# Assignment 1: LO Models

**Exercise 1** 1) Draw the feasible set of the LO program

$$\begin{array}{ll}\max\limits_{x_1,x_2} & x_2 \\ \text{s.t.} & \\ & x_1 - 2x_2 \leq 0 \\ & 2x_1 - 3x_2 \leq 2 \\ & x_1 - x_2 \leq 3 \\ & -x_1 + 2x_2 \leq 2 \\ & -2x_1 + x_2 \leq 0 \end{array}$$

and find the optimal value and an optimal solution to the problem.

2) Now assume that the objective is replaced with $\cos(\phi)x_1 + \sin(\phi)x_2$, where $\phi$ is chosen at random, according to the uniform distribution on $[0, 2\pi]$. What is the probability to get, as an optimal solution, the same point as in the original problem?

**Exercise 2** [rucksack problem] There are $n$ goods available to you; the maximal available volume of good $j$ is $v_j \geq 0$, and the value of good $j$ per unit of volume is $c_j \geq 0$. You have a rucksack of volume $v$ and want to fill it with goods to get as large total value of the rucksack as possible. Build an LO model of the resulting problem and present an elementary scheme for generating optimal solution.

**Exercise 3** A 24/7 calling center works as follows: every agent works 5 days in a row and has two days rest, e.g., every week works Tuesday-Saturday and rests on Sunday and Monday. The numbers of agents working every day of a week should be at least given numbers $d_1, ..., d_7$. The manager wants to meet this requirement with the minimal possible total number of agents employed, by deciding what will be the days off of the agents. Assuming that $d_i$ are large, so that we can ignore integrality restrictions, formulate manager's problem as an LO program.

**Exercise 4** The water supply system in a town includes pump station, tank and a distribution network. At every given hour, the pump station can pump the water partly into tank, and partly – directly into the distribution network. To pump a gallon of water, it takes one unit of electrical energy, and the cost of energy consumed in hour $t$ is $c_t$ dollars per unit (usually, for night hours $c_t$ is less than for day hours). The demand for water in hour $t$, $0 \leq t \leq 23$, is $d_t$ gallons, and this demand can be partly satisfied from the tank, and partly – from the station, no matter what are the parts. At the beginning of hour 0, the tank is empty, same as it should be empty at the end of hour 23. The capacity of the tank is $C$ gallons. We want to decide how much water should be pumped every hour $t$, $0 \leq t \leq 23$, into the network and into the tank in order to meet the demand at the lowest possible energy cost. Build an LO model of the situation.

**Exercise 5** Run experiment as follows:

1. Pick at random in the segment $[0, 1]$ two "true" parameters $\theta_0^*$ and $\theta_1^*$ of the regression model

$$y = \theta_0 + \theta_1 x;$$

2. Generate a sample of $N = 1000$ observation errors $\xi_i \sim P$, where $P$ is a given distribution, and then generate observations $y_i$ according to

$$y_i = \theta_0^* + \theta_1^* x_i + \xi_i, \; x_i = i/N;$$

3. Estimate the parameters from the observations according to the following three estimation schemes:

$$\begin{array}{ll}
\text{uniform fit}: & [\theta_{0,\infty};\theta_{1,\infty}] = \text{argmin}_\theta \max\limits_{1\le i\le N} |y_i - [\theta_0 + \theta_1 x_i]| \\
\text{least squares fit}: & [\theta_{0,2};\theta_{1,2}] = \text{argmin}_\theta \sum\limits_{1\le i\le N} (y_i - [\theta_0 + \theta_1 x_i])^2 \\
\ell_1 \text{ fit}: & [\theta_{0,1};\theta_{1,1}] = \text{argmin}_\theta \sum\limits_{1\le i\le N} |y_i - [\theta_0 + \theta_1 x_i]|
\end{array}$$

   and compare the estimates with the true values of the parameters.

Run 3 series of experiments:

- $P$ is the uniform distribution on $[-1, 1]$;
- $P$ is the standard Gaussian distribution with the density $\frac{1}{\sqrt{2\pi}} \exp\{-t^2/2\}$;
- $P$ is the Cauchy distribution with the density $\frac{1}{\pi(1+t^2)}$.

Try to explain the results you get. When doing so, you can think about a simpler problem, where you are observing $N$ times a scalar parameter $\theta^*$ according to $y_i = \theta^* + \xi_i$, $1 \le i \le N$, and then use the above techniques to estimate $\theta^*$.

# Assignment 2: What can be reduced to LO

**Exercise 6** In the below list, mark by **P** the polyhedral sets, and by **PR** – the polyhedral representations. For those polyhedral sets which in the below list are *not* given by polyhedral representations, point out their polyhedral representations.

1. $X = \{[x_1; x_2] : x_1 + x_2 \leq 0\}$
2. $X = \{[x_1; x_2] : \max[x_1, x_2] \leq 0\}$
3. $X = \{[x_1, x_2] : \max[x_1, x_2] \geq 0\}$
4. $X = \{[x_1; x_2] : \min[x_1, x_2] \leq 0\}$
5. $X = \{[x_1; x_2] : \min[x_1, x_2] \geq 0\}$
6. $X = \{[x_1; x_2] : x_1^2 + x_2^2 \leq 1\}$
7. $X = \{[x_1; x_2] : x_1^2 + x_2^2 \leq -1\}$
8. $X = \{[x_1; x_2] : -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1, x_1^2 + x_2^2 \leq 1\}$
9. $X = \{[x_1; x_2] : -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1, x_1^2 + x_2^2 \leq 2\}$
10. $X = \{[x_1; x_2] : |x_1| + |x_2| \leq 1\}$
11. $X = \{[x_1; x_2] : |x_1| - |x_2| \leq 1\}$
12. $X = \{[x_1; x_2] : |x_1| + x_2 \leq 1\}$
13. $X = \{[x_1; x_2] : |x_1| - x_2 \leq 1\}$
14. $X = \{[x_1; x_2] : x_1 - |x_2| \leq 1\}$
15. $X = \{[x_1; x_2] : -x_1 - |x_2| \leq 1\}$

**Exercise 7** Represent the projection $X$ of the polyhedral set

$$Y = \{x \in \mathbf{R}^3 : -1 \leq x_1 + x_2 \leq 1, -1 \leq x_2 + x_3 \leq 1, -1 \leq x_1 + x_3 \leq 1, x_1 + x_2 + x_3 \leq 2\}$$

onto the $x_1, x_2$-plane by a system of linear inequalities in the variables $x_1, x_2$.

**Exercise 8** In the below list, mark by **P** the polyhedrally representable functions, and build their polyhedral representations.

1. $f(x_1, x_2) \equiv 0$
2. $f(x_1, x_2) = x_1 - x_2$
3. $f(x_1, x_2) = \max[x_1, x_2]$
4. $f(x_1, x_2) = \min[x_1, x_2]$
5. $f(x_1, x_2) = 1 - \max[x_1, x_2]$
6. $f(x_1, x_2) = 1 - \min[x_1, x_2]$

7. $f(x_1, x_2) = \begin{cases} \max[x_1, x_2], & \max[x_1, x_2] \leq 1 \\ +\infty, & \text{otherwise} \end{cases}$

8. $f(x_1, x_2) = \begin{cases} \max[x_1, x_2], & \min[x_1, x_2] \leq 1 \\ +\infty, & \text{otherwise} \end{cases}$

9. $f(x_1, x_2) = \begin{cases} \min[x_1, x_2], & x_2 \geq 2, x_1 \leq 0 \\ +\infty, & \text{otherwise} \end{cases}$

10. $f(x_1, x_2, x_3) = \max[x_1, x_2] + \max[x_1, x_3]$

11. $f(x_1, x_2, x_3) = \max[x_1, x_2] - \max[x_1, x_3]$

12. $f(x_1, x_2, x_3) = \max[x_1, x_2] + \min[x_1, x_3]$

13. $f(x_1, x_2, x_3) = \max[x_1, x_2] - \min[x_1, x_3]$

14. $f(x_1, x_2) = \max[x_1 + \max[x_2, x_3], x_3 + \max[x_1, x_2]]$

15. $f(x_1, x_2) = \max[|x_1|, |x_2|]$

16. $f(x_1, x_2) = |\max[x_1, x_2]|$

**Exercise 9** In the below list, some problems can be posed as LO programs. Identify these problems and reformulate them as LO programs.

1. $$\min_{x_1, x_2} \{\max[|2x_1 + 3x_2|, |x_1 - x_2|] : |x_1| + 2\max[x_1, x_2] \leq 1\}$$

2. $$\max_{x_1, x_2} \{\max[|2x_1 + 3x_2|, |x_1 - x_2|] : |x_1| + 2\max[x_1, x_2] \leq 1\}$$

3. $$\max_{x_1, x_2} \{2\min[x_1 + x_2, 2x_2] - |x_1 - x_2| : \max[|x_1 - 2x_2|, |x_2|] - 2x_2 \leq 1 - |x_1|\}$$

# Assignment 3: Linear and Affine subspaces, Convexity

**Exercise 10** In the below list, mark the sets which are linear subspaces and point out their dimensions.

1. $\mathbf{R}^n$
2. $\{0\}$
3. $\emptyset$
4. $\{x \in \mathbf{R}^n : \sum_{i=1}^{n} i x_i = 0\}$
5. $\{x \in \mathbf{R}^n : \sum_{i=1}^{n} i x_i^2 = 0\}$
6. $\{x \in \mathbf{R}^n : \sum_{i=1}^{n} i x_i = 1\}$
7. $\{x \in \mathbf{R}^n : \sum_{i=1}^{n} i x_i^2 = 1\}$

**Exercise 11** Point out a linear and an affine bases in the linear subspace

$$\{x \in \mathbf{R}^n : \sum_{i=1}^{n} x_i = 0\}$$

and the orthogonal complement to this subspace.

**Exercise 12** In the below list, mark sets which are affine subspaces and point out their affine dimensions.

1. $\mathbf{R}^n$
2. $\{a\}$
3. $\emptyset$
4. $\{x \in \mathbf{R}^n : \sum_{i=1}^{n} i x_i = 0\}$
5. $\{x \in \mathbf{R}^n : \sum_{i=1}^{n} i x_i^2 = 0\}$
6. $\{x \in \mathbf{R}^n : \sum_{i=1}^{n} i x_i = 1\}$
7. $\{x \in \mathbf{R}^n : \sum_{i=1}^{n} i x_i^2 = 1\}$

**Exercise 13** Point out the linear subspace parallel to the affine subspace

$$M = \{x \in \mathbf{R}^n : \sum_{i=1}^{n} x_i = 1\} \subset \mathbf{R}^n,$$

and an affine basis in $M$.

**Exercise 14** In the below list, point out the dimensions of the sets and mark those sets which are convex.

1. $\mathbf{R}^n$
2. $\{0\}$
3. $\{x \in \mathbf{R}^n : \sum_{i=1}^n ix_i = 0\}$
4. $\{x \in \mathbf{R}^n : \sum_{i=1}^n ix_i \leq 0\}$
5. $\{x \in \mathbf{R}^n : \sum_{i=1}^n ix_i \geq 0\}$
6. $\{x \in \mathbf{R}^n : \sum_{i=1}^n ix_i^2 = 1\}$
7. $\{x \in \mathbf{R}^n : \sum_{i=1}^n ix_i^2 \leq 1\}$
8. $\{x \in \mathbf{R}^n : \sum_{i=1}^n ix_i^2 \geq 1\}$
9. $\{x \in \mathbf{R}^2 : |x_1| + |x_2| \leq 1\}$
10. $\{x \in \mathbf{R}^2 : |x_1| - |x_2| \leq 1\}$
11. $\{x \in \mathbf{R}^2 : -|x_1| - |x_2| \leq 1\}$

**Exercise 15** For the sets to follow, point out their linear and affine spans and their convex hulls:

1. $X = \{[0; 1], [1; 1], [2; 1]\}$
2. $X = \{[0; 0]; [1; 0]; [1; 1]; [0; 1]\}$
3. $X = \{x \in \mathbf{R}^2 : x_2 = 0, |x_1| \leq 1\}$
4. $X = \{x \in \mathbf{R}^2 : x_2 = 1, |x_1| \leq 1\}$
5. $X = \{x \in \mathbf{R}^2 : |x_1| - |x_2| = 1\}$

**Exercise 16** Mark by **T** those of the following claims which always are true:

1. The linear image $Y = \{y = Ax : x \in X\}$ of a linear subspace $X$ is a linear subspace
2. The linear image $Y = \{y = Ax : x \in X\}$ of an affine subspace $X$ is an affine subspace
3. The linear image $Y = \{y = Ax : x \in X\}$ of a convex set $X$ is convex
4. The affine image $Y = \{y = Ax + b : x \in X\}$ of a linear subspace $X$ is a linear subspace
5. The affine image $Y = \{y = Ax + b : x \in X\}$ of an affine subspace $X$ is an affine subspace
6. The affine image $Y = \{y = Ax + b : x \in X\}$ of a convex set $X$ is convex
7. The intersection of two linear subspaces in $\mathbf{R}^n$ is a linear subspace
8. The intersection of two affine subspaces in $\mathbf{R}^n$ is an affine subspace

9. The intersection of two affine subspaces in $\mathbf{R}^n$, when nonempty, is an affine subspace
10. The intersection of two convex sets in $\mathbf{R}^n$ is a convex set
11. The intersection of two convex sets in $\mathbf{R}^n$, when nonempty, is a convex set

**Exercise 17** Given are $n$ distinct from each other sets

$$E_1 \subset E_2 \subset ... \subset E_n$$

in $\mathbf{R}^{100}$. How large can be $n$, if

1. every one of $E_i$ is a linear subspace
2. every one of $E_i$ is an affine subspace
3. every one of $E_i$ is a convex set

# Assignment 4: Extreme points

**Exercise 18** Let $X$ be a nonempty convex set in $\mathbf{R}^n$ and $x \in X$. Prove that $x$ is an extreme point of $X$

1. if and only if the set $X \backslash \{x\}$ is convex
2. if and only if for every representation $x = \sum_{i=1}^{k} \lambda_i x_i$ of $x$ as a convex combination of points from $X$ with positive coefficients $\lambda_i$ it holds $x_i = x$, $i = 1, ..., k$

**Exercise 19** Let $X = \{x \in \mathbf{R}^n : a_i^T x \le b_i,\ 1 \le i \le m\}$ be a polyhedral set. Prove that

1. If $X'$ is a face of $X$, then there exists a linear function $e^T x$ such that
$$X' = \operatorname*{Argmax}_{x \in X} e^T x := \{x \in X : e^T x = \sup_{x' \in X} e^T x'\}.$$
2. If $v$ is a vertex of $X$, then there exists a linear function $e^T x$ such that $v$ is the unique maximizer of this function on $X$.

**Exercise 20** Describe all extreme points of the following convex sets:

1. $X = \mathrm{Conv}\{1, 2, 3, 4, 5\}$
2. $X = \mathrm{Conv}\{[0; 0], [1; 1], [1; 0], [0.5; 0.5]\}$
3. $X = \{x \in \mathbf{R}^n : 0 \le x_i \le 1,\ 1 \le i \le n, \sum_{i=1}^{n} x_i \le 3/2\}$
4. $X = \{x \in \mathbf{R}^n : \|x\|_2 \le 1\}$
5. $X = \{x \in \mathbf{R}^n : x \ge 0, \sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} a_i x_i = 1\}$, where $a_1 < a_2 < ... < a_n$.

**Exercise 21** Point out the recessive cones and the extreme points of the polyhedral sets

1. $X = \{x \in \mathbf{R}^3 : x_1 \ge 0, x_2 \ge 0, x_3 \ge 0\}$
2. $X = \{x \in \mathbf{R}^2 : x_1 \ge 0, x_2 \ge 0, x_1 + x_2 \le 2\}$
3. $X = \{x \in \mathbf{R}^3 : x_1 \ge 0, x_2 \ge 0, x_1 + x_2 \le 2\}$
   Think what is the general form of the result on $\mathrm{Ext}(X)$ you got.
4. $X = \{x \in \mathbf{R}^3 : x_1 \ge 0, x_2 \ge 0, x_1 + x_2 \le 2, x_3 \ge 0\}$

# Assignment 5: Cones and Structure of Polyhedral Sets

**Exercise 22** Prove that if a nonempty polyhedral set $X$ is represented as

$$X = \mathrm{Conv}(\{v_1, ..., v_N\}) + \mathrm{Cone}\,(\{r_1, ..., r_M\}),$$

then $\mathrm{Rec}(X) = \mathrm{Cone}\,(\{r_1, ..., r_M\})$
Hint: Assuming that a recessive direction $e$ of $X$ does not belong to $\mathrm{Cone}\,(\{r_1, ..., r_M\})$, use the Homogeneous Farkas Lemma to verify that there exists $d$ such that $d^T e < 0$ and $d^T r_j \geq 0$ for all $j$, and think of whether the linear function $d^T x$ of $x$ is bounded below on $X$.

**Exercise 23** Prove that if $K$ is a polyhedral cone, then the dual cone $K_*$ is so, and $(K_*)_* = K$.

**Exercise 24** Prove that if $K = \{x \in \mathbf{R}^n : a_i^T x \geq 0, i = 1, ..., m\}$, then $K_* = \mathrm{Cone}\,(\{a_1, ..., a_m\})$.

**Exercise 25** Prove that if $K$ is a polyhedral cone and $d$ is a generator of an extreme ray of $K$, then in every representation

$$d = d_1 + ... + d_M,\ d_i \in K\ \forall i$$

$d_i$ are nonnegative multiples of $d$.

**Exercise 26** Let

$$K = \mathrm{Cone}\,(\{r_1, ..., r_M\}).$$

Prove that if $R$ is an extreme ray of $K$, then one of $r_j$ can be chosen as a generator of $R$. What is the "extreme point" analogy of this statement?

**Exercise 27** Let $K_1, ..., K_m$, $M_1, ..., M_m$ be polyhedral cones in $\mathbf{R}^n$. Prove that

1. $M_1 + ... + M_m$ is a polyhedral cone in $\mathbf{R}^n$.

2. $(K_1 \cap K_2 \cap ... \cap K_m)_* = (K_1)_* + ... + (K_m)_*$.

**Exercise 28** For the polyhedral cones to follow, point out a base (if it exists), extreme rays (if they exist) and a polyhedral representation of the dual cone (you may skip the derivations and present the results only).

1. $K = \{0\} \subset \mathbf{R}$
2. $K = \mathbf{R}$
3. $K = \{x \in \mathbf{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 \leq x_2\}$
4. $K = \{x \in \mathbf{R}^3 : x_1 + x_2 \geq 0, x_2 + x_3 \geq 0, x_1 + x_3 \geq 0\}$
5. $K = \{x \in \mathbf{R}^3 : x_1 + x_2 \geq x_3, x_2 + x_3 \geq x_1, x_1 + x_3 \geq x_2\}$

**Exercise 29** For the polyhedral sets $X$ to follow, find their representations in the form of

$$X = \mathrm{Conv}(\{v_1, ..., v_N\}) + \mathrm{Cone}\,(\{r_1, ..., r_M\})$$

(you may skip the derivation and present the results only)

1. $X = \{x \in \mathbf{R}^n : x \geq 0, \sum_i x_i \leq 1\}$
2. $X = \{x \in \mathbf{R}^n : x \geq 0, \sum_i x_i \geq 1\}$
3. $X = \{x \in \mathbf{R}^n : x \geq 0, 1 \leq \sum_i x_i \leq 2\}$
4. $X = \{x \in \mathbf{R}^3 : x \geq 0, x_1 + x_2 - x_3 \geq 0\}$.
5. $X = \{x \in \mathbf{R}^3 : x \geq 0, x_1 + x_2 - x_3 \geq 1\}$.

# Assignment 6: General Theorem on Alternative

**Exercise 30** 1) Prove Gordan Theorem on Alternative:

> *A system of strict homogeneous linear inequalities $Ax < 0$ in variables $x$ has a solution iff the system $A^T\lambda = 0, \lambda \geq 0$ in variables $\lambda$ has only trivial solution $\lambda = 0$.*

2) Prove Motzkin Theorem on Alternative:

> *A system $Ax < 0$, $Bx \leq 0$ of strict and nonstrict homogeneous linear inequalities has a solution iff the system $A^T\lambda + B^T\mu = 0, \lambda \geq 0, \mu \geq 0$ in variables $\lambda, \mu$ has no solution with $\lambda \neq 0$.*

**Exercise 31** For the systems of constraints to follow, write them down equivalently in the standard form $Ax < b, Cx \leq d$ and point out their solvability status ("solvable – unsolvable") along with the corresponding certificates.

1. $x \leq 0$ $(x \in \mathbf{R}^n)$
2. $x \leq 0$ & $\sum_{i=1}^n x_i > 0$ $(x \in \mathbf{R}^n)$
3. $-1 \leq x_i \leq 1$, $1 \leq i \leq n, \sum_{i=1}^n x_i \geq n$ $(x \in \mathbf{R}^n)$
4. $-1 \leq x_i \leq 1$, $1 \leq i \leq n, \sum_{i=1}^n x_i > n$ $(x \in \mathbf{R}^n)$
5. $-1 \leq x_i \leq 1$, $1 \leq i \leq n, \sum_{i=1}^n ix_i \geq \frac{n(n+1)}{2}$ $(x \in \mathbf{R}^n)$
6. $-1 \leq x_i \leq 1$, $1 \leq i \leq n, \sum_{i=1}^n ix_i > \frac{n(n+1)}{2}$ $(x \in \mathbf{R}^n)$
7. $x \in \mathbf{R}^2$, $|x_1| + x_2 \leq 1, x_2 \geq 0, x_1 + x_2 = 1$
8. $x \in \mathbf{R}^2$, $|x_1| + x_2 \leq 1, x_2 \geq 0, x_1 + x_2 > 1$
9. $x \in \mathbf{R}^4$, $x \geq 0$, sum of two largest entries in $x$ does not exceed 2, $x_1 + x_2 + x_3 \geq 3$
10. $x \in \mathbf{R}^4$, $x \geq 0$, sum of two largest entries in $x$ does not exceed 2, $x_1 + x_2 + x_3 > 3$

**Exercise 32** Let $(\mathcal{S})$ be the following system of linear inequalities in variables $x \in \mathbf{R}^3$:

$$x_1 \leq 1, x_1 + x_2 \leq 1, x_1 + x_2 + x_3 \leq 1 \qquad (\mathcal{S})$$

In the following list, point out which inequalities are, and which are not consequences of the system, and certify your claims as explained in examples in items 1 and 2.

1. $3x_1 + 2x_2 + x_3 \leq 4$
2. $3x_1 + 2x_2 + x_3 \leq 2$
3. $3x_1 + 2x_2 \leq 3$
4. $3x_1 + 2x_2 \leq 2$
5. $3x_1 + 3x_2 + x_3 \leq 3$
6. $3x_1 + 3x_2 + x_3 \leq 2$

Make a generalization: prove that a linear inequality $px_1 + qx_2 + rx_3 \leq s$ is a consequence of $(\mathcal{S})$ if and only if $s \geq p \geq q \geq r \geq 0$.

**Exercise 33** Is the inequality $x_1 + x_2 \leq 1$ a consequence of the system $x_1 \leq 1, x_1 \geq 2$? If yes, can it be obtained by taking a legitimate weighted sum of inequalities from the system and the identically true inequality $0^T x \leq 1$, as it is suggested by the Inhomogeneous Farkas Lemma?

**Exercise 34** Certify the correct statements in the following list:

1. The polyhedral set $X = \{x \in \mathbf{R}^3 : x \geq [1/3; 1/3; 1/3], \sum_{i=1}^3 x_1 \leq 1\}$ is nonempty.
2. The polyhedral set $X = \{x \in \mathbf{R}^3 : x \geq [1/3; 1/3; 1/3], \sum_{i=1}^3 x_1 \leq 0.99\}$ is empty.
3. The linear inequality $x_1 + x_2 + x_3 \geq 2$ is violated somewhere on the polyhedral set $X = \{x \in \mathbf{R}^3 : x \geq [1/3; 1/3; 1/3], \sum_{i=1}^3 x_i \leq 1\}$.
4. The linear inequality $x_1 + x_2 + x_3 \geq 2$ is violated somewhere on the polyhedral set $X = \{x \in \mathbf{R}^3 : x \geq [1/3; 1/3; 1/3], \sum_{i=1}^3 x_i \leq 0.99\}$.
5. The linear inequality $x_1 + x_2 \leq 3/4$ is satisfied everywhere on the polyhedral set $X = \{x \in \mathbf{R}^3 : x \geq [1/3; 1/3; 1/3], \sum_{i=1}^3 x_i \leq 1.05\}$.
6. The polyhedral set $Y = \{x \in \mathbf{R}^3 : x_1 \geq 1/3, x_2 \geq 1/3, x_3 \geq 1/3\}$ is not contained in the polyhedral set $X = \{x \in \mathbf{R}^3 : x \geq [1/3; 1/3; 1/3], \sum_{i=1}^3 x_i \leq 1\}$.
7. The polyhedral set $Y = \{x \in \mathbf{R}^3 : x \geq [1/3; 1/3; 1/3], \sum_{i=1}^3 x_i \leq 1\}$ is contained in the polyhedral set $X = \{x \in \mathbf{R}^3 : x_1 + x_2 \leq 2/3, x_2 + x_3 \leq 2/3, x_1 + x_3 \leq 2/3\}$.
8. The polyhedral set $X = \{x \in \mathbf{R}^3 : x \geq [1/3; 1/3; 1/3], \sum_{i=1}^3 x_i \leq 1\}$ is bounded.
9. The polyhedral set $X = \{x \in \mathbf{R}^3 : x_1 \geq 1/3, x_2 \geq 1/3, \sum_{i=1}^3 x_i \leq 1\}$ is unbounded.

**Exercise 35** Consider the LO program

$$\mathrm{Opt} = \max_x \{x_1 : x_1 \geq 0, x_2 \geq 0, ax_1 + bx_2 \leq c\} \qquad (P)$$

where $a, b, c$ are parameters. Answer the following questions and certify your answers:

1. Let $c = 1$. Is the problem feasible?
2. Let $a = b = 1, c = -1$. Is the problem feasible?
3. Let $a = b = 1, c = -1$. Is the problem bounded?
4. Let $a = b = c = 1$. Is the problem bounded?
5. Let $a = 1, b = -1, c = 1$. Is the problem bounded?
6. Let $a = b = c = 1$. Is it true that $\mathrm{Opt} \geq 0.5$?
7. Let $a = b = 1, c = -1$. Is it true that $\mathrm{Opt} \leq 1$?
8. Let $a = b = c = 1$. Is it true that $\mathrm{Opt} \leq 1$?
9. Let $a = b = c = 1$. Is it true that $x_* = [1; 1]$ is an optimal solution of $(P)$?
10. Let $a = b = c = 1$. Is it true that $x_* = [1/2; 1/2]$ is an optimal solution of $(P)$?
11. Let $a = b = c = 1$. Is it true that $x_* = [1; 0]$ is an optimal solution of $(P)$?

# Assignment 7: More on Duality

**Exercise 36** Write down problems dual to the following LO programs:

1. $$\max_{x\in\mathbf{R}^3}\left\{x_1+2x_2+3x_3:\begin{array}{l}x_1-x_2+x_3=0\\ x_1+x_2-x_3\geq 100\\ x_1\leq 0\\ x_2\geq 0\\ x_3\geq 0\end{array}\right\}$$

2. $\max\limits_{x\in\mathbf{R}^n}\left\{c^Tx : Ax=b, x\geq 0\right\}$

3. $\max\limits_{x\in\mathbf{R}^n}\left\{c^Tx : Ax=b, \underline{u}\leq x\leq \overline{u}\right\}$

4. $\max\limits_{x,y}\left\{c^Tx : Ax+By\leq b, x\leq 0, y\geq 0\right\}$

**Exercise 37** Consider a primal-dual pair of LO programs

$$\max_x\left\{c^Tx:\begin{array}{c}Px\leq p\\ Qx\geq q\\ Rx=r\end{array}\right\} \tag{P}$$

$$\min_{\lambda=[\lambda_\ell,\lambda_g,\lambda_e]}\left\{p^T\lambda_\ell+q^T\lambda_g+r^T\lambda_e:\begin{array}{l}\lambda_\ell\geq 0\\ \lambda_g\leq 0\\ P^T\lambda_\ell+Q^T\lambda_g+R^T\lambda_e=c\end{array}\right\} \tag{D}$$

Assume that both problems are feasible, and that the primal problem does contain inequality constraints. Prove that the feasible set of at least one of these problems is unbounded.

**Exercise 38** For positive integers $k\leq n$, let $s_k(x)$ be the sum of the $k$ largest entries in a vector $x\in\mathbf{R}^n$, e.g., $s_2([1;1;1])=1+1=2$, $s_2([1;2;3])=2+3=5$. Find a polyhedral representation of $s_k(x)$.
<u>Hint:</u> Take into account that the extreme points of the set $\{x\in\mathbf{R}^n : 0\leq x_i\leq 1, \sum_i x_i=k\}$ are exactly the 0/1 vectors from this set, and derive from this that

$$s_k(x)=\max_y\left\{y^Tx : 0\leq y_i\leq 1\,\forall i, \sum_i y_i=k\right\}.$$

**Exercise 39** Consider scalar linear constraint

$$a^Tx\leq b \tag{1}$$

with uncertain data $a\in\mathbf{R}^n$ ($b$ is certain) varying in the set

$$\mathcal{U}=\{a:|a_i-a_i^*|/\delta_i\leq 1, 1\leq i\leq n, \sum_{i=1}^n|a_i-a_i^*|/\delta_i\leq k\} \tag{2}$$

where $a_i^*$ are given "nominal data," $\delta_i>0$ are given quantities, and $k\leq n$ is an integer (in literature, this is called "budgeted uncertainty"). Rewrite the Robust Counterpart

$$a^Tx\leq b\,\forall a\in\mathcal{U} \tag{RC}$$

in a tractable LO form (that is, write down an explicit system $(S)$ of linear inequalities in variables $x$ and additional variables such that $x$ satisfies (RC) if and only if $x$ can be extended to a feasible solution of $(S)$).

# Assignment 8: Simplex Method

**Exercise 40** Solve the LO program

$$\begin{array}{rrrrrrrl} \max & -6x_1 & -5x_2 & -2x_3 & -x_4 & -2x_5 & -x_6 & \\ & x_1 & +x_2 & +x_3 & & & & = 2 \\ & & & & x_4 & +x_5 & +x_6 & = 3 \\ & x_1 & & & +x_4 & & & = 1 \\ & & x_2 & & & +x_5 & & = 2 \\ & & & & x \geq 0 & & & \end{array}$$

by the Primal Simplex Method, the initial basis being $\{1, 2, 5, 6\}$.

**Exercise 41** Four students were solving a maximization LO program by the Primal Simplex Method and arrived at intermediate tableaus as follows:

A.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| 16 | 0 | $-2$ | 0 | 0 | 4 | 0 |
| $x_3 = 5$ | 0 | $-2$ | 1 | 0 | 5 | 0 |
| $x_1 = 2$ | 1 | 3 | 0 | 0 | $-6$ | 0 |
| $x_4 = 3$ | 0 | 1 | 0 | 1 | 4 | 0 |
| $x_6 = -1$ | 0 | 1 | 0 | 0 | 2 | 1 |

B.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| 16 | 0 | $-2$ | 0 | 0 | 4 | $-1$ |
| $x_3 = 5$ | 0 | $-2$ | 1 | 0 | 5 | 0 |
| $x_1 = 2$ | 1 | 3 | 0 | 0 | $-6$ | 0 |
| $x_4 = 3$ | 0 | 1 | 0 | 1 | 4 | 0 |
| $x_6 = 1$ | 0 | 1 | 0 | 0 | 2 | 1 |

C.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| 16 | 0 | $-2$ | 0 | 0 | 4 | 0 |
| $x_3 = 5$ | 0 | $-2$ | 1 | 0 | 5 | 0 |
| $x_1 = 2$ | 1 | 3 | 0 | 0 | $-6$ | 0 |
| $x_4 = 3$ | 0 | 1 | 0 | 1 | 4 | 0 |
| $x_6 = 1$ | 0 | 1 | 0 | 0 | 2 | 1 |

D.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| 16 | 0 | $-2$ | 0 | 0 | 4 | 0 |
| $x_3 = 5$ | 0 | $-2$ | 1 | 0 | 5 | 0 |
| $x_1 = 2$ | 1 | 3 | 0 | 0 | $-6$ | $-1$ |
| $x_4 = 3$ | 0 | 1 | 0 | 1 | 4 | 0 |
| $x_6 = 1$ | 0 | 1 | 0 | 0 | 2 | 1 |

It is known that exactly one of the students did not make a mistake. Identify the correct tableau and complete the solution process.

**Exercise 42** Four students were solving a maximization LO program by the Dual Simplex Method and arrived at intermediate tableaus as follows:

A.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| 16 | 0 | −2 | 0 | 0 | −4 | 0 |
| $x_3 = 5$ | 0 | −2 | 1 | 0 | 5 | 0 |
| $x_1 = 2$ | 1 | 3 | 0 | 0 | −6 | 0 |
| $x_4 = 3$ | 0 | 1 | 0 | 1 | 4 | 0 |
| $x_6 = -1$ | 0 | 1 | 0 | 0 | −2 | 1 |

B.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| 16 | 0 | −2 | 0 | 0 | −4 | −1 |
| $x_3 = 5$ | 0 | −2 | 1 | 0 | 5 | 0 |
| $x_1 = 2$ | 1 | 3 | 0 | 0 | −6 | 0 |
| $x_4 = 3$ | 0 | 1 | 0 | 1 | 4 | 0 |
| $x_6 = -1$ | 0 | 1 | 0 | 0 | −2 | 1 |

C.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| 16 | 0 | −2 | 0 | 0 | 4 | 0 |
| $x_3 = 5$ | 0 | −2 | 1 | 0 | 5 | 0 |
| $x_1 = 2$ | 1 | 3 | 0 | 0 | −6 | 0 |
| $x_4 = 3$ | 0 | 1 | 0 | 1 | 4 | 0 |
| $x_6 = -1$ | 0 | 1 | 0 | 0 | −2 | 1 |

D.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| 16 | 0 | −2 | 0 | 0 | −4 | 0 |
| $x_3 = 5$ | 0 | −2 | 1 | 0 | 5 | 0 |
| $x_1 = 2$ | 1 | 3 | 0 | 0 | −6 | −1 |
| $x_4 = 3$ | 0 | 1 | 0 | 1 | 4 | 0 |
| $x_6 = -1$ | 0 | 1 | 0 | 0 | −2 | 1 |

It is known that exactly one of the students did not make a mistake. Identify the correct tableau and complete the solution process.

**Exercise 43** 1) Consider a parametric LO program:

$$\max_{x,s}\left\{c^T x - M\sum_{i=1}^m s_i : Ax + s = b, x \geq 0, s \geq 0\right\} \qquad (P_M)$$

with $b \geq 0$, along with LO program

$$\max_x \left\{c^T x : Ax = b, x \geq 0\right\} \qquad (P)$$

with $m \times n$ matrix $A$ of rank $m$. Prove that if $(P)$ is solvable, then there exists $M_* > 0$ such that whenever $M > M_*$, problem $(P_M)$ is solvable, the optimal values of $(P_M)$ and $(P)$ are the same, and all optimal solutions $(x, s)$ to $(P_M)$ are of the form $(x, 0)$, where $x$ is an optimal solution to $(P)$.

2) The result of 1) suggests the following single-phase "Big $M$"-implementation of the PSM: Given a standard from LO program $(P)$ with $b \geq 0$ (the latter can always be achieved by multiplying appropriate equality constraints by $-1$), we associate with it program $(P_M)$ for which we can easily point out an initial basis comprised of (indexes of) the slack variables $s$ along with the associated basic feasible solution $(x = 0, s = b)$. We then solve the resulting problem $(P_M)$ thinking of $M$ as about a large constant. From the description of the PSM it follows that $M$ will affect only the subsequent vectors of reduced costs (and the decisions we make based on these costs) and will appear linearly in the reduced costs. Now, making decisions based on reduced costs requires to compare them with zero, and here we think about $M$ as about large constant, meaning that reduced cost of the form $a + pM$ ($a$ and $p$ are known reals) should be treated as positive when $p > 0$ and as negative when $p < 0$; when $p = 0$, the decision is made based on what $a$ is. Assuming that $(P)$ is solvable, by 1) the problem $(P_M)$ for all large enough values of $M$ is solvable, and every optimal solution to this problem is $(x, 0)$, where $x$ is an optimal solution to $(P)$. In other words, when $(P)$ is solvable, running the big $M$ method, modulo highly unlikely cycling, will result in arriving at a basic feasible solution with the $s$-components equal to 0 and the reduced costs of the form $a_i + p_i M$ where for every $i$ either $a_i \leq 0$ and $p_i \leq 0$, or $p_i < 0$, meaning that the current basic solution is feasible for $(P)$ and optimal for all problems $(P_M)$ with large enough values of $M$ and therefore is optimal for $(P)$.

Use Big $M$ version of the PSM to solve the problem

$$
\begin{array}{lllll}
\max & -2x_1 + x_2 + x_3 - 6x_4 & & & \\
x_1 & +x_2 & & = & 2 \\
 & & x_3 \quad +x_4 & = & 2 \\
x_1 & \quad\quad +x_3 & & = & 2 \\
 & x \geq 0 & & &
\end{array}
$$

# Assignment 9: Flows

**Exercise 44** Recall the Transportation problem:

> The is a single product, $p$ suppliers with positive supplies $a_1, ..., a_p$ of the product and $q$ customers with positive demands $b_1, ..., b_q$ for the product, with the total supply equal to the total demand:
> $$\sum_{i=1}^{p} a_i = \sum_{j=1}^{q} b_j.$$
> The cost of shipping a unit of product from supplier $i$ to customer $j$ is a real $c_{ij}$. Find the cheapest shipment from the suppliers to the customers, i.e., find the amounts $x_{ij} \geq 0$ of product to be shipped from supplier $i$ to customer $j$ in such a way that every supplier $i$ ships to the customers totally $a_i$ units of product, every customer $j$ gets totally $b_j$ units of product, and the total transportation cost $\sum_{i,j} c_{ij} x_{ij}$ is as small as possible.

1. Pose the problem as an uncapacitated Network Flow problem on an appropriate graph.
2. Show that the problem is solvable
3. Prove that there exists an optimal solution $x^*$ to the problem with the following properties:
   a) the total number of nonzero shipments is $\leq p + q - 1$
   b) if $i_1, i_2$ are two distinct suppliers and $j_1, j_2$ are two distinct customers, then among the four shipments $x^*_{i_\mu, j_\nu}$, $1 \leq \mu, \nu \leq 2$, at least one is equal to zero.

**Exercise 45** Prove that every solvable capacitated Network Flow problem with nonzero vector of external supplies is equivalent to a capacitated Network flow problem where the vector of external supplies has at most 2 nonzero entries summing up to 0.

**Exercise 46** Consider a transportation problem

$$\text{Opt} = \min_{x_{ij}} \left\{ \sum_{i=1}^{p} \sum_{j=1}^{q} c_{ij} x_{ij} : \begin{array}{l} \sum_j x_{ij} = a_i, \, 1 \leq i \leq p \\ \sum_{i=1}^{p} x_{ij} = b_j, \, 1 \leq j \leq q \\ x_{ij} \geq 0 \end{array} \right\}$$

where $a_i \geq 0$, $b_i \geq 0$ and $\sum_i a_i = \sum_j b_j$.

Assume that supplies $a_i$ and the demands $b_j$ are somehow decreased (but remain nonnegative), while the new total supply is equal to the new total demand. Is it true that the optimal value in the problem cannot increase?

# Assignment 10: Miscellaneous problems

**Exercise 47** The Maximal Flow problem is as follows:

> *Given an oriented graph $G = (\mathcal{N}, \mathcal{E})$ with arc capacities $\{u_\gamma > 0 : \gamma \in \mathcal{E}\}$ and two selected nodes: source $\underline{i}$ and sink $\bar{i}$, find the largest flow from source to sink, that is, find the largest $s$ such that the vector of external supplies "$s$ at the source, $-s$ at the sink, zero at all remaining nodes" fits a flow $f$ satisfying the conservation law and obeying the bounds $0 \le f_\gamma \le u_\gamma$ for all $\gamma \in \mathcal{E}$.*

1. Write down the Maximal Flow problem as an LO program in variables $s$ (external supply at the source) and $f$ (flow in the network) and write down the dual problem

2. Let us define a *cut* as a partition of the set $\mathcal{N}$ of nodes of $G$ into two non-overlapping subsets $\underline{I}$ and $\bar{I} = \mathcal{N}\backslash\underline{I}$ such that the source is in $\underline{I}$, and the sink is in $\bar{I}$. For a cut $(\underline{I}, \bar{I})$, let the capacity $U(\underline{I}, \bar{I})$ of the cut be defined as

$$U(\underline{I}, \bar{I}) = \sum_{\gamma=(i,j)\in\mathcal{E}: i\in\underline{I}, j\in\bar{I}} u_\gamma.$$

   Prove that is $(s, f)$ is a feasible solution to the LO reformulation of the Maximal Flow problem, then for every cut $(\underline{I}, \bar{I})$ one has

$$s \le U(\underline{I}, \bar{I}).$$

   Prove that the inequality here is an equality if and only if the flow $f_\gamma$ in every arc $\gamma$ which starts in $\underline{I}$ and ends in $\bar{I}$ ("forward arc of the cut") is equal to $u_\gamma$, and the flow $f_\gamma$ in every arc $\gamma$ which starts in $\bar{I}$ and ends in $\underline{I}$ ("backward arc of the cut") is zero; if it is the case, $s$ is the optimal value in the Maximal Flow problem.

3. * Invoking the LP Duality Theorem, prove the famous *Max Flow - Min Cut Theorem: If the Max Flow problem is solvable, then its optimal value is equal to the minimum, over all cuts, capacity of a cut.*

**Exercise 48** Consider the *Assignment problem* as follows:

> *There are $p$ jobs and $p$ workers. When job $j$ is carrie out by worker $i$, we get profit $c_{ij}$. We want to associate jobs with workers in such a way that every work is assigned to exactly one worker, and no worker is assigned two or more jobs. Under these restrictions, we seek to maximize the total profit.*

1. Let $G$ be a graph with $2p$ nodes, $p$ of them representing the workers, and remaining $p$ representing the jobs. Every worker node $i$, $1 \le i \le p$, is linked by an arc $(i, p+j)$ with every job node $p+j$, $1 \le j \le p$, the capacity of the arc is $+\infty$, and the transportation cost is $-c_{i,p+j}$. The vector of external supplies has entries indexed by the worker nodes equal to 1 and entries indexed by the job nodes equal to -1.

   Prove that the basic feasible solutions $f = \{f_{i,p+j}\}_{1\le i,j\le p}$ of the resulting Network Flow problem are exactly the assignements, that is, flows given by permutations $i \mapsto \sigma(i)$ of the index set $\{1, 2, ..., p\}$ according to $f_{i,p+\sigma(i)} = 1$ and $f_{i,p+j} = 0$ when $j \ne \sigma(i)$.

2. Extract from the previous item that the Assignment problem can be reduced to a Network Flow problem (more exactly, to finding an optimal basic feasible solution in a Network Flow problem).

Pay attention to the following two facts:

• every spanning tree of the graph in question has $2p-1$ arcs, while every basic feasible solution has just $p$ nonzero entries, meaning that when $p>1$, all basic feasible solutions are degenerate.

• the flows $\{f_{i,p+j}\}_{i,j=1}^{p}$ on $G$ can be associated with $p\times p$ matrices $F_{ij}=f_{i,p+j}$, and with this interpretation, the feasible set of the above Network Flow problem is nothing but the set of all double stochastic $p\times p$ matrices. Thus, the fact established in the first item: "the extreme points of the feasible set of the problem (i.e., the vertices of its feasible set) are exactly the assignments" recoves anew the Birkhoff Theorem: "The extreme points of the set of double stochastic matrices are exactly the permutation matrices."

**Exercise 49** Consider the following modification of Assignment problem:

> (!) *There are $q$ jobs and $p\geq q$ workers; assigning worker $i$ with job $j$, we get profit $c_{ij}$. We want to assign every job with an exactly one worker in such a way that no worker is assigned to two or more jobs (although some of the workers can be not assigned to jobs at all) and want to maximize the total profit under this assignment.*

1. Reformulate the problem as an assignment problem.

2. Assume that all $c_{ij}$ are either 0 or 1; let us say that $c_{ij}=1$ means that worker $i$ knows how to do job $j$, and $c_{ij}=0$ means that worker $i$ does not know how to do job $j$, and that an assignment ("exactly one worker for every job, no worker with more than one job assigned") is good if every job is assigned to a worker which knows how to do this job.

   Consider the network with $p+q+2$ nodes: the source (node 0), $p$ worker nodes $(1,2,...,p)$, $q$ job nodes $(p+1,...,p+q)$ and the sink (node $p+q+1$). The arcs, every one of capacity 1, are:
   • $p$ arcs "source $\mapsto$ worker node (i.e., arcs $(0,i)$, $1\leq i\leq p$);
   • the arcs "worker $i\mapsto$ "job $j$ which the worker $i$ knows how to do" (i.e., arcs $(i,p+j)$, $1\leq i\leq p$, $1\leq j\leq q$, corresponding to the pairs $(i,j)$ with $c_{ij}=1$);
   • $q$ arcs "job node $\mapsto$ sink" (i.e., arcs $(p+j,p+q+1)$, $1\leq j\leq q$)
   along with the Maximal flow problem on this network.

   (a) Prove that the existence of a good assignment is equivalent to the fact that the magnitude of the maximal flow in the above Max Flow problem is equal to $q$.

   (b) Use the Max Flow – Min Cut Theorem to prove the following fact:

   > (!!) *In (!) with zero/one $c_{ij}$, an assignment with profit $q$ (i.e., an assignment where every job $j$ is assigned to a worker which knows how to do this job) exists if and only if for every subset $S$ of the set $\{1,...,q\}$ of jobs the total number of workers which know how to do a job from $S$ is at least the cardinality of $S$.*
   >
   > Note: (!!) is called the *Marriage Lemma*, according to the following interpretation: there are $q$ young ladies and $p$ young men, some of the ladies being acquainted with some of the men. When it is possible to select for every one

of the ladies a bridegroom from the above group of men in such a way that different ladies get different bridegrooms, and every lady is acquianted with her bridegroom? The answer is: it is possible if and only if for every set $S$ of the ladies, the total number of men acquainted each with at least one of the ladies from $S$ is at least the cardinality of $S$.