eliminating this term, we come to a conservative approximation of (38) as follows:

(39)

$$
\begin{array}{l}
(S_+\delta C + S_+\delta DF)^T(S_+\delta C + S_+\delta DF) \\
+\left[\begin{array}{cc} I & F^T \end{array}\right]\left[\begin{array}{c|c} \overbrace{\begin{array}{c}\delta C^T Q\mathbf{C} + \mathbf{C}^T Q\delta C \\ +\mathbf{C}^T Q\mathbf{C}\end{array}}^{J_{00}[\Sigma]} & \overbrace{\begin{array}{c}\mathbf{C}^T(L+Q\mathbf{D}) + \delta C^T(L+Q\mathbf{D}) \\ +\mathbf{C}^T Q\delta D\end{array}}^{J_{01}[\Sigma]} \\ \hline \underbrace{\begin{array}{c}(L+Q\mathbf{D})^T\mathbf{C} + (L+Q\mathbf{D})^T\delta C \\ +\delta D^T Q\mathbf{C}\end{array}}_{J_{10}[\Sigma]} & \underbrace{\begin{array}{c}\mathbf{D}^T Q\mathbf{D} + L^T\mathbf{D} + \mathbf{D}^T L + R \\ +\delta D^T Q\mathbf{D} + \mathbf{D}^T Q\delta D \\ +L^T\delta D + \delta D^T L\end{array}}_{J_{11}[\Sigma]}\end{array}\right]\left[\begin{array}{c} I \\ F \end{array}\right] \\
\hfill \prec ([A+BF]^T H + H[A+BF]) \\
\hfill \forall \Sigma \equiv \left[\begin{array}{cc} A = \mathbf{A}+\delta A & B = \mathbf{B}+\delta B \\ C = \mathbf{C}+\delta C & D = \mathbf{D}+\delta D \end{array}\right] \in \mathcal{U}_\rho.
\end{array}
$$

Note that the matrices $J_{ij}[\Sigma]$ are affine in Σ.

Observe that (39) is exactly the semi-infinite MI

$$
\begin{array}{rl}
[A+BF]^T H + H[A+BF] - J_{00}[\Sigma] - F^T J_{10}[\Sigma] - J_{01}[\Sigma]F & \\
-(S_+\delta C + S_+\delta DF)^T(S_+\delta C + S_+\delta DF) - F^T J_{11}[\Sigma]F & \succ \quad 0 \\
\forall \Sigma \equiv \left[\begin{array}{cc} A = \mathbf{A}+\delta A & B = \mathbf{B}+\delta B \\ C = \mathbf{C}+\delta C & D = \mathbf{D}+\delta D \end{array}\right] \in \mathcal{U}_\rho. &
\end{array} \tag{40}
$$

Now assume that $J_{11}[\Sigma] \succ 0$. Note that this assumption is quite natural—the matrix $J_{11}[\Sigma]$ should be positive semidefinite already to make feasible (16) with $\rho = 0$. Let

$$
\mathbf{K} = J_{11}^{-1}[\Sigma], \qquad \delta J_{11}[\delta\Sigma] = J_{11}[\Sigma + \delta\Sigma] - J_{11}[\Sigma].
$$

We claim that the following relations hold true:

$$
\mathbf{K} - \mathbf{K}\delta J_{11}[\delta\Sigma]\mathbf{K} \succ 0 \qquad \forall \Sigma \equiv \Sigma + \delta\Sigma \in \mathcal{U}_\rho \tag{41a}
$$

$$\Updownarrow$$

$$
J_{11}[\Sigma] \succ 0 \qquad \forall \Sigma \in \mathcal{U}_\rho \tag{41b}
$$

$$\Downarrow$$

(41c) $$[\mathbf{K} - \mathbf{K}\delta J_{11}[\delta\Sigma]\mathbf{K}]^{-1} \succeq [J_{11}[\Sigma]]^{-1} \succ 0 \qquad \forall \Sigma \equiv \Sigma + \delta\Sigma \in \mathcal{U}_\rho.$$

Indeed, the equivalence between (41a) and (41b) follows from the identity

$$\mathbf{K} - \mathbf{K}\delta J_{11}[\delta\Sigma]\mathbf{K} = \mathbf{K}J_{11}[\Sigma - \delta\Sigma]\mathbf{K}$$

(which is readily given by the definition of $\mathbf{K}$), combined with the fact that $\mathcal{U}_\rho$ is symmetric with respect to Σ. To see that (41b) implies (41c), observe, first, that

(42) $$X \succ \pm Y \Rightarrow [X^{-1} - X^{-1}YX^{-1}]^{-1} \succeq X + Y \succ 0.$$

Indeed, assuming $X \succ \pm Y$ and setting $Z = X^{-1/2}YX^{-1/2}$ (so that $I \succ \pm Z$), we have

$$\begin{aligned}(X+Y)^{-1} - [X^{-1} - X^{-1}YX^{-1}] &= [X^{1/2}(I+Z)X^{1/2}]^{-1} - X^{-1/2}[I-Z]X^{-1/2} \\ &= X^{-1/2}[(I+Z)^{-1} - (I-Z)]X^{-1/2} \\ &= X^{-1/2}Z(I+Z)^{-1}ZX^{-1/2} \succeq 0,\end{aligned}$$

hence $(X+Y)^{-1} \succeq [X^{-1} - X^{-1}YX^{-1}]$ and thus $[X^{-1} - X^{-1}YX^{-1}]^{-1} \succeq X + Y \succ 0$, as required in (42). Now let $\delta\Sigma$ be such that $\Sigma + \delta\Sigma \in \mathcal{U}_\rho$. Since $\mathcal{U}_\rho$ is symmetric with respect to Σ, we have $\Sigma - \delta\Sigma \in \mathcal{U}_\rho$ as well. In the case of (41b) it follows that $J_{11}[\Sigma \pm \delta\Sigma] \succ 0$ or, which is the same, $J_{11}[\Sigma] \succ \pm\delta J_{11}[\delta\Sigma]$. Applying (42), we arrive at (41c).

By (41), in the case of (41a) the semi-infinite MI

$$\begin{aligned}&[A+BF]^T H + H[A+BF] - J_{00}[\Sigma] - F^T J_{10}[\Sigma] - J_{01}[\Sigma]F \\ &- (S_+\delta C + S_+\delta DF)^T(S_+\delta C + S_+\delta DF) - F^T[\mathbf{K} - \mathbf{K}\delta J_{11}[\delta\Sigma]\mathbf{K}]^{-1}F \quad \succ \quad 0 \\ &\forall \Sigma \equiv \Sigma + \delta\Sigma \equiv \left[\begin{array}{cc} A = \mathbf{A} + \delta A & B = \mathbf{B} + \delta B \\ C = \mathbf{C} + \delta C & D = \mathbf{D} + \delta D \end{array}\right] \in \mathcal{U}_\rho\end{aligned}$$

is a conservative approximation of (40), which in turn is a conservative approximation of (17). Applying the Schur complement lemma, the resulting semi-infinite MI can be rewritten as

(43)
$$\left[\begin{array}{c|c|c} \begin{array}{c}[A+BF]^T H + H[A+BF] \\ -J_{00}[\Sigma] - F^T J_{10}[\Sigma] - J_{01}[\Sigma]F\end{array} & (S_+\delta C + S_+\delta DF)^T & F^T \\ \hline (S_+\delta C + S_+\delta DF) & I & \\ \hline F & & \mathbf{K} - \mathbf{K}\delta J_{11}[\delta\Sigma]\mathbf{K} \end{array}\right] \succ 0$$
$$\forall \Sigma \equiv \Sigma + \delta\Sigma \equiv \left[\begin{array}{cc} A = \mathbf{A} + \delta A & B = \mathbf{B} + \delta B \\ C = \mathbf{C} + \delta C & D = \mathbf{D} + \delta D \end{array}\right] \in \mathcal{U}_\rho.$$

Note that the validity of this semi-infinite LMI automatically implies (41a). Further, the matrix in the left-hand side of the resulting semi-infinite LMI is affine in Σ, so that we can apply the scheme from section 4.1 to build a computationally tractable conservative approximation of this semi-infinite LMI. The approximation is the following system of LMIs in matrix variables $F, \{Y_\ell\}$:

(44)

$$
Y_\ell \succeq \pm \overbrace{\left[\begin{array}{c|c|c}
\begin{array}{c}[dA_\ell + dB_\ell F]^T H + H[dA_\ell + dB_\ell F] \\ -dC_\ell^T Q\mathbf{C} - \mathbf{C}^T Q dC_\ell \\ -\left\{dC_\ell^T[L+Q\mathbf{D}] + \mathbf{C}^T Q dD_\ell\right\} F \\ -F^T\left\{dC_\ell^T[L+Q\mathbf{D}] + \mathbf{C}^T Q dD_\ell\right\}^T\end{array} & \begin{array}{c}dC_\ell^T S_+ \\ +F^T dD_\ell^T S_+\end{array} & \\
\hline
\begin{array}{c}S_+ dC_\ell \\ +S_+ dD_\ell F\end{array} & & \\
\hline
 & & \begin{array}{c}-\mathbf{K} dD_\ell^T[L+Q\mathbf{D}]\mathbf{K} \\ -\mathbf{K}[L+Q\mathbf{D}]^T dD_\ell \mathbf{K}\end{array}
\end{array}\right]}^{\mathcal{D}^\ell[F]}, \quad \ell = 1,\dots,L,
$$

$$
\rho \sum_{\ell=1}^{L} Y_\ell \prec \left[\begin{array}{c|c|c}
\begin{array}{c}[\mathbf{A}+\mathbf{B}F]^T H + H[\mathbf{A}+\mathbf{B}F] - \mathbf{C}^T Q\mathbf{C} \\ -F^T[L+Q\mathbf{D}]^T\mathbf{C} - \mathbf{C}^T[L+Q\mathbf{D}]F\end{array} & & F^T \\
\hline
 & I & \\
\hline
F & & \mathbf{K}
\end{array}\right].
$$

We arrive at the following result.

PROPOSITION 4.4. *Assume that the matrix*

$$\mathbf{K}^{-1} \equiv \mathbf{D}^T Q\mathbf{D} + L^T\mathbf{D} + \mathbf{D}^T L + R \tag{45}$$

is positive definite. Then

(i) *The system of LMIs* (36a), (36b), (44) *and the LMI*

$$H \prec G \tag{46}$$

in matrix variables $G, F, \{X_\ell, Y_\ell\}$ is a conservative approximation of the system (16), (17), (18) *associated with Problem* 2B. *In particular, the efficiently computable supremum $\widehat{\rho}$ of those $\rho \geq 0$ for which the approximating system is solvable is a lower bound on the optimal value $\rho^\star_{2\mathrm{B}}$ of Problem* 2B.

(ii) *If either*

(a) *C, D are certain (i.e., $dC_\ell = 0$, $dD_\ell = 0$ for all ℓ)*

or

(b) *$Q = 0$ and D is certain,*

then

$$1 \leq \frac{\rho^\star_{2\mathrm{B}}}{\widehat{\rho}} \leq \vartheta(\mu), \tag{47}$$

provided that $\rho^\star_{2\mathrm{B}} > 0$. Here $\vartheta(\mu)$ is the function from Proposition 4.1 *and*

$$\mu = \max\left[\max_{\ell\geq 1, G} \mathrm{Rank}(\mathcal{B}^\ell[G]), \max_{\ell\geq 1, F}\mathrm{Rank}(\mathcal{D}^\ell[F])\right];$$

see (36a), (44) *for the definitions of $\mathcal{B}^\ell[G]$ and $\mathcal{D}^\ell[F]$.*

Tractable conservative approximation of Problem 3B looks exactly like the one for Problem 2B, with the only difference that the LMI (46) should now be replaced with the LMIs

$$0 \preceq G \prec H.$$

The properties of this approximation are completely similar to those established in Proposition 4.4.

4.5. Approximating Problem 3C. Now consider Problem 3C. The system of MIs to be approximated is now comprised of the semi-infinite MI (17) in matrix variables F, H and the LMI $0 \prec H \preceq (1+\delta)\mathbf{Z}_{req}$. The system in question can be rewritten equivalently as

(48)
$$\begin{array}{rcl} \left[\begin{array}{cc} I & F^T \end{array}\right]\left[\begin{array}{cc} C^T & \\ D^T & I \end{array}\right]\left[\begin{array}{cc} Q & L \\ L^T & R \end{array}\right]\left[\begin{array}{cc} C & D \\ & I \end{array}\right]\left[\begin{array}{c} I \\ F \end{array}\right] & \prec & [(A+BF)^T H + H(A+BF)] \\ & & \forall \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \in \mathcal{U}_\rho, \\ 0 \prec H & \preceq & (1+\delta)\mathbf{Z}_{req}. \end{array}$$

We can assume that $\mathbf{Z}_{req} \succ 0$—otherwise the system clearly is unsolvable. Let us use the standard change of variables $(H, F) \mapsto (U = H^{-1}, V = FH^{-1})$. Multiplying both sides of (48) from the right and from the left by H^{-1}, we rewrite (48) in the new variables as

(49)
$$\begin{array}{rcl} \left[\begin{array}{cc} U & V^T \end{array}\right]\left[\begin{array}{cc} C^T & \\ D^T & I \end{array}\right]\underbrace{\left[\begin{array}{cc} Q & L \\ L^T & R \end{array}\right]}_{\mathfrak{P}}\left[\begin{array}{cc} C & D \\ & I \end{array}\right]\left[\begin{array}{c} U \\ V \end{array}\right] & \prec & [AU + UA^T + BV + V^T B^T] \\ & & \forall \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \in \mathcal{U}_\rho, \\ U & \succeq & (1+\delta)^{-1}\mathbf{Z}_{req}^{-1}. \end{array}$$

Setting $M \equiv \begin{bmatrix} M_{yy} & M_{yu} \\ M_{yu}^T & M_{uu} \end{bmatrix} = \mathfrak{P}^{1/2}$ (recall that we are in the case of $\mathfrak{P} \succeq 0$) and applying the Schur complement lemma, we can rewrite the latter system equivalently as

(50)
$$\left[\begin{array}{c|c|c} \begin{array}{c} AU + UA^T \\ +BV + V^T B^T \end{array} & \begin{array}{c} UC^T M_{yy} \\ +V^T [M_{yy} D + M_{yu}]^T \end{array} & \begin{array}{c} UC^T M_{yu} \\ +V^T [M_{yu}^T D + M_{uu}]^T \end{array} \\ \hline \begin{array}{c} M_{yy} CU \\ +[M_{yy} D + M_{yu}]V \end{array} & I_p & \\ \hline \begin{array}{c} M_{yu}^T CU \\ +[M_{yu}^T D + M_{uu}]V \end{array} & & I_m \end{array}\right] \succeq 0$$
$$\forall \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \in \mathcal{U}_\rho,$$
$$U \succeq (1+\delta)^{-1}\mathbf{Z}_{req}^{-1}.$$

System (50) is in the form of (26); applying the construction from section 4.1, we end up with a tractable conservative approximation of (49), which is the following system of LMIs in matrix variables $U, V, \{X_\ell\}$:

(51)
$$U \succeq (1+\delta)^{-1}\mathbf{Z}_{req}^{-1},$$
$$X_\ell \succeq \pm \underbrace{\left[\begin{array}{c|c|c} \begin{array}{c} dA_\ell U + U dA_\ell^T \\ +dB_\ell V + V^T dB_\ell^T \end{array} & \begin{array}{c} U dC_\ell^T M_{yy} \\ +V^T dD_\ell^T M_{yy} \end{array} & \begin{array}{c} U dC_\ell^T M_{yu} \\ +V^T dD_\ell^T M_{yu} \end{array} \\ \hline \begin{array}{c} M_{yy} dC_\ell U \\ +M_{yy} dD_\ell V \end{array} & 0_{p\times p} & \\ \hline \begin{array}{c} M_{yu}^T dC_\ell U \\ +M_{yu}^T dD_\ell V \end{array} & & 0_{m\times m} \end{array}\right]}_{\mathcal{E}^\ell[U,V]}, \quad \ell = 1, \ldots, L,$$

$$\rho \sum_{\ell=1}^{L} X_\ell \preceq \left[\begin{array}{c|c|c} \begin{array}{c}\mathbf{A}U + U\mathbf{A}^T\\ +\mathbf{B}V + V^T\mathbf{B}^T\end{array} & \begin{array}{c}U\mathbf{C}^T M_{yy}\\ +V^T[M_{yy}\mathbf{D} + M_{yu}]^T\end{array} & \begin{array}{c}U\mathbf{C}^T M_{yu}\\ +V^T[M_{yu}^T\mathbf{D} + M_{uu}]^T\end{array} \\ \hline \begin{array}{c}M_{yy}\mathbf{C}U\\ +[M_{yy}\mathbf{D} + M_{yu}]V\end{array} & I_p & \\ \hline \begin{array}{c}M_{yu}^T\mathbf{C}U\\ +[M_{yu}^T\mathbf{D} + M_{uu}]V\end{array} & & I_m \end{array}\right].$$

We arrive at the following.

PROPOSITION 4.5. *Assume that the supply matrix* $\mathfrak{P} = \begin{bmatrix} Q & L \\ L^T & R \end{bmatrix}$ *is positive semidefinite and that* $\mathbf{Z}_{req} \succ 0$. *Then the system of LMIs* (51) *in matrix variables* $U, V, \{X_\ell\}$ *is a conservative approximation of the system associated with Problem* 3C. *In particular, the efficiently computable supremum* $\widehat{\rho}$ *of those* $\rho \geq 0$ *for which the approximating system is solvable is a lower bound on the optimal value* $\rho^\star_{3\mathrm{C}}$ *of Problem* 3C. *For this lower bound, one has*

$$1 \leq \frac{\rho^\star_{3\mathrm{C}}}{\widehat{\rho}} \leq \vartheta(\mu), \tag{52}$$

provided that $\rho^\star_{3\mathrm{C}} > 0$. *Here* $\vartheta(\mu)$ *is the function from Proposition* 4.1 *and*

$$\mu = \max_{\ell \geq 1, U, V} \mathrm{Rank}(\mathcal{E}^\ell[U, V]);$$

see (51).

4.6. Simplifying approximating systems. A severe practical disadvantage of the tractable approximations of Problems 1, 2A, 2B, 3A, 3B, and 3C we have built is that the sizes of these approximations, although polynomial in the sizes m, n, p, L of the underlying dynamical system and uncertainty set, are quite large. For example, approximation (31) has a single $(m+n+p)\times(m+n+p)$ symmetric matrix variable X_ℓ and two $(m+n+p)\times(m+n+p)$ LMIs per every basic perturbation in the data, so that the design dimension of the approximation is of order of $L(m+n+p)^2$, a quantity which typically is prohibitively large for practical computations. We are about to demonstrate that under favorable circumstances the sizes of the approximating systems can be reduced dramatically. For the sake of simplicity, we restrict our considerations to the case of the approximation (31) associated with Problem 1; the approximations associated with other problems can be processed in a completely similar fashion.

System (31) is of the generic form

$$\mathcal{P}(x) \succeq 0, \tag{53a}$$

$$U_\ell \succeq \pm \mathcal{Q}_\ell(x), \qquad \ell = 1, \dots, M, \tag{53b}$$

$$V_\ell \succeq \pm R_\ell, \qquad \ell = 1, \dots, N, \tag{53c}$$

$$\rho \left[\sum_\ell U_\ell + \sum_\ell V_\ell\right] \preceq \mathcal{S}(x), \tag{53d}$$

where

- x is the collection of the original design variables (for (31), $x = Z$);
- U_ℓ, V_ℓ are additional $K \times K$ matrix variables (for (31), $K = m + n + p$, $M + N = L$, the U-variables are those of X_ℓ for which $\mathcal{A}^\ell[Z]$ indeed depends on Z, while the V-variables correspond to those of X_ℓ for $\mathcal{A}^\ell[Z]$ in fact does not depend on Z);

- $\mathcal{P}(x)$, $\mathcal{Q}_\ell(x)$, $\mathcal{S}(x)$ are affine functions of x taking values in the spaces of symmetric matrices of appropriate sizes, and R_ℓ are given $K \times K$ symmetric matrices.

Note that in the situations we are interested in, the ranks of the matrices $\mathcal{Q}_\ell(x)$, R_ℓ are small, provided that the ranks of basic perturbation matrices $dA_\ell, dB_\ell, dC_\ell, dD_\ell$ are small (as indeed is the case in applications). The undesirable large sizes of the approximating system (53) come exactly from the necessity to introduce large-size "matrix bounds" U_ℓ, V_ℓ on the small rank matrices $\mathcal{Q}_\ell(x)$, R_ℓ.

Note that in our applications all we are interested in are the x-components of the feasible solutions of (53). Thus, for our purposes (53) can be replaced with any *x-equivalent* system of LMIs—a system of LMIs $\mathcal{L}(x,y) \succeq 0$ in the original variables x and additional variables y such that the set of x-components of feasible solutions to the latter system is exactly the same as the set of x-components of feasible solutions of (53). What we intend to do is to demonstrate that under favorable circumstances we can build a system of LMIs which is x-equivalent to (53), while being "much smaller" than the latter system. The key to our construction is given by the following two observations.

LEMMA 4.6 (see [3, Lemma 3.1 and Proposition 2.1]). (i) *Let a, b be two nonzero vectors. A symmetric matrix X satisfies the relation*

$$X \succeq \pm[ab^T + ba^T]$$

if and only if there exists positive λ such that

$$X \succeq \lambda aa^T + \frac{1}{\lambda} bb^T.$$

(ii) *Let A be a $n \times n$ symmetric matrix of rank $k > 0$, so that $A = P^T \widehat{A} P$ for appropriately chosen $k \times k$ matrix $\widehat{A}$ and $k \times n$ matrix P of rank k. A symmetric matrix X satisfies the relation*

$$X \succeq \pm A$$

if and only if there exists $k \times k$ symmetric matrix $\widehat{X}$ such that

$$\begin{aligned} X &\succeq P^T \widehat{X} P, \\ \widehat{X} &\succeq \pm \widehat{A}. \end{aligned} \tag{54}$$

Now assume that the matrices $\mathcal{Q}_\ell(x)$ are of the from

$$\mathcal{Q}_\ell(x) = a_\ell b_\ell^T(x) + b_\ell(x) a_\ell^T, \tag{55}$$

where $a_\ell \neq 0$, $b_\ell(x) \not\equiv 0$ are, respectively, a vector and an affine vector-valued function of x. Let also

$$R_\ell = P_\ell^T \widehat{R}_\ell P_\ell : \quad \widehat{R}_\ell = \widehat{R}_\ell^T \in \mathbf{S}^{k_\ell}, \qquad k_\ell = \mathrm{Rank}(R_\ell) > 0.$$

Applying Lemma 4.6, we see that (53) is x-equivalent to the following system of constraints in the original variables x and the additional variables $\lambda_\ell \geq 0$, $\widehat{V}_\ell \in \mathbf{S}^{k_\ell}$:

$$\begin{aligned} &\mathcal{P}(x) \succeq 0, \\ &\widehat{V}_\ell \succeq \pm \widehat{R}_\ell, \qquad \ell = 1, \ldots, N, \\ &\rho \left[\sum_\ell \left[\lambda_\ell a_\ell a_\ell^T + \tfrac{1}{\lambda_\ell} b_\ell(x) b_\ell^T(x) \right] + \sum_\ell P_\ell^T \widehat{V}_\ell P_\ell \right] \preceq \mathcal{S}(x) \end{aligned}$$

(where $\frac{1}{0}bb^T$ is 0 for $b = 0$ and is undefined for $b \neq 0$). The resulting system, via the Schur complement lemma, is x-equivalent to the system of LMIs

$$\mathcal{P}(x) \succeq 0, \tag{56a}$$

$$\left[\begin{array}{c|cccc} X - \sum_{\ell=1}^{M} \lambda_\ell a_\ell a_\ell^T & b_1(x) & b_2(x) & \dots & b_M(x) \\ \hline b_1^T(x) & \lambda_1 & & & \\ b_2^T(x) & & \lambda_2 & & \\ \vdots & & & \ddots & \\ b_M^T(x) & & & & \lambda_M \end{array}\right] \succeq 0, \tag{56b}$$

$$\widehat{V}_\ell \succeq \pm \widehat{R}_\ell, \qquad \ell = 1, \dots, N, \tag{56c}$$

$$\rho \left[X + \sum_\ell P_\ell^T \widehat{V}_\ell P_\ell \right] \preceq \mathcal{S}(x) \tag{56d}$$

in the original variables x and additional scalar variables $\{\lambda_\ell\}_{\ell=1}^M$ and matrix variables X, $\{\widehat{V}_\ell\}_{\ell=1}^N$.

System (56) is x-equivalent to our original system (53) and is usually much better suited for numerical processing than the original system. Indeed, as compared to (53), in (56) there are

- a single $K \times K$ matrix variable X and M scalar variables $\{\lambda_\ell\}_{\ell=1}^M$ instead of M $K \times K$ matrix variables U_ℓ;
- $k_\ell \times k_\ell$ matrix variables $\widehat{V}_\ell$ instead of $K \times K$ matrix variables V_ℓ, and $k_\ell \times k_\ell$ LMIs (56c) instead of $K \times K$ LMIs (53c) (recall that k_ℓ are assumed to be small as compared to K);
- a single LMI (56b) instead of M LMIs (53b). Although the size of LMI (56b) is larger than those of LMIs (53b), the LMI is of very simple arrow structure and is extremely sparse.

It remains to understand what should be required from the uncertainty set $\mathcal{U}_\rho$ in order to ensure that the approximations associated with Problems 1, 2A, 2B, 3A, 3B, and 3C possess property (55) and thus admit the outlined simplification. The corresponding requirements are as follows:

A. In the case of Problems 1, 2A, 3A, it suffices to assume the following:
   A.1. The parts $[A, B]$ and $[C, D]$ of the matrix $\Sigma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ are perturbed independently (i.e., for every ℓ exactly one of the matrices $[dA_\ell, dB_\ell]$, $[dC_\ell, dD_\ell]$ is nonzero).
   A.2. The basic perturbations of the part $[A, B]$ of Σ are of ranks ≤ 1.
   Note that under these assumptions the quantity μ in Propositions 4.2, 4.3 and the above quantities k_ℓ satisfy the relation

$$k_\ell \leq \mu \leq 2 \max \left[1, \max_\ell (\mathrm{Rank}(dC_\ell) + \mathrm{Rank}(dD_\ell)) \right].$$

B. In the case of Problems 2B, 3B, it suffices to assume the following.
   B.1. The parts A, B, C, D of Σ are perturbed independently (i.e., for every ℓ exactly one of the matrices dA_ℓ, dB_ℓ, dC_ℓ, dD_ℓ is nonzero).
   B.2. The basic perturbations of the parts A, B, C, D of Σ are of ranks ≤ 1 and
      i. either $Q = 0$

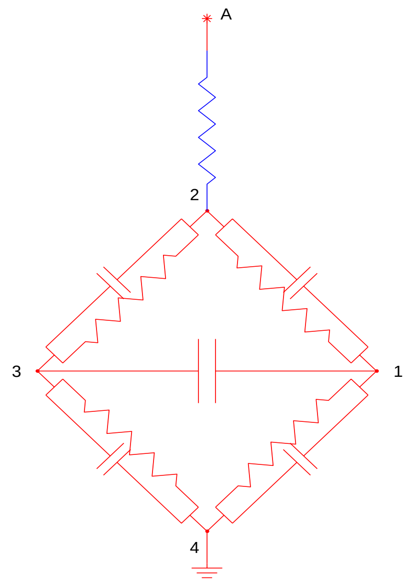


FIG. 1. *"Bridge."*

ii. or D is certain.

Note that under these assumptions the quantity μ in Proposition 4.4 and the above quantities k_ℓ satisfy the relation

$$k_\ell \le \mu \le 2.$$

C. In the case of Problem 3C, it suffices to assume that

C.1. the basic perturbations $d\Sigma_\ell$ are of ranks ≤ 1.

Note that under these assumptions the quantity μ in Proposition 4.5 equals 2.

Note that the sets A.1–A.2, B.1–B.2, C.1 of the assumptions are satisfied in the simplest case of the *interval uncertainty*—every entry in Σ, independently of other entries, runs through a given interval. In this case, $k_\ell \le \mu = 2$, and the corresponding "tightness bound" $\vartheta(\mu)$ (see (32), (37), (47), (52)) becomes $\frac{\pi}{2}$.

5. Illustrating examples. Here we present three simple illustrations of the proposed approach. The first two of them correspond to the positive-real case, while the third has to do with the linear-quadratic case.

5.1. Positive-real case. Consider the simple RC circuit ("bridge") presented in Figure 1. The input is the outer voltage applied between the node A and the ground, the output is the current through the circuit. The state variables are the potentials at the nodes 1, 2, 3 (normalized by the condition that the potential of the ground is identically zero). Applying the Kirchoff laws, the description of the system becomes

$$\begin{aligned} \dot{z}(t) &= A_{c,r}z(t) + B_{c,r}u(t), \\ y(t) &= C_r z(t) + D_r u(t), \end{aligned} \tag{57}$$

where we have the following:

- $c \in \mathbf{R}^{10}$ is the vector of capacitances of the capacitors in the 10 arcs of the circuit (9 "visible arcs" and the external arc from node 2 via point A to the ground; for arc i with no capacitor, $c_i = 0$).
- $r \in \mathbf{R}^{10}$ is the vector of conductances of the resistors in the 10 arcs of the circuit (for arc i with no resistor, $r_i = 0$).
- the matrix $\Sigma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is given by

$$\Sigma = \Sigma_{c,r} \equiv \left[\begin{array}{c|c} -[P^T\mathrm{Diag}\{c\}P]^{-1}[P^T\mathrm{Diag}\{r\}P] & [P^T\mathrm{Diag}\{c\}P]^{-1}[P^T\mathrm{Diag}\{r\}J], \\ \hline -[P^T\mathrm{Diag}\{r\}J], & J^T\mathrm{Diag}\{r\}J, \end{array}\right],$$

where $\mathrm{Diag}\{p\}$ denotes the diagonal matrix with diagonal entries given by vector p and

- P is the *incidence matrix*. The rows of P are indexed by the 10 arcs in the circuit, the columns are indexed by the 3 nonground nodes 1, 2, 3 and the element P_{ij} is equal to $+1$, -1 or 0 depending on whether node # j starts arc # i, ends this arc, or is not incident to the arc. For our circuit, P is as follows (R stands for arcs with resistors, C for arcs with capacitors):

Arcs			Nodes		
Origin	Destination	Type	1	2	3
1	2	R	1	−1	0
1	2	C	1	−1	0
2	3	R	0	1	−1
2	3	C	0	1	−1
3	4	R	0	0	1
3	4	C	0	0	1
4	1	R	−1	0	0
4	1	C	−1	0	0
1	3	C	1	0	−1
2	→ A → 4	R	0	1	0

- $J = (0, \dots, 0, 1)^T \in \mathbf{R}^{10}$ "points" to the external arc (which in our enumeration is the last of the 10 arcs of the circuit).

We treat as the uncertain parameters the capacitances of the capacitors and the conductances of the resistors (except for the "outer" resistor in the external arc; it represents the inner resistance of the outer supply and is assumed to be certain) and assume that every one of these parameters can vary, independently of others, by at most ρ times the nominal value of the parameter, where ρ is the uncertainty level in question. The nominal values of the data are given in Table 1. Here is the nominal instance (entries are rounded to 4 digits after the dot):

$$\Sigma = \left[\begin{array}{rrr|r} -0.5005 & -50.0000 & -0.4995 & 50.0000 \\ 0.1000 & -101.1000 & 0.0000 & 100.0000 \\ -0.4995 & -50.0000 & -0.5005 & 50.0000 \\ \hline 0 & -100.0000 & 0 & 100.0000 \end{array}\right].$$

The elements of the matrix $\Sigma_{c,r}$ are nonlinear functions of the "physical data" c, r, so that an interval uncertainty in the latter data is not equivalent to a box uncertainty in $\Sigma_{c,r}$. We neglect this phenomenon by linearizing $\Sigma_{r,c}$ at the nominal data, thus

Table 1
Nominal values for the bridge circuit.

Element	Nominal value	Element	Nominal value
R_{12}	1.2	C_{12}	1.0
R_{23}	1.0	C_{23}	1.0
R_{34}	1.0	C_{34}	1.0
R_{41}	1.0	C_{41}	1.0
R_{2A}	100	C_{13}	1000

arriving at a box uncertainty set with $L = 9$ basic perturbation matrices, according to the number of uncertain capacitances and conductances in the circuit. Note that for our particular circuit, the resulting uncertainty affects only the $[A, B]$-part of Σ, and the basic perturbation matrices $[dA_\ell, dB_\ell]$ are of rank 1.

Recall that the supply in the SISO positive-real case is $2yu$, i.e.,

$$\mathfrak{P} = \left[\begin{array}{cc} Q = 0 & L = 1 \\ L^T = 1 & R = 0 \end{array}\right];$$

for our RC circuit, the supply is nothing but (twice) the electrical power pumped into the circuit by the external voltage.

We have carried out two experiments with the outlined system: the first deals with extracting the energy stored in the circuit, and the second with moving the circuit from the zero initial state to a given state.

Extracting available energy. The question we are addressing is to find the largest level $\rho^\star_{av}$ of uncertainty for which the "performance" Θ of the "ideal extracting feedback" $\mathbf{F}_{av}$ (see D.4) corresponding to the nominal instance is at least $1 - \epsilon$, i.e., this feedback still allows, for every perturbed instance and every initial state ζ of the circuit, to extract at least $(1 - \epsilon)$-part of the nominal available storage $\zeta^T \mathbf{Z}_{av} \zeta$. In our experiment, we set $\epsilon = 0.1$. Solving the conservative approximation

$$\max_{\rho, G, H, \{X_\ell, Y_\ell\}} \{\rho : \ (\rho, G, H, \{X_\ell, Y_\ell\}) \text{ satisfies } (36)\}$$

of the associated Problem 2A, we end up with a lower bound

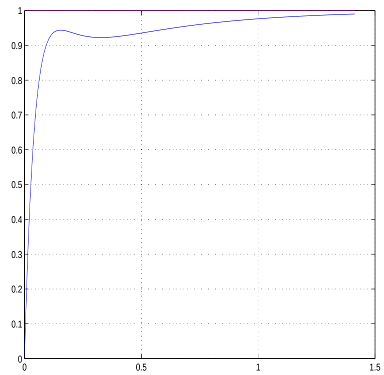
$$\widehat{\rho} = 1.1\text{e}{-3}$$

on $\rho^\star_{av}$; in other words, we can be sure that with 0.11% perturbations of the uncertain capacitances and conductances, the nominal feedback $\mathbf{F}_{av}$ still allows us to extract at least 90% of the nominal available storage, whatever is the initial state of the circuit. A natural question arises, How conservative is our bound? Recall that there are two reasons for it to be conservative:

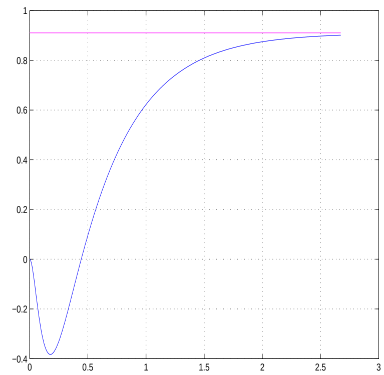
- First, the bound comes from solving a conservative approximation of Problem 2A rather than from solving the problem itself; according to Proposition 4.3, the true optimal value in the problem is at most $\frac{\pi}{2}$ times larger than the bound (recall that we are in the situation of $Q = 0$ and $\mu = 2$).
- Second, and worse, even the true optimal value in Problem 2A is a lower bound on $\rho^\star_{av}$, since the problem comes from the *sufficient* condition, stated by Proposition 3.2, for "good" performance of the nominal feedback $\mathbf{F}_{av}$ under data perturbations. Note that we have no idea how conservative this sufficient condition is.

TABLE 2
Performance of the nominal feedback $\mathbf{F}_{av}$ versus uncertainty level.

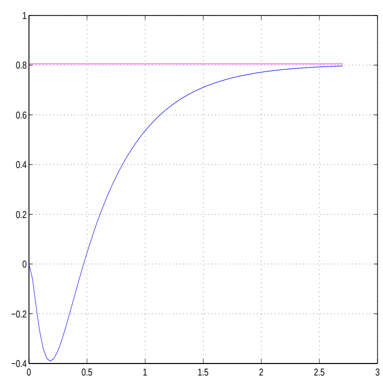
ρ	$1.2\widehat{\rho} = 1.3\mathtt{e}{-3}$	$2.2\widehat{\rho} = 2.3\mathtt{e}{-3}$	$3\widehat{\rho} = 3.2\mathtt{e}{-3}$
Θ	0.893	0.805	0.736



$\rho = 0$
$\Theta = 1$



$\rho = \widehat{\rho} = 1.1\mathtt{e}{-3}$
$\Theta = 0.910$



$\rho = 2.2\widehat{\rho} = 2.3\mathtt{e}{-3}$
$\Theta = 0.805$

FIG. 2. *Sample plots of* $\frac{E_{av}(t)}{z^T(0)\mathbf{Z}_{av}z(0)}$.

In spite of these pessimistic considerations, the experiment shows that our bound is pretty tight. Looking through all $2^L = 512$ "extreme" perturbations of the data, and playing with the initial state of the circuit, we found out that the worst-case (with respect to relative perturbations of the uncertain entries in c, r of level ρ and initial states) performance Θ of the ideal nominal feedback is *at most* as given in Table 2. In particular, we see that with the level of perturbations $1.2\widehat{\rho}$, the worst-case performance of the ideal nominal feedback is less than 0.9 ($\equiv 1 - \epsilon$) times the nominal available storage. It follows that $\rho^\star_{av} \leq 1.2\widehat{\rho}$, i.e., our bound $\widehat{\rho}$ is within 20% margin of the quantity of interest.

Figure 2 represents three sample plots of the extracted energy $E_{av}(t)$ as a function of time for the feedback $\mathbf{F}_{av}$.

TABLE 3
Price of the nominal feedback $\mathbf{F}_{req}$ versus uncertainty level.

ρ	$1.2\widehat{\rho} = 5.5\mathtt{e}{-4}$	$2.2\widehat{\rho} = 1.0\mathtt{e}{-3}$	$3\widehat{\rho} = 1.4\mathtt{e}{-3}$
Γ	1.056	1.105	1.148

Moving the circuit to a given state. Now let us try to find the largest uncertainty level $\rho^{\star}_{req}$ for which the "price" Γ of the "ideal driving feedback" $\mathbf{F}_{req}$ (see D.4) corresponding to the nominal instance is at most $1+\delta$, i.e., this feedback still allows, for every perturbed instance and every target state ζ of the circuit, to move the circuit from the zero initial state to the state ζ while pumping into the circuit at most $(1+\delta)$ times the nominal required energy $\zeta^T\mathbf{Z}_{req}\zeta$. In our experiment, we set $\delta = 0.1$. Solving the conservative approximation of the associated Problem 3A (see the end of section 4.3), we end up with a lower bound

$$\widehat{\rho} = 4.6\mathtt{e}{-4}$$

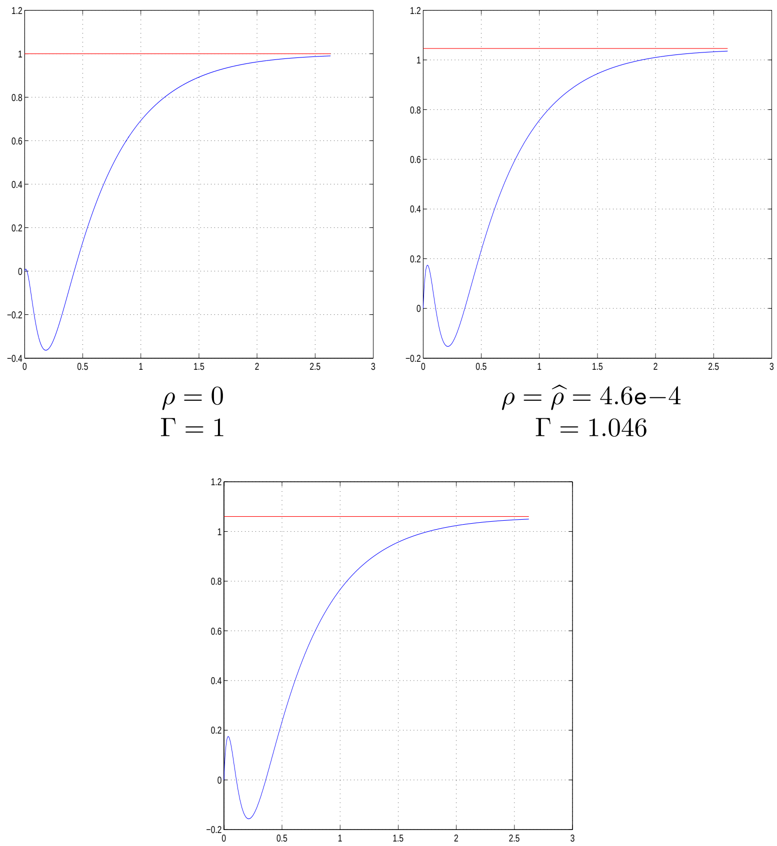
on $\rho^{\star}_{req}$; thus, we can be sure that with 0.046% perturbations of the uncertain capacitances and conductances, the ideal nominal feedback $\mathbf{F}_{req}$ still allows us to move the circuit from the zero state to (any) target one while pumping into the circuit at most 110% of the nominal required energy. It turns out that our bound is perhaps not as tight as in the previous case, but still is good enough. Indeed, looking at the data in Table 3, which represent *lower* bounds on the price of the ideal nominal driving feedback $\mathbf{F}_{req}$ under different levels of perturbations, we see that with the perturbations of the level $2.2\widehat{\rho}$ the price of moving the circuit to certain target state ζ by the feedback $\mathbf{F}_{req}$ can be larger than 1.1 ($\equiv 1+\delta$) times the nominal required energy $\zeta^T\mathbf{Z}_{req}\zeta$; hence $\rho^{\star}_{req} \leq 2.2\widehat{\rho}$. Note that, in the case in question, the conservative approximation of Problem 3A contributes to the ratio $\rho^{\star}_{req}/\widehat{\rho} \approx 2.2$ a factor $\leq \frac{\pi}{2} = 1.57$; the remaining factor in the ratio (which is at least $2.2/1.57 \approx 1.4$) comes from the conservativeness of the sufficient condition expressed in Proposition 3.3 and underlying Problem 3A.

Figure 3 presents three sample plots of the pumped energy $E_{req}(t)$ as a function of time for the feedback $\mathbf{F}_{req}$.

5.2. Linear-quadratic case. Consider the mechanical system shown on Figure 4; it consists of 5 material points in a two-dimensional plane linked to each other by elastic springs as shown on the figure; the points can slide without friction along the respective axes $01,\dots,05$. The nominal data for the system are given in Table 4. The system is controlled by two external forces acting at the masses 1 and 5. The first 5 components of the state vector are the shifts x_i of the points from their equilibrium positions along the lines of motion, and the next 5 components are the linear velocities $\dot{x}_i$ of the points; these velocities are the outputs of the system. With respect to these states, the dynamical system in question is

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} &= \left[\begin{array}{c|c} & I_5 \\ \hline -M^{-1}E & \end{array}\right]\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + Bu, \\ y &= \dot{x}, \end{aligned} \tag{58}$$

where M is the diagonal matrix with the masses $m(i)$ of the points as the diagonal entries, E is the *stiffness matrix* readily given by the rigidities of the springs and the equilibria positions of the points, and B is the 10×2 matrix with two nonzero



$\rho = 0$
$\Gamma = 1$

$\rho = \widehat{\rho} = 4.6\text{e}{-4}$
$\Gamma = 1.046$

$\rho = 2.2\widehat{\rho} = 1.0\text{e}{-3}$
$\Gamma = 1.105$

FIG. 3. *Sample plots of* $\frac{E_{req}(t)}{z^T(0)\mathbf{Z}_{req}z(0)}$.

entries $B_{5,1} = m^{-1}(1)$ and $B_{10,2} = m^{-1}(5)$. Here is the nominal instance (entries are rounded to 4 digits after the dot):

$$\Sigma = \left[\begin{array}{cccccccccc|cc}
 & & & & & 1 & & & & & & \\
 & & & & & & 1 & & & & & \\
 & & & & & & & 1 & & & & \\
 & & & & & & & & 1 & & & \\
 & & & & & & & & & 1 & & \\
-2.5647 & -1.0797 & & & -1.0890 & & & & & & & 1.9637 \\
-0.6038 & -0.8206 & -0.4766 & & & & & & & & & \\
 & -0.6009 & -1.5044 & -0.4808 & & & & & & & & \\
 & & -0.4300 & -1.1142 & -0.5131 & & & & & & & \\
-0.6190 & & & -0.4626 & -0.8352 & & & & & & 1.1161 & \\
\hline
 & & & & & 1 & & & & & & \\
 & & & & & & 1 & & & & & \\
 & & & & & & & 1 & & & & \\
 & & & & & & & & 1 & & & \\
 & & & & & & & & & 1 & &
\end{array}\right].$$

We are interested to bring the system from the equilibrium to a given state while

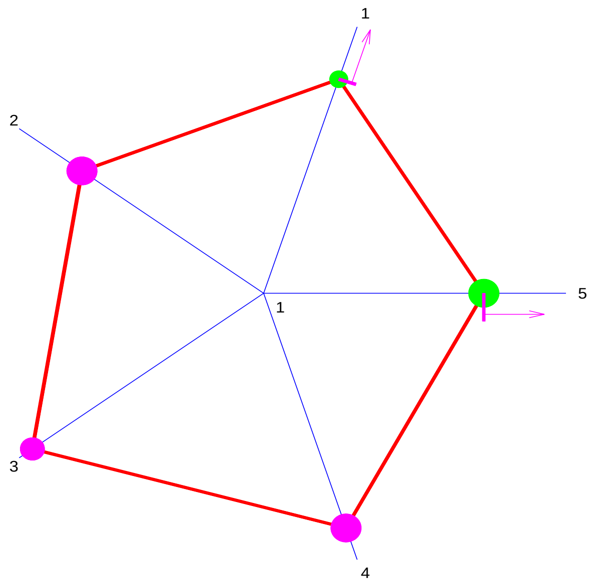


FIG. 4. 5 *masses linked by elastic springs*

TABLE 4
The nominal data.

Point	Mass	Distance to the origin at equilibrium	Spring	Rigidity
1	0.5093	0.8034	1 - 2	1.461
2	0.9107	0.7430	2 - 3	1.369
3	0.7224	0.9456	3 - 4	1.088
4	0.8077	0.8810	4 - 5	1.203
5	0.8960	0.7282	5 - 1	1.468

minimizing the cost functional

$$\int_0^\infty \left[\sum_{i=1}^{5} (\dot{x}_i)^2(t) + \sum_{i=1}^{2} u_i^2(t) \right] dt,$$

which is equivalent to the providing required supply problem with the supply matrix

$$\mathfrak{P} = \left[\begin{array}{c|c} Q = I_5 & L = 0_{5\times 2} \\ \hline L^T = 0_{2\times 5} & R = I_2 \end{array} \right].$$

In our experiment, we treat as uncertain parameters the masses of the points and the rigidities of the springs and assume that every one of these parameters can vary, independently of others, by at most ρ times the nominal value of the parameter. Note that the perturbations affect only the $[A, B]$-part of the matrix Σ of the system and that the dependence of Σ on the masses and rigidities is nonlinear (although both M and E in (58) are affine in the parameters). As in the previous example, we neglect this phenomenon by linearizing Σ at the nominal data, and end up with a box uncertainty set with $L = 10$ basic perturbation matrices, according to the number of uncertain parameters; all these perturbation matrices turn out to be of rank 1. The outlined model underlies two numerical experiments we are about to report.

Designing robust feedback with "nearly optimal" performance. For the nominal system, there exists the ideal state feedback $u = \mathbf{F}z$ which moves the system from the equilibrium to (any) given initial state ζ at the minimum possible cost $\zeta^T \mathbf{Z}_{req} \zeta$. What we are interested in now is to find the largest uncertainty level for which there still exists an instance-independent state feedback with a given *performance index* $1+\delta$; the latter means that the feedback allows to move every instance of the perturbed system from the equilibrium to (any) given state ζ at the cost at most $(1+\delta)$ times the "ideal nominal cost" $\zeta^T \mathbf{Z}_{req} \zeta$. In our experiment, we set $\delta = 0.1$ and get the desired feedback by solving the conservative approximation (50) of Problem 3C associated with the outlined model. As a result, we get

(a) state feedback with the matrix

$$F = \begin{bmatrix} -0.0396 & 0.0220 & -0.3685 & -0.8069 & -0.4099 & 0.0152 & -0.3694 & 0.0647 & -0.0498 & 1.3167 \\ -0.3993 & -0.6453 & -0.4886 & -0.2269 & -0.0322 & 1.1859 & -0.5896 & -0.2165 & -0.3263 & 0.0268 \end{bmatrix},$$

which is slightly different from the ideal nominal feedback

$$\mathbf{F} = \begin{bmatrix} -0.0281 & 0.0289 & -0.4196 & -0.8948 & -0.4551 & 0.0063 & -0.3897 & 0.0628 & -0.0558 & 1.3826 \\ -0.4467 & -0.7133 & -0.5466 & -0.2423 & -0.0311 & 1.2269 & -0.6375 & -0.2570 & -0.3520 & 0.0111 \end{bmatrix},$$

and

(b) the "safe" uncertainty level $\widehat{\rho} = 0.0048$, which is a lower bound on the optimal value $\rho^\star_{3C}$ in Problem 3C.

What we know about F and $\widehat{\rho}$ from their origin is the following:

- The performance index of the state feedback $u = Fz$ is no worse than $1+\delta$, provided that the level of perturbations does not exceed 0.48% (which is our $\widehat{\rho}$). Note that this statement remains true even for dynamical perturbations.
- The true optimal value $\rho^\star_{3C}$ in Problem 3C is at most $\frac{\pi}{2}$ times larger than $\widehat{\rho}$ (see Proposition 4.5; note that our basic perturbation matrices are of rank 1, so that the quantity μ in (52) equals 2 by item C of Section 4.6).

What we are interested in now is how conservative are our results, specifically, what is the actual value of the ratio $\rho^\star_{3C}/\widehat{\rho}$. An even more important question is as follows. The optimal value $\rho^\star_{3C}$ of Problem 3C is itself no more than a lower bound on the supremum $\rho^\star$ of those perturbation levels for which there still exists a state feedback with performance index $1+\delta = 1.1$ (since what underlies Problem 3C is no more than a sufficient condition for good performance under uncertainty). How large is the ratio $\rho^\star/\widehat{\rho}$, or, in other words, how far is the robustness of our feedback F from the "ideal" robustness compatible with the prescribed performance index 1.1? It turns out that the answers to these questions are quite assuring. Indeed, looking at a large enough number of randomly perturbed instances with different perturbation levels and computing the required supply for these instances, one can find out that already at the perturbation level $1.2\widehat{\rho} = 0.0058$ there exist perturbed instances Σ and target states ζ such that Σ cannot be moved from the equilibrium to the state ζ at the cost $\leq 1.1\zeta^T \mathbf{Z}_{req} \zeta$. It follows that

$$\rho^\star_{3C} \leq \rho^\star < 1.2\widehat{\rho},$$

which is much better than we could expect.

Lyapunov stability analysis. Here we use the data yielded by the previous experiment for illustrating another application of the proposed approach, namely, estimating the level of perturbations which keep the closed-loop system stable. This problem was the subject of Example 4 in section 3.1, where it was shown that the problem can be posed as the one of finding the supremum of those uncertainty levels for which all perturbed instances of the system share a common dissipativity certificate. As our sample closed-loop system, we used the outlined mechanical system equipped with the state feedback F found in the previous experiment. Our uncertainty model for the matrix

$$\widehat{A} = A + BF$$

of the closed-loop system is as follows: we use the aforementioned "physical" model of perturbations in $[A, B]$ and assume, in addition, that the entries in F also are subject to perturbations. Since we have no physical model of the controller, we assume that the entries F_{ij} in F can vary, independently of each other (and independently of the perturbations in $[A, B]$), in the intervals $[F_{ij}^{\mathrm{c}} - \rho|F_{ij}^{\mathrm{c}}|, F_{ij}^{\mathrm{c}} + \rho|F_{ij}^{\mathrm{c}}|]$, where ρ is the uncertainty level, and F_{ij}^{c} are the "nominal" values as computed in the previous experiment.

As in the previous cases, we linearized the dependence of $\widehat{A}$ on the perturbations, thus arriving at a box model of perturbations in the matrix of the closed-loop system. Then we solved the conservative approximation (31) of Problem 1 associated with system (14) and the supply matrix (15). Since we were interested solely in the stability of the closed-loop system under perturbations and did not care of any kind of performance, we looked for the common dissipativity certificate Z in a pretty wide "matrix interval" $\mathcal{I} = \{Z : 10^{-7}\mathbf{Z} \preceq Z \preceq \mathbf{Z}\}$, which in the situation of Example 4 basically means that we do not impose restrictions on Z except for being positive definite.

The results of our experiment are as follows. The solution of (31) yields a level of perturbations $\widehat{\rho} = 0.041$ and a positive definite matrix Z, which is a common Lyapunov stability certificate for all perturbed instances of the matrix $\widehat{A}$ of the closed-loop system when the level of perturbations is $\widehat{\rho}$. Thus, we can be sure that the closed-loop system remains stable whatever are 4.1% perturbations of the physical parameters of our mechanical system and 4.1% perturbations of the coefficient in the feedback matrix, even when these perturbations are dynamical. A natural question is, How conservative is this conclusion? Note that, a priori, there is no reason to be too optimistic in this respect, since the existence of a common Lyapunov stability certificate, as a sufficient condition for stability, may be quite conservative already by itself, and we are dealing with conservative approximation of this condition. However, the experiment demonstrates that we are lucky: simulating about 1,000 random perturbations of the closed-loop system at different uncertainty levels, it turns out that at the uncertainty level $1.6\widehat{\rho} = 0.065$ there already exist perturbations which make the closed-loop system unstable. Thus, the closed-loop system definitely survives perturbations not exceeding 4.1% and can be crushed by 6.5% perturbations.

6. Conclusions. We have developed techniques for specifying the magnitudes of dynamic perturbations in the parameters of a linear system which preserve a desired property of the system (such as positive-realness, nonexpansiveness, etc.). The standard sufficient condition for this is the solvability of an associated *infinite* system $\mathcal{S}$ of linear matrix inequalities. The latter condition, however, is usually NP-hard to verify, so that one is forced to look for *efficiently verifiable* sufficient conditions for $\mathcal{S}$ to be solvable. We propose such a condition and demonstrate that in many cases it is *provably tight, within an absolute constant factor*, $\frac{\pi}{2}$ in most cases (for details, see

Propositions 4.2, 4.3, 4.4, 4.5). This "guaranteed tightness" is a specific (and, to the best of our knowledge, unique) feature of the paper.

Recently, it turned out that the matrix cube theorem, which underlies all our developments, can be extended to the complex case and even with a model of uncertainty richer than the interval one. These extensions could then imply corresponding extensions of the results we have presented here.

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