Extended Matrix Cube Theorems with Applications to μ-Theory in Control

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Abstract

We study semi-infinite systems of Linear Matrix Inequalities which are generically NP-hard. For these systems, we introduce computationally tractable approximations and derive quantitative guarantees of their quality. As applications, we discuss the problem of maximizing a Hermitian quadratic form over the complex unit cube and the problem of bounding the complex structured singular value. With the help of our complex Matrix Cube Theorem we demonstrate that the standard scaling upper bound on $\mu(M)$ is a tight upper bound on the largest level of structured perturbations of the matrix M for which all perturbed matrices share a common Lyapunov certificate for the (discrete time) stability.

1 Introduction

Numerous applications of Semidefinite Programming, especially those in Robust Optimization (see, e.g., [1, 7, 8, 5, 2] and references therein) require processing of *semi-infinite* systems of Linear Matrix Inequalities (LMIs) of the form

$$\mathcal{A}[x, \Delta] \succeq 0 \quad \forall \Delta \in \gamma \boldsymbol{\Delta}, \tag{1}$$

where x is the vector of design variables, $\Delta \in \mathbf{R}^N$ represents perturbations of the data, $\mathcal{A}[x, \Delta]$ is a symmetric $m \times m$ matrix which is "bi-affine", i.e., affine in x for Δ fixed, and affine in Δ for x fixed, $\boldsymbol{\Delta} \subset \mathbf{R}^N$ is the set of "data perturbations of magnitude not exceeding 1", and $\gamma \geq 0$ is the "uncertainty level". As a simple and instructive example of this type, consider the following Lyapunov Stability Analysis problem. We are given a "nominal" linear dynamical system

$$\dot{z}(t) = S_\star z(t). \tag{2}$$

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The $m\times m$ matrix $S_\star$ of the system is partitioned into rectangular blocks $S_\star^p$, $p=1,...,k$, of sizes $\ell_p\times r_p$. In "real life", the blocks are affected by perturbations $S_\star^p\mapsto S_\star^p+\Delta_p$ which we assume to be norm-bounded: $\|\Delta_p\|\leq\gamma$, where $\|\cdot\|$ is the standard matrix norm (maximal singular value). Except for this norm-boundedness assumption, the perturbations are "completely free" and may even depend on time. With this uncertainty model, the actual description of the dynamical system becomes

$$\dot{z}(t)=\left[S_\star+\sum_{p=1}^{k}L_p^T\Delta_p(t)R_p\right]z(t), \tag{3}$$

(the matrices $R_p\in\mathbf{R}^{r_p\times m}$, $L_p\in\mathbf{R}^{\ell_r\times m}$ are readily given by the positions of the blocks); all we know about the perturbations $\Delta_p(t)\in\mathbf{R}^{\ell_p\times r_p}$ is that they are measurable functions of t such that $\|\Delta_p(t)\|\leq\gamma$ for all t.

A basic question pertaining to a dynamical system is whether it is stable, i.e., whether all its trajectories tend to 0 as $t\to\infty$. The standard *sufficient* stability condition is that all matrices S we can get from $S_\star$ by the perturbations in question share a common *Lyapunov stability certificate*, which is a positive definite matrix X such that

$$SX+XS^T\prec 0.$$

By homogeneity reasons, the existence of such a common Lyapunov stability certificate is equivalent to the existence of a positive definite solution to the semi-infinite LMI

$$\begin{array}{r}\mathcal{A}[X,\Delta]\equiv -I-[S_\star X+XS_\star^T]+\sum\limits_{p=1}^{k}\left[L_p^H\Delta_p[R_pX]+[R_pX]^T\Delta_p^TL_p\right]\succeq 0\\ \forall\Delta_p\in\mathbf{R}^{\ell_p\times r_p}, \|\Delta_p\|\leq\gamma,\ p=1,...,k,\end{array} \tag{4}$$

which is of the generic form (1).

The Lyapunov Stability Analysis example is instructive in two ways: it demonstrates the importance of semi-infinite LMIs and suggests specific ways of representing the perturbations Δ, the set $\boldsymbol{\Delta}$ and the matrix-valued function $\mathcal{A}[x,\Delta]$ appearing in (1). Namely, in this example

1) A perturbation Δ is a collection of "perturbation blocks" – matrices Δ_p of given sizes $\ell_p\times r_p$, $p=1,...,k$;

2) The mapping $\mathcal{A}[x,\Delta]$ is of the form

$$\mathcal{A}[x,\Delta]=A[x]+\sum_{p=1}^{k}\left[L_p^T\Delta_pR_p[x]+R_p^T[x]\Delta_p^TL_p\right], \tag{5}$$

where $A[x]$ is a symmetric $m\times m$ matrix, $L_p\in\mathbf{R}^{\ell_p\times m}$, $R_p[x]\in\mathbf{R}^{r_p\times m}$ and $A[x]$, $R_p[x]$ are affine in x;

3) The set $\boldsymbol{\Delta}$ of "perturbations of magnitude ≤ 1" is comprised of all collections $(\Delta_1,...,\Delta_k)$ such that $\Delta_p\in\mathbf{R}^{\ell_p\times r_p}$, $\|\Delta_p\|\leq 1$ and, besides this, matrices Δ_p, for prescribed values of p, are restricted to be scalar (i.e., of the form $\delta_pI_{r_p}$; of course, $\ell_p=r_p$ for indices p in question).

In fact, in our motivating example there was no need for scalar perturbations; nevertheless, there are many reasons to introduce them rather than to allow all perturbations to be of "full size". The simplest of these reasons is that with scalar perturbations, the outlined perturbation model allows to represent in the form of (5) every (affine in x and in Δ) function $\mathcal{A}[x, \Delta]$ with symmetric matrix values; to this end it suffices to treat every entry Δ_p in $\Delta \in \mathbf{R}^N$ as a scalar matrix perturbation $\Delta_p I_m$. We see that when scalar perturbations are allowed, items 1) and 2) above do not restrict the "expressive abilities" of the perturbation model (provided that we restrict ourselves to affine perturbations). What does restrict generality, is the part of item 3) which says that the only restriction on $\Delta = (\Delta_1, ..., \Delta_k) \in \boldsymbol{\Delta}$, except for the requirement for some of the perturbation blocks Δ_p to be scalar matrices, is the common norm bound $\|\Delta_p\| \leq 1$ on all perturbation blocks. This assumption provides (1) with a specific structure which, as we shall see, allows for a productive processing of (1).

It makes sense to assume once for ever that the matrices Δ_p are square. This does not restrict generality, since we can always enforce $\ell_p = r_p$ by adding to L_p or to R_p a number of zero rows; it is easily seen that this modification does not affect anything except for simplifying notation. From now on, we denote the common value of ℓ_p and r_p by d_p.

Typical problems associated with a semi-infinite LMI of the form (1) are to find a point in the feasible set of the LMI and to minimize a linear objective over this feasible set. These are convex problems with "implicitly defined" feasible set; basically all we need in order to solve such a problem efficiently is a *feasibility oracle* capable to solve efficiently the *analysis problem* as follows: *Given x, check whether x is feasible for* (1) (for details on relations between "analysis and synthesis" in Convex Optimization, see [9] or [3], Chapter 5). Note that with model 1) – 3), the analysis problem for (1) is the "Matrix Cube" problem as follows:

> *Given a symmetric $m \times m$ matrix A, $d_p \times m$ matrices L_p, R_p, $p = 1, ..., k$, and $\gamma \geq 0$, check whether all matrices of the form*
>
> $$A + \gamma \sum_{p=1}^{k} [L_p^T \Delta_p R_p + R_p^T \Delta_p^T L_p],$$
>
> *where $\|\Delta_p\| \leq 1$ for all p and $\Delta_p = \delta_p I_{d_p}$ for prescribed values of p, are positive semidefinite.*

Unfortunately, the Matrix Cube problem, same as the majority of other semi-infinite LMIs known from the literature, in general is NP-hard. However, it was found in [4] that *when all perturbations are scalar*, the problem admits a computationally tractable approximation which is tight within a factor of $\vartheta = O(1)\sqrt{\max\limits_p d_p}$. Specifically,

> *Given a Matrix Cube problem with scalar perturbation blocks, one can build an explicit system $\mathcal{S}$ of LMIs in variables u of size polynomial in m and $\sum\limits_p d_p$ with the following property: if $\mathcal{S}$ is feasible, the answer in the Matrix Cube problem is affirmative; if $\mathcal{S}$ is infeasible, then the answer in the Matrix Cube problem with the perturbation level γ replaced by $\vartheta\gamma$ is negative. Besides this, if the data A, $R_1, ..., R_k$ in the Matrix Cube problem depend affinely on*

a decision vector x, while $L_1, ..., L_k$ are independent of x (cf. (5)), *then $\mathcal{S}$ is a system of LMIs in x, u.*

As it is shown in [4], this result allows to build tight approximations of several important NP-hard problems of the form (1).

The goal of this paper is to extend the approach and the results of [4] from the case of purely scalar perturbations onto the more general perturbation model 1) – 3). We consider both the outlined model of *real* perturbations and its complex-valued counterpart (which is important for some of control applications); in both real and complex cases, we build computationally tractable approximations of the respective Matrix Cube problems and demonstrate that these approximations are tight within a factor $O(1)\sqrt{d^s}$, where d^s is the maximum of sizes d_p of *scalar* perturbation blocks in Δ; surprisingly, the "full size" perturbation blocks, however large they are, do not affect the quality of the approximation.

The rest of the paper is organized as follows. In the remaining part of Introduction, we fix the notation to be used. Section 2 deals with the technically slightly more difficult complex case version of the Matrix Cube problem; the real case of the problem is considered in Section 3. In concluding sections 4, 5 we illustrate our main results by their applications to the problem of maximizing a positive definite quadratic form over the "complex cube" $\{z \in \mathbf{C}^m : |z_p| \leq 1, p = 1, ..., m\}$, and to the problem of bounding from above an important Control entity – the complex structured singular value.

Notation we use is as follows:

• $\mathbf{C}^{m\times n}$, $\mathbf{R}^{m\times n}$ stand for the spaces of complex, respectively, real $m \times n$ matrices. As always, we write $\mathbf{C}^n$ and $\mathbf{R}^n$ as shorthands for $\mathbf{C}^{n\times 1}$, $\mathbf{R}^{n\times 1}$, respectively.

For $A \in \mathbf{C}^{m\times n}$, A^T stands for the transpose, and A^H for the conjugate transpose of A:

$$(A^H)_{rs} = A^*_{sr},$$

where z^* is the conjugate of $z \in \mathbf{C}$.

Both $\mathbf{C}^{m\times n}$, $\mathbf{R}^{m\times n}$ are equipped with the inner product

$$\langle A, B\rangle = \mathrm{Tr}(AB^H) = \sum_{r,s} A_{rs}B^*_{rs}.$$

The norm associated with this inner product is denoted by $\|\cdot\|_2$.

We use the notation I_m, $O_{m\times n}$ for the unit $m\times m$, respectively, the zero $m\times n$ matrices.

• $\mathbf{H}^m$, $\mathbf{S}^m$ are real vector spaces of $m \times m$ Hermitian, respectively, real symmetric matrices. Both are Euclidean spaces w.r.t. the inner product $\langle\cdot,\cdot\rangle$.

For a Hermitian/real symmetric $m \times m$ matrix A, $\lambda(A)$ is the vector of eigenvalues $\lambda_r(A)$ of A taken with their multiplicities in the non-ascending order:

$$\lambda_1(A) \geq \lambda_2(A) \geq ... \geq \lambda_m(A).$$

For a $m \times n$ matrix A, $\sigma(A) = (\sigma_1(A), ..., \sigma_n(A))^T$ is the vector of singular values of A:

$$\sigma_r(A) = \lambda_r^{1/2}(A^H A),$$

and

$$\|A\| = \sigma_1(A) = \max\{\|Ax\|_2 : x \in \mathbf{C}^n, \|x\|_2 \le 1\}$$

(by evident reasons, when A is real, one can replace $\mathbf{C}^n$ in the right hand side with $\mathbf{R}^n$).

For Hermitian/real symmetric matrices A, B, we write $A \succeq B$ ($A \succ B$) to express that $A - B$ is positive semidefinite (resp., positive definite). We denote by $\mathbf{H}^n_+$ ($\mathbf{S}^n_+$) the cones of positive semidefinite Hermitian (resp., positive semidefinite real symmetric) $n \times n$ matrices.

For $X \succeq 0$, $X^{1/2}$ denotes the positive semidefinite square root of X (uniquely defined by the relations $X^{1/2} \succeq 0$, $(X^{1/2})^2 = X$).

• On many occasions in this paper we use the term "efficient computability" of various quantities. An appropriate definition of this notion does exist[1)], but for our purposes here it suffices to agree that all "LMI-representable" quantities – those which can be represented as optimal values in semidefinite programs

$$\min_x \left\{ c^T x : A(x) \equiv A_0 + \sum_{i=1}^N x_i A_i \succeq 0 \right\}, \qquad [A_i \in \mathbf{S}^K]$$

or generalized eigenvalue problems

$$\min_{x,\omega} \left\{ \omega : \; A(x) \equiv A_0 + \sum_{i=1}^N x_i A_i \succeq 0, B(x) \equiv B_0 + \sum_{i=1}^N x_i B_i \preceq \omega A(x) \right\} \qquad [A_i, B_i \in \mathbf{S}^K]$$

are efficiently computable functions of the data $c, A_0, ..., A_N$, resp., $A_0, ..., A_N, B_0, ..., B_N$.

2 Matrix Cube Theorem, Complex case

The "Complex Matrix Cube" problem is as follows:

CMC: *Let m, d_1,...,d_k be positive integers, and $A \in \mathbf{H}^m_+$, $L_p, R_p \in \mathbf{C}^{d_p \times m}$ be given matrices, $L_p \neq 0$. Let also a partition $\{1, 2, ..., k\} = I_{\rm s}^{\rm r} \cup I_{\rm s}^{\rm c} \cup I_{\rm f}^{\rm c}$ of the index set $\{1, ..., k\}$ into three non-overlapping sets be given. With these data, we associate a parametric family of "matrix boxes"*

$$\mathcal{U}[\gamma] = \left\{ A + \gamma \sum_{p=1}^k [L_p^H \Delta_p R_p + R_p^H \Delta_p^H L_p] : \begin{array}{c} \Delta_p \in \mathbf{\Delta}_p, \|\Delta_p\| \le 1, \\ p = 1, ..., k \end{array} \right\} \subset \mathbf{H}^m, \tag{6}$$

where $\gamma \geq 0$ is the parameter and

$$\mathbf{\Delta}_p = \begin{cases} \{\delta I_{d_p} : \delta \in \mathbf{R}\}, & p \in I_{\rm s}^{\rm r} \text{ ["real scalar perturbations"]} \\ \{\delta I_{d_p} : \delta \in \mathbf{C}\}, & p \in I_{\rm s}^{\rm c} \text{ ["complex scalar perturbations"]} \\ \mathbf{C}^{d_p \times d_p}, & p \in I_{\rm f}^{\rm c} \text{ ["full size complex perturbations"]} \end{cases} . \tag{7}$$

Given $\gamma \geq 0$, check whether

$$\mathcal{U}[\gamma] \subset \mathbf{H}^m_+ \tag{$\mathcal{I}[\gamma]$}$$

[1)]For a definition which fits best of all the contents of the paper, see [3], Chapter 5.

Remark 2.1 *In the sequel, we always assume that $d_p > 1$ for $p \in I_{\rm s}^{\rm c}$. Indeed, one-dimensional complex scalar perturbations can always be regarded as full size complex perturbations.*

It is well-known that the CMC problem is, in general, NP-hard. Our goal is to build a "computationally tractable" *sufficient condition* for the validity of ($\mathcal{I}[\gamma]$) and to understand how "conservative" is this condition.

Consider, along with predicate ($\mathcal{I}[\gamma]$), the predicate

$$\begin{array}{lrcl} \multicolumn{4}{l}{\exists Y_p \in \mathbf{H}^m,\ p = 1, ..., k \text{ such that :}} \\ (a) & Y_p & \succeq & L_p^H \Delta_p R_p + R_p^H \Delta_p^H L_p \quad \forall (\Delta_p \in \boldsymbol{\Delta}_p, \|\Delta_p\| \le 1),\ p = 1, ..., k, \\ (b) & A - \gamma \sum\limits_{p=1}^{k} Y_p & \succeq & 0. \end{array} \qquad (\mathcal{II}[\gamma])$$

Our main result is as follows:

Theorem 2.1 [The Complex Matrix Cube Theorem] *One has:*

(i) *Predicate ($\mathcal{II}[\gamma]$) is stronger than ($\mathcal{I}[\gamma]$) – the validity of the former predicate implies the validity of the latter one.*

(ii) ($\mathcal{II}[\gamma]$) *is computationally tractable – the validity of the predicate is equivalent to the solvability of the system of LMIs*

$$\begin{array}{lrcl} (s.\mathbf{R}) & Y_p \pm \left[L_p^H R_p + R_p^H L_p\right] & \succeq & 0,\ p \in I_{\rm s}^{\rm r}, \\ (s.\mathbf{C}) & \left[\begin{array}{cc} Y_p - V_p & L_p^H R_p \\ R_p^H L_p & V_p \end{array}\right] & \succeq & 0,\ p \in I_{\rm s}^{\rm c}, \\ (f.\mathbf{C}) & \left[\begin{array}{cc} Y_p - \lambda_p L_p^H L_p & R_p^H \\ R_p & \lambda_p I_{d_p} \end{array}\right] & \succeq & 0,\ p \in I_{\rm f}^{\rm c} \\ (*) & A - \gamma \sum\limits_{p=1}^{k} Y_p & \succeq & 0. \end{array} \qquad (8)$$

in the matrix variables $Y_p \in \mathbf{H}^m$, $p = 1, ..., k$, $V_p \in \mathbf{H}^m$, $p \in I_{\rm s}^{\rm c}$, and the real variables λ_p, $p \in I_{\rm f}^{\rm c}$.

(iii) *"The gap" between ($\mathcal{I}[\gamma]$) and ($\mathcal{II}[\gamma]$) can be bounded solely in terms of the maximal size*

$$d^{\rm s} = \max\{d_p : p \in I_{\rm s}^{\rm r} \cup I_{\rm s}^{\rm c}\} \qquad (9)$$

of the scalar perturbations (here the maximum over an empty set by definition is 0). Specifically, there exists a universal function $\vartheta_{\mathbf{C}}(\cdot)$ such that

$$\vartheta_{\mathbf{C}}(\nu) \le 4\pi\sqrt{\nu},\ \nu \ge 1, \qquad (10)$$

and

$$\textit{if } (\mathcal{II}[\gamma]) \textit{ is not valid, then } (\mathcal{I}[\vartheta_{\mathbf{C}}(d^{\rm s})\gamma]) \textit{ is not valid.} \qquad (11)$$

Corollary 2.1 *The efficiently computable supremum* $\widehat{\gamma}$ *of those* $\gamma \geq 0$ *for which the system of LMIs* (8) *is solvable is a lower bound on the supremum* $\gamma^\star$ *of those* $\gamma \geq 0$ *for which* $\mathcal{U}[\gamma] \subset \mathbf{H}^m_+$*, and this lower bound is tight within the factor* $\vartheta_{\mathbf{C}}(d^{\rm s})$*:*

$$\widehat{\gamma} \leq \gamma^\star \leq \vartheta_{\mathbf{C}}(d^{\rm s})\widehat{\gamma}. \tag{12}$$

Remark 2.2 *From the proof of Theorem 2.1 it follows that* $\vartheta_{\mathbf{C}}(0) = \frac{4}{\pi}$*,* $\vartheta_{\mathbf{C}}(1) = 2$*. Thus,*

- *when there are no scalar perturbations:* $I^{\rm r}_{\rm s} = I^{\rm c}_{\rm s} = \emptyset$*, the factor* ϑ *in the implication*

  $$\neg(\mathcal{II}[\gamma]) \Rightarrow \neg(\mathcal{I}[\vartheta\gamma]) \tag{13}$$

  can be set to $\frac{4}{\pi} = 1.27...$

- *when there are no complex scalar perturbations (cf. Remark 2.1) and all real scalar perturbations are non-repeated* ($I^{\rm c}_{\rm s} = \emptyset$*,* $d_p = 1$ *for all* $p \in I^{\rm r}_{\rm s}$*), the factor* ϑ *in* (13) *can be set to* 2.

Remark 2.3 *From the proof of the Matrix Cube Theorem 2.1 it follows that its statement remains intact when in the definition* (6) *of the matrix box, the restrictions* $\|\Delta_p\| \leq 1$*,* $p \in I^{\rm c}_{\rm f}$*, on the norms of full size perturbations are replaced with the restrictions* $\|\Delta_p\|^{(p)} \leq 1$*, where* $\|\cdot\|^{(p)}$ *are norms on* $\mathbf{C}^{d_p\times d_p}$ *such that* $\|\Delta_p\| \leq \|\Delta_p\|^{(p)}$ *for all* $\Delta_p \in \mathbf{C}^{d_p\times d_p}$ *and* $\|\Delta_p\| = \|\Delta_p\|^{(p)}$ *whenever* Δ_p *is a rank 1 matrix (e.g., one can set* $\|\cdot\|^{(p)}$ *to be the Frobenius norm* $\|\cdot\|_2$ *of a matrix).*

The following simple observation is crucial when applying Theorem 2.1 in the context of semi-infinite bi-affine LMIs of the form (1).

Remark 2.4 *Assume that the data* $A, R_1, ..., R_k$ *of the Matrix Cube problem are affine in a vector of parameters* x*, while the data* $L_1, ..., L_k$ *are independent of* x *(cf.* (5)*). Then* (8) *is a system of LMIs in the variables* Y_p*,* V_p*,* λ_p *and* x*.*

2.1 Proof of Theorem 2.1

Item (i) is evident. We prove item (ii); item (iii) is proved in Section 2.1.2.

2.1.1 Proof of Theorem 2.1.(ii)

The equivalence between the validity of ($\mathcal{II}[\gamma]$) and the solvability of (8) is readily given by the following facts (the first of them is perhaps new):

Lemma 2.1 *Let* $B \in \mathbf{C}^{m\times m}$ *and* $Y \in \mathbf{H}^m$*. Then the relation*

$$Y \succeq \delta B + \delta^* B^H \quad \forall(\delta \in \mathbf{C}, |\delta| \leq 1) \tag{14}$$

is satisfied if and only if

$$\exists V \in \mathbf{H}^m : \quad \begin{bmatrix} Y - V & B^H \\ B & V \end{bmatrix} \succeq 0. \tag{15}$$

Lemma 2.2 [see [6]] *Let $L \in \mathbf{C}^{\ell\times m}$ and $R \in \mathbf{C}^{r\times m}$.*

(i) *Assume that L, R are nonzero. A matrix $Y \in \mathbf{H}^m$ satisfies the relation*

$$Y \succeq L^H U R + R^H U^H L \quad \forall (U \in \mathbf{C}^{\ell\times r} : \|U\| \le 1) \tag{16}$$

if and only if there exists a positive real λ such that

$$Y \succeq \lambda L^H L + \lambda^{-1} R^H R. \tag{17}$$

(ii) *Assume that L is nonzero. A matrix $Y \in \mathbf{H}^m$ satisfies* (16) *if and only if there exists $\lambda \in \mathbf{R}$ such that*

$$\left[\begin{array}{cc} Y - \lambda L^H L & R^H \\ R & \lambda I_r \end{array}\right] \succeq 0. \tag{18}$$

Lemmas 2.1, 2.2 $\Rightarrow$ Theorem 2.1.(ii). All we need to prove is that a collection of matrices Y_p satisfies the constraints in $(\mathcal{II}[\gamma])$ if and only if it can be extended by properly chosen V_p, $p \in I_{\rm f}^{\rm c}$, and λ_p, $p \in I_{\rm s}^{\rm c}$, to a feasible solution of (8). This is immediate, since matrices Y_p, $p \in I_{\rm f}^{\rm c}$, satisfy the corresponding constraints $(\mathcal{II}[\gamma].a)$ if and only if these matrices along with some matrices V_p satisfy (8.s.**C**)) (Lemma 2.1), while matrices Y_p, $p \in I_{\rm s}^{\rm c}$, satisfy the corresponding constraints $(\mathcal{II}[\gamma].a)$ if and only if these matrices along with some reals λ_p satisfy (8.f.**C**) (Lemma 2.2.(ii)). ∎

Proof of Lemma 2.1. "if" part: Assume that V is such that $\left[\begin{array}{cc} Y - V & B^H \\ B & V \end{array}\right] \succeq 0$. Then, for every $\xi \in \mathbf{C}^n$ and every $\delta \in \mathbf{C}$, $|\delta| = 1$, we have

$$0 \le \left[\begin{array}{c} \xi \\ -\delta^*\xi \end{array}\right]^H \left[\begin{array}{cc} Y - V & B^H \\ B & V \end{array}\right] \left[\begin{array}{c} \xi \\ -\delta^*\xi \end{array}\right] = \xi^H (Y - V)\xi + \xi^H V \xi - \xi^H[\delta B + \delta^* B^H]\xi,$$

so that $Y \succeq \delta B + \delta^* B^H$ for all $\delta \in \mathbf{C}$, $|\delta| = 1$, which, by evident convexity reasons, implies (14).

"only if" part: Let $Y \in \mathbf{H}^m$ satisfy (14). Assume, on the contrary to what should be proved, that there does not exist $V \in \mathbf{H}^m$ such that $\left[\begin{array}{cc} Y - V & B^H \\ B & V \end{array}\right] \succeq 0$, and let us lead this assumption to a contradiction. Observe that our assumption means that the optimization program

$$\min_{t,V} \left\{ t : \left[\begin{array}{cc} tI_m + Y - V & B^H \\ B & V \end{array}\right] \succeq 0 \right\} \tag{19}$$

has no feasible solutions with $t \le 0$; since problem (19) is clearly solvable, its optimal value is therefore positive. Now, our problem is a conic problem[2)] on the (self-dual) cone of positive semidefinite Hermitian matrices; since the problem clearly is strictly feasible, the Conic Duality Theorem says that dual problem

$$\max_{Z \in \mathbf{H}^m, W \in \mathbf{C}^{m\times m}} \left\{ -2\Re\left\{\mathrm{Tr}(W^H B)\right\} - \mathrm{Tr}(ZY) : \begin{array}{ll} \left[\begin{array}{cc} Z & W^H \\ W & Z \end{array}\right] \succeq 0, & (a) \\ \mathrm{Tr}(Z) = 1 & (b) \end{array} \right\} \tag{20}$$

[2)] For background on conic problems and conic duality, see [10], Chapter 4, or [3], Chapter 2.

is solvable with the same – positive – optimal value as the one of (19). In (20), we can easily eliminate the W-variable; indeed, constraint (20.a), as it is well-known, is equivalent to the fact that $Z \succeq 0$ and $W = Z^{1/2}XZ^{1/2}$ with $X \in \mathbf{C}^{m\times m}$, $\|X\| \leq 1$. With this parameterization of W, the W-term in the objective of (20) becomes $-2\Re\{\mathrm{Tr}(X^H Z^{1/2}BZ^{1/2})\}$; as it is well-known, the maximum of the latter expression in X, $\|X\| \leq 1$, equals to $2\|\sigma(Z^{1/2}BZ^{1/2})\|_1$. Since the optimal value in (20) is positive, we arrive at the following intermediate conclusion:

(*) *There exists $Z \in \mathbf{H}^m$, $Z \succeq 0$, such that*

$$2\|\sigma(Z^{1/2}BZ^{1/2})\|_1 > \mathrm{Tr}(ZY) = \mathrm{Tr}(Z^{1/2}YZ^{1/2}). \tag{21}$$

The desired contradiction is now readily given by the following simple observation:

Lemma 2.3 *Let $S \in \mathbf{H}^m$, $C \in \mathbf{C}^{m\times m}$ be such that*

$$S \succeq \delta C + \delta^* C^H \quad \forall(\delta \in \mathbf{C}, |\delta| = 1). \tag{22}$$

Then $2\|\sigma(C)\|_1 \leq \mathrm{Tr}(S)$.

To see that Lemma 2.3 yields the desired contradiction, note that the matrices $S = Z^{1/2}YZ^{1/2}$, $C = Z^{1/2}BZ^{1/2}$ satisfy the premise of the lemma by (14), and for these matrices the conclusion of the lemma contradicts (21).

Proof of Lemma 2.3: As it was already mentioned,

$$\|\sigma(C)\|_1 = \max_X \left\{\Re\{\mathrm{Tr}(XC^H)\} : \|X\| \leq 1\right\}.$$

Since the extreme points of the set $\{X \in \mathbf{C}^{m\times m} : \|X\| \leq 1\}$ are unitary matrices, the maximizer X_* in the right hand side can be chosen to be unitary: $X_*^H = X_*^{-1}$; thus, X_* is a unitary similarity transformation of a diagonal unitary matrix. Applying appropriate unitary rotation $A \mapsto U^H AU$, $U^H = U^{-1}$, to all matrices involved, we may assume that X_* itself is diagonal. Now we are in the situation as follows: we are given matrices C, S satisfying (22) and a *diagonal* unitary matrix X_* such that $\|\sigma(C)\|_1 = \Re\{\mathrm{Tr}(X_* C^H)\}$. In other words,

$$\|\sigma(C)\|_1 = \Re\left\{\sum_{\ell=1}^m (X_*)_{\ell\ell} C_{\ell\ell}^*\right\} \leq \sum_{\ell=1}^m |C_{\ell\ell}| \tag{23}$$

(the concluding inequality comes from the fact that X_* is unitary). On the other hand, let e_ℓ be the standard basic orths in $\mathbf{C}^m$. By (22), we have

$$\delta C_{\ell\ell} + \delta^* C_{\ell\ell}^* = e_\ell^H[\delta C + \delta^* C^H]e_\ell \leq e_\ell^H S e_\ell = S_{\ell\ell} \quad \forall(\delta \in \mathbf{C}, |\delta| = 1),$$

whence, maximizing in δ, $2|C_{\ell\ell}| \leq S_{\ell\ell}$, $\ell = 1, ..., m$, which combines with (23) to imply that $2\|\sigma(C)\|_1 \leq \mathrm{Tr}(S)$. ∎

Proof of Lemma 2.2. (i), "if" part: Let (17) be valid for certain $\lambda > 0$. Then for every $\xi \in \mathbf{C}^m$ one has

$$\begin{array}{rcl}\xi^H Y\xi & \geq & \lambda\xi^H L^H L\xi + \lambda^{-1}\xi^H R^H R\xi \geq 2\sqrt{\xi^H L^H L\xi}\sqrt{\xi^H R^H R\xi} = 2\|L\xi\|_2\|R\xi\|_2 \\ & & \Downarrow \\ \multicolumn{3}{l}{\forall (U, \|U\| \leq 1):} \\ \xi^H Y\xi & \geq & 2|[L\xi]^H U[R\xi]| \geq 2\Re\{[L\xi]^H U[R\xi]\} = \xi^H[L^H U R + R^H U^H L]\xi,\end{array}$$

as claimed.

(i), "only if" part: Assume that Y satisfies (16) and L, R are nonzero; we prove that then there exists $\lambda > 0$ such that (17) holds true. First, observe that w.l.o.g. we may assume that L and R are of the same sizes $r \times n$ (to reduce the general case to this particular one, it suffices to add several zero rows either to L (when $\ell < r$), or to R (when $\ell > r$)). We have the following chain of equivalences:

$$\begin{array}{rc}
 & (16) \\
 & \Updownarrow \\
\forall \xi \in \mathbf{C}^m: & \xi^H Y\xi \geq 2\|L\xi\|_2\|R\xi\|_2 \\
 & \Updownarrow \\
\forall (\xi \in \mathbf{C}^n, \eta \in \mathbf{C}^r): & \|\eta\|_2 \leq \|L\xi\|_2 \Rightarrow \xi^H Y\xi - \eta^H R\xi - \xi^H R^H \eta \geq 0 \\
 & \Updownarrow \\
\forall (\xi \in \mathbf{C}^m, \eta \in \mathbf{C}^r): & \xi^H L^H L\xi - \eta^H\eta \geq 0 \Rightarrow \xi^H Y\xi - \eta^H R\xi - \xi^H R^H \eta \geq 0 \\
 & \Updownarrow \\
\exists (\lambda \geq 0): & \left[\begin{array}{cc} Y & R^H \\ R & \end{array}\right] - \lambda \left[\begin{array}{cc} L^H L & \\ & -I_r \end{array}\right] \succeq 0 \quad [\mathcal{S}\text{-Lemma}] \\
 & \Updownarrow \\
(a) & \left[\begin{array}{cc} Y - \lambda L^H L & R^H \\ R & \lambda I_r \end{array}\right] \succeq 0
\end{array} \qquad (24)$$

Recall that $\mathcal{S}$-Lemma we have referred to is the following extremely useful statement:

> *Let P, Q be real symmetric matrices of the same size such that $\bar{x}^T P\bar{x} > 0$ for certain $\bar{x}$. Then the implication*
>
> $$x^T P x \geq 0 \Rightarrow x^T Q x \geq 0$$
>
> *holds true if and only if there exists $\lambda \geq 0$ such that $Q \succeq \lambda P$.*

From this "real symmetric case" statement one can immediately derive its "Hermitian" analogy (the one we have actually used), since Hermitian quadratic forms on $\mathbf{C}^m$ can be treated as real quadratic forms on $\mathbf{R}^{2m}$.

Condition (24.a), in view of $R \neq 0$, clearly implies that $\lambda > 0$. Therefore, by the Schur Complement Lemma (SCL), (24.a) is equivalent to

$$Y - \lambda L^H L - \lambda^{-1} R^H R \succeq 0,$$

as claimed.

(ii): When $R \neq 0$, (ii) is clearly equivalent to (i) and thus is already proved. When $R = 0$, it is evident that (18) can be satisfied by properly chosen $\lambda \in \mathbf{R}$ if and only if $Y \succeq 0$, which is exactly what is stated by (16) when $R = 0$. ∎

2.1.2 Proof of Theorem 2.1.(iii)

In order to prove (iii), it suffices to prove the following statement:

Claim 2.1 *Assume that $\gamma \geq 0$ is such that the predicate $(\mathcal{II}[\gamma])$ is not valid. Then the predicate $(\mathcal{I}[\vartheta_{\mathbf{C}}(d^{\mathrm{s}})\gamma])$, with appropriately defined function $\vartheta_{\mathbf{C}}(\cdot)$ satisfying (10), is also not valid.*

We are about to prove Claim 2.1. The case of $\gamma = 0$ is trivial, so that from now on we assume that $\gamma > 0$ and that all matrices L_p, R_p are nonzero (the latter assumption, of course, does not restrict generality). *From now till the end of Section 2.1, we assume that we are under the premise of Claim 2.1, i.e., the predicate $(\mathcal{II}[\gamma])$ is not valid.*

2.1.3 First step: duality

Consider the optimization program

$$\min_{\substack{t,\{Y_p\in\mathbf{H}^m\}_{p\in I_{\mathrm{s}}^{\mathrm{r}}},\\ \{U_p,V_p\in\mathbf{H}^m\}_{p\in I_{\mathrm{s}}^{\mathrm{c}}},\\ \{\lambda_p,\nu_p\in\mathbf{R}\}_{p\in I_{\mathrm{f}}^{\mathrm{c}}}}} \left\{ t : \begin{array}{ll} Y_p \pm \underbrace{[L_p^H R_p + R_p^H L_p]}_{2A_p, A_p = A_p^H} \succeq 0,\ p \in I_{\mathrm{s}}^{\mathrm{r}}, & (a) \\ \left[\begin{array}{cc} U_p & R_p^H L_p \\ L_p^H R_p & V_p \end{array}\right] \succeq 0,\ p \in I_{\mathrm{s}}^{\mathrm{c}}, & (b) \\ \left[\begin{array}{cc} \lambda_p & 1 \\ 1 & \nu_p \end{array}\right] \succeq 0,\ p \in I_{\mathrm{f}}^{\mathrm{c}}, & (c) \\ tI + A - \gamma\Big[\sum\limits_{p\in I_{\mathrm{s}}^{\mathrm{r}}} Y_p + \sum\limits_{p\in I_{\mathrm{s}}^{\mathrm{c}}} [U_p + V_p] & \\ \quad + \sum\limits_{p\in I_{\mathrm{f}}^{\mathrm{c}}} [\lambda_p L_p^H L_p + \nu_p R_p^H R_p]\Big] \succeq 0 & (d) \end{array} \right\}. \tag{25}$$

Introducing "bounds" $Y_p = U_p + V_p$ for $p \in I_{\mathrm{s}}^{\mathrm{c}}$ and $Y_p \succeq \lambda_p L_p^H L_p + \nu_p R_p^H R_p$ for $p \in I_{\mathrm{f}}^{\mathrm{c}}$ and then eliminating the variables U_p, $p \in I_{\mathrm{s}}^{\mathrm{c}}$, ν_p, $p \in I_{\mathrm{f}}^{\mathrm{c}}$, we convert (25) into the equivalent problem

$$\min_{\substack{t,\{Y_p\in\mathbf{H}^m\}_{p=1}^k,\\ \{V_p\in\mathbf{H}^m\}_{p\in I_{\mathrm{s}}^{\mathrm{c}}},\\ \{\lambda_p\in\mathbf{R}\}_{p\in I_{\mathrm{f}}^{\mathrm{c}}}}} \left\{ t : \begin{array}{l} Y_p \pm [L_p^H R_p + R_p^H L_p] \succeq 0,\ p \in I_{\mathrm{s}}^{\mathrm{r}}, \\ \left[\begin{array}{cc} Y_p - V_p & R_p^H L_p \\ L_p^H R_p & V_p \end{array}\right] \succeq 0,\ p \in I_{\mathrm{s}}^{\mathrm{c}}, \\ \left[\begin{array}{cc} Y_p - \lambda_p L_p^H L_p & R_p^H \\ R_p & \lambda_p I_{d_p} \end{array}\right] \succeq 0,\ p \in I_{\mathrm{f}}^{\mathrm{c}}, \\ tI + A - \gamma \sum\limits_{p=1}^k Y_p \succeq 0 \end{array} \right\}.$$

By (already proved) item (ii) of Theorem 2.1, predicate $(\mathcal{II}[\gamma])$ is valid if and only if the latter problem, and thus problem (25), admits a feasible solution with $t \leq 0$. We are in the situation when $(\mathcal{II}[\gamma])$ is *not* valid; consequently, (25) does not admit feasible solutions with $t \leq 0$. Since the problem clearly is solvable, it means that the optimal value in the problem is positive. Problem (25) is a conic problem on the product of cones of Hermitian and real symmetric positive semidefinite matrices. Since (25) is strictly feasible and bounded below, the Conic Duality Theorem implies that the conic dual problem of (25) is solvable with the same positive optimal value. Taking into account that the cones associated with (25) are self-dual, the dual problem, after straightforward simplifications, becomes the conic problem

$$
\begin{array}{lrcl}
\text{maximize} & \multicolumn{3}{l}{-2\gamma\left[\sum\limits_{p\in I_{\rm s}^{\rm r}}\mathrm{Tr}([P_p-Q_p]A_p)+\sum\limits_{p\in I_{\rm s}^{\rm c}}\Re\{\mathrm{Tr}\left(S_pR_p^HL_p\right)\}+\sum\limits_{p\in I_{\rm f}^{\rm c}}w_p\right]-\mathrm{Tr}(ZA)}\\
\text{s.t.} & & & \\
(a.1) & P_p,Q_p & \succeq & 0,\ p\in I_{\rm s}^{\rm r},\\
(a.2) & P_p+Q_p & = & Z,\ p\in I_{\rm s}^{\rm r};\\
(b) & \left[\begin{array}{cc} Z & S_p^H\\ S_p & Z\end{array}\right] & \succeq & 0,\ p\in I_{\rm s}^{\rm c};\\
(c) & \left[\begin{array}{cc}\mathrm{Tr}(L_pZL_p^H) & w_p\\ w_p & \mathrm{Tr}(R_pZR_p^H)\end{array}\right] & \succeq & 0,\ p\in I_{\rm f}^{\rm c};\\
(d) & Z\succeq 0,\mathrm{Tr}(Z) & = & 1.
\end{array}
\tag{26}
$$

in matrix variables $Z\in\mathbf{H}^m_+$, $P_p,Q_p\in\mathbf{H}^m$, $p\in I_{\rm s}^{\rm r}$, $S_p\in\mathbf{C}^{m\times m}$, $p\in I_{\rm s}^{\rm c}$, and real variables w_p, $p\in I_{\rm f}^{\rm c}$. Using (26.c), we can eliminate the variables w_p, thus coming the following equivalent reformulation of the dual problem:

$$
\begin{array}{lrcl}
\text{maximize} & \multicolumn{3}{l}{2\gamma\Big[-\sum\limits_{p\in I_{\rm s}^{\rm r}}\mathrm{Tr}([P_p-Q_p]A_p)-\sum\limits_{p\in I_{\rm s}^{\rm c}}\Re\{\mathrm{Tr}\left(S_pR_p^HL_p\right)\}}\\
& \multicolumn{3}{l}{\quad+\sum\limits_{p\in I_{\rm f}^{\rm c}}\underbrace{\sqrt{\mathrm{Tr}(L_pZL_p^H)}}_{\|L_pZ^{1/2}\|_2}\underbrace{\sqrt{\mathrm{Tr}(R_pZR_p^H)}}_{\|R_pZ^{1/2}\|_2}\Big]-\mathrm{Tr}(ZA)}\\
\text{s.t.} & & & \\
(a.1) & P_p,Q_p & \succeq & 0,\ p\in I_{\rm s}^{\rm r},\\
(a.2) & P_p+Q_p & = & Z,\ p\in I_{\rm s}^{\rm r};\\
(b) & \left[\begin{array}{cc} Z & S_p^H\\ S_p & Z\end{array}\right] & \succeq & 0,\ p\in I_{\rm s}^{\rm c};\\
(c) & Z\succeq 0,\mathrm{Tr}(Z) & = & 1.
\end{array}
\tag{27}
$$

Next we eliminate the variables S_p, Q_p, R_p. It is clear that

1. (27.a) is equivalent to the fact that $P_p=Z^{1/2}\widehat{P}_pZ^{1/2}$, $Q_p=Z^{1/2}\widehat{Q}_pZ^{1/2}$ with $\widehat{P}_p,\widehat{Q}_p\succeq 0$, $\widehat{P}_p+\widehat{Q}_p=I_m$. With this parameterization of P_p,Q_p, the corresponding terms in the objective become $-2\gamma\,\mathrm{Tr}([\widehat{P}_p-\widehat{Q}_p](Z^{1/2}A_pZ^{1/2}))$. Note that the matrices A_p are Hermitian (see (25)), and observe that if $A\in\mathbf{H}^m$, then

$$\max_{P,Q\in\mathbf{H}^m}\{\mathrm{Tr}([P-Q]A):0\preceq P,Q,P+Q=I_m\}=\|\lambda(A)\|_1\equiv\sum_\ell|\lambda_\ell(A)|$$

   (w.l.o.g., we may assume that A is Hermitian and diagonal, in which case the relation becomes evident). In view of this observation, partial optimization in P_p, Q_p in (27) allows to replace in the objective of the problem the terms $-2\gamma\,\mathrm{Tr}([P_p-Q_p]A_p)$ with $2\gamma\|\lambda(Z^{1/2}A_pZ^{1/2})\|_1$ and to eliminate the constraints (27.a).

2. Same as in the proof of Lemma 2.1, constraints $(27.b)$ are equivalent to the fact that $S_p = -Z^{1/2}U_pZ^{1/2}$ with $\|U_p\| \le 1$. With this parameterization, the corresponding terms in the objective become $2\gamma\Re\{\mathrm{Tr}(U_p(Z^{1/2}R_p^HL_pZ^{1/2}))\}$, and the maximum of this expression in U_p, $\|U_p\| \le 1$, equals to $2\gamma\|\sigma(Z^{1/2}R_p^HL_pZ^{1/2})\|_1$. With this observation, partial optimization in S_p in (27) allows to replace in the objective of the problem the terms $-2\gamma\Re\{\mathrm{Tr}(S_pR_p^HL_p)\}$ with $2\gamma\|\sigma(Z^{1/2}R_p^HL_pZ^{1/2})\|_1$ and to eliminate the constraints $(27.b)$.

After the above reductions, problem (27) becomes

$$
\begin{array}{ll}
\text{maximize} & 2\gamma\Big[\sum\limits_{p\in I_{\rm s}^{\rm r}}\|\lambda(Z^{1/2}A_pZ^{1/2})\|_1 + \sum\limits_{p\in I_{\rm s}^{\rm c}}\|\sigma(Z^{1/2}R_p^HL_pZ^{1/2})\|_1 \\
& \qquad + \sum\limits_{p\in I_{\rm f}^{\rm c}}\|L_pZ^{1/2}\|_2\|R_pZ^{1/2}\|_2\Big] - \mathrm{Tr}(ZA) \\
\text{s.t.} & Z \succeq 0, \mathrm{Tr}(Z) = 1.
\end{array}
\tag{28}
$$

Recall that we are in the situation when the optimal value in problem (26), and thus in problem (28), is positive. Thus, we arrive at an intermediate conclusion as follows.

Lemma 2.4 *Under the premise of Claim 1, there exists $Z \in \mathbf{H}^m$, $Z \succeq 0$, such that*

$$
\begin{array}{l}
2\gamma\Big[\sum\limits_{p\in I_{\rm s}^{\rm r}}\|\lambda(Z^{1/2}A_pZ^{1/2})\|_1 + \sum\limits_{p\in I_{\rm s}^{\rm c}}\|\sigma(Z^{1/2}R_p^HL_pZ^{1/2})\|_1 \\
\qquad + \sum\limits_{p\in I_{\rm f}^{\rm c}}\|L_pZ^{1/2}\|_2\|R_pZ^{1/2}\|_2\Big] > \mathrm{Tr}(Z^{1/2}AZ^{1/2}).
\end{array}
\tag{29}
$$

Here the Hermitian matrices A_p are given by

$$
2A_p = L_p^HR_p + R_p^HL_p,\ p \in I_{\rm s}^{\rm r}. \tag{30}
$$

2.1.4 Second step: probabilistic interpretation of (29)

The major step in completing the proof of Theorem 2.1.(iii) is based on a probabilistic interpretation of (29). This step is described next.

Preliminaries. Let us define a standard Gaussian vector ξ in $\mathbf{R}^n$ (notation: $\xi \in \mathcal{N}_{\mathbf{R}}^n$) as a real Gaussian random n-dimensional vector with zero mean and unit covariance matrix; in other words, ξ_ℓ are independent Gaussian random variables with zero mean and unit variance, $\ell = 1, ..., n$. Similarly, we define a standard Gaussian vector ξ in $\mathbf{C}^n$ (notation: $\xi \in \mathcal{N}_{\mathbf{C}}^n$) as a complex Gaussian random n-dimensional vector with zero mean and unit (complex) covariance matrix. In other words, $\xi_\ell = \alpha_\ell + i\alpha_{n+\ell}$, where $\alpha_1, ..., \alpha_{2n}$ are independent real Gaussian random variables with zero means and variances $\frac{1}{2}$, and i is the imaginary unit.

We shall use the facts established in the next three propositions.

Proposition 2.1 *Let ν be a positive integer, and let $\vartheta_{\mathbf{S}}(\nu)$, $\vartheta_{\mathbf{H}}(\nu)$ be given by the relations*

$$
\begin{array}{rcll}
\vartheta_{\mathbf{S}}^{-1}(\nu) &=& \min\limits_{\alpha}\left\{\mathbf{E}_\xi\left\{|\sum\limits_{\ell=1}^{\nu}\alpha_\ell\xi_\ell^2|\right\} : \quad \alpha \in \mathbf{R}^\nu, \|\alpha\|_1 = 1\right\} & [\xi \in \mathcal{N}_{\mathbf{R}}^\nu], \\
\vartheta_{\mathbf{H}}^{-1}(\nu) &=& \min\limits_{\alpha}\left\{\mathbf{E}_\zeta\left\{|\sum\limits_{\ell=1}^{\nu}\alpha_\ell|\zeta_\ell|^2|\right\} : \quad \alpha \in \mathbf{R}^\nu, \|\alpha\|_1 = 1\right\} & [\zeta \in \mathcal{N}_{\mathbf{C}}^\nu].
\end{array}
\tag{31}
$$

Then

(i) *Both* $\vartheta_{\mathbf{S}}(\cdot)$, $\vartheta_{\mathbf{H}}(\cdot)$ *are nondecreasing functions such that*

$$\begin{array}{lrcl} (a.1) & \vartheta_{\mathbf{S}}(1) & = & 1,\ \vartheta_{\mathbf{S}}(2)=\frac{\pi}{2},\\ (a.2) & \vartheta_{\mathbf{S}}(\nu) & \leq & \frac{\pi}{2}\sqrt{\nu},\ \nu\geq 1;\\ (b.1) & \vartheta_{\mathbf{H}}(1) & = & 1,\ \vartheta_{\mathbf{H}}(2)=2,\\ (b.2) & \vartheta_{\mathbf{H}}(\nu) & \leq & \vartheta_{\mathbf{S}}(2\nu)\leq \pi\sqrt{\nu/2},\ \nu\geq 1. \end{array} \tag{32}$$

(ii) *For every* $A\in\mathbf{S}^n$, *one has*

$$\mathbf{E}_\xi\left\{|\xi^TA\xi|\right\}\geq \|\lambda(A)\|_1\vartheta_{\mathbf{S}}^{-1}(\mathrm{Rank}(A))\quad [\xi\in\mathcal{N}_{\mathbf{R}}^n], \tag{33}$$

and for every $A\in\mathbf{H}^n$ *one has*

$$\mathbf{E}_\zeta\left\{|\zeta^TA\zeta|\right\}\geq \|\lambda(A)\|_1\vartheta_{\mathbf{H}}^{-1}(\mathrm{Rank}(A))\quad [\zeta\in\mathcal{N}_{\mathbf{C}}^n]. \tag{34}$$

Proof. The function $\vartheta_{\mathbf{S}}(\cdot)$ was introduced in [4], where (32.a) and (33) were proved as well. From the definition of $\vartheta_{\mathbf{H}}(\cdot)$ it is clear that this function is nondecreasing. To establish (34), note that the distribution of a random complex Gaussian vector is invariant under unitary transformations of $\mathbf{C}^n$, hence it suffices to verify (34) in the particular case when A is a diagonal Hermitian matrix with $\mathrm{Rank}(A)$ nonzero diagonal entries (the nonzero eigenvalues of A), and the remaining diagonal entries equal to 0. But in this case (34) is readily given by the definition of $\vartheta_{\mathbf{H}}(\cdot)$.

It remains to verify (32.b). The relation $\vartheta_{\mathbf{H}}(1)=1$ is evident. Further, we clearly have

$$\vartheta_{\mathbf{H}}^{-1}(2)=\min_{\beta\in[0,1]}\psi(\beta),\quad \psi(\beta)=\mathbf{E}_\zeta\left\{|\beta|\zeta_1|^2-(1-\beta)|\zeta_2|^2|\right\},\ \zeta\in\mathcal{N}_{\mathbf{C}}^2.$$

The function $\psi(\beta)$ is convex in $\beta\in[0,1]$ and is symmetric: $\psi(1-\beta)=\psi(\beta)$. It follows that its minimum is achieved at $\beta=\frac{1}{2}$; direct computation demonstrates that $\psi(1/2)=1/2$, which completes the proof of (32.b.1).

It remains to prove the first inequality in (32.b.2). Given $\alpha\in\mathbf{R}^\nu$, $\|\alpha\|_1=1$, let $\widetilde{\alpha}=(\alpha^T,\alpha^T)^T\in\mathbf{R}^{2\nu}$. Now, if $\zeta=\eta+i\omega$ is a standard Gaussian vector in $\mathbf{C}^\nu$, then the vector $\xi=2^{1/2}(\eta^T,\omega^T)^T$ is a standard Gaussian vector in $\mathbf{R}^{2\nu}$. We now have

$$\begin{array}{rcl}\mathbf{E}_\zeta\left\{|\sum\limits_{\ell=1}^{\nu}\alpha_\ell|\zeta_\ell|^2|\right\} & = & \mathbf{E}_\zeta\left\{|\sum\limits_{\ell=1}^{\nu}\alpha_\ell[\eta_\ell^2+\omega_\ell^2]|\right\}=\frac{1}{2}\mathbf{E}_\xi\left\{|\sum\limits_{\ell=1}^{2\nu}\widetilde{\alpha}_\ell\xi_\ell^2|\right\}\geq\frac{1}{2}\|\widetilde{\alpha}\|_1\vartheta_{\mathbf{S}}^{-1}(2\nu)\\ & = & \vartheta_{\mathbf{S}}^{-1}(2\nu),\end{array}$$

whence $\vartheta_{\mathbf{H}}^{-1}(\nu)\geq\vartheta_{\mathbf{S}}^{-1}(2\nu)$, and the desired inequality follows. ■

Proposition 2.2 *For every* $A\in\mathbf{C}^{n\times n}$ *one has*

$$\mathbf{E}_\zeta\left\{|\zeta^HA\zeta|\right\}\geq\|\sigma(A)\|_1\frac{1}{4}\vartheta_{\mathbf{H}}^{-1}(2\,\mathrm{Rank}(A))\quad[\zeta\in\mathcal{N}_{\mathbf{C}}^n]. \tag{35}$$

Proof. Let $\widehat{A}=\left[\begin{array}{cc} & A\\ A^H & \end{array}\right]$, so that $\widehat{A}\in\mathbf{H}^{2n}$, $\mathrm{Rank}(\widehat{A})=2\,\mathrm{Rank}(A)$ and the eigenvalues of $\widehat{A}$ are $\pm\sigma_\ell(A)$, $\ell=1,...,n$. Let also $\zeta=(\eta^T,\omega^T)^T$ be a standard Gaussian vector in

$\mathbf{C}^{2n}$ partitioned into two n-dimensional blocks, so that η, ω are independent standard Gaussian vectors in $\mathbf{C}^n$. We have

$$
\begin{array}{rcl}
\zeta^H \widehat{A}\zeta & = & 2\Re\{\eta^H A\omega\} \\
& = & \Re\Big\{[(\eta+\omega)^H A(\eta+\omega) - \eta^H A\eta - \omega^H A\omega] \\
& & \quad +i[(\eta - i\omega)^H A(\eta - i\omega) - \eta^H A\eta - \omega^H A\omega]\Big\} \\
& & \hfill \text{[polarization identity]} \\
\Rightarrow \mathbf{E}_\zeta\left\{|\zeta^H \widehat{A}\zeta|\right\} & \leq & \mathbf{E}_{\eta,\omega}\left\{|(\eta+\omega)^H A(\eta+\omega)|\right\} + \mathbf{E}_{\eta,\omega}\left\{|(\eta - i\omega)^H A(\eta - i\omega)|\right\} \\
& & +2\mathbf{E}_\eta\left\{|\eta^H A\eta|\right\} + 2\mathbf{E}_\omega\{|\zeta^H A\zeta|\}.
\end{array}
\tag{36}
$$

Since η, ω are independent standard Gaussian vectors in $\mathbf{C}^n$, the vectors $2^{-1/2}(\eta+\omega)$ and $2^{-1/2}(\eta - i\omega)$ also are standard Gaussian. Therefore (36) implies that

$$
\mathbf{E}_\zeta\left\{|\zeta^H \widehat{A}\zeta|\right\} \leq 8\mathbf{E}_\eta\left\{|\eta^H A\eta|\right\}. \tag{37}
$$

Since $\widehat{A}$ is a Hermitian matrix of rank $2\,\mathrm{Rank}(A)$ and $\|\lambda(\widehat{A})\|_1 = 2\|\sigma(A)\|_1$, the left hand side in (37) is at least $2\|\sigma(A)\|_1 \vartheta_{\mathbf{H}}^{-1}(2\,\mathrm{Rank}(A))$, and (37) implies (35). ■

Proposition 2.3 (i) *Let $L, R \in \mathbf{C}^{d\times n}$, and let ζ be a standard Gaussian vector in $\mathbf{C}^n$. Then*

$$
\mathbf{E}_\zeta\left\{\|L\zeta\|_2\|R\zeta\|_2\right\} \geq \frac{\pi}{4}\|L\|_2\|R\|_2. \tag{38}
$$

(ii) *Let $L, R \in \mathbf{R}^{d\times n}$, and let ξ be a standard Gaussian vector in $\mathbf{R}^n$. Then*

$$
\mathbf{E}_\xi\left\{\|L\xi\|_2\|R\xi\|_2\right\} \geq \frac{2}{\pi}\|L\|_2\|R\|_2. \tag{39}
$$

Proof. (i): There is nothing to prove when L or R are zero matrices; thus, assume that both L and R are nonzero.

Let us demonstrate first that it suffices to verify (38) in the case when both L and R are rank 1 matrices. Let $L^H L = U^H \mathrm{Diag}\{\lambda\} U$ be the eigenvalue decomposition of $L^H L$, so that U is a unitary matrix and $\lambda \geq 0$. We have

$$
\begin{array}{rcl}
\mathbf{E}\left\{\|L\xi\|_2\|R\xi\|_2\right\} & = & \mathbf{E}\left\{\sqrt{\xi^H L^H L\xi}\|R\xi\|_2\right\} \\
& = & \mathbf{E}\Big\{((U\xi)^H \mathrm{Diag}\{\lambda\} \underbrace{(U\xi)}_{\zeta})^{1/2} \underbrace{\|RU^H\zeta\|_2}_{\phi(\zeta)\geq 0}\Big\} \\
& = & \mathbf{E}\left\{\phi(\zeta)\sqrt{\sum_{\ell=1}^n \lambda_\ell|\zeta_\ell|^2}\right\} = \Phi(\lambda), \\
\Phi(x) & = & \mathbf{E}\left\{\phi(\zeta)\sqrt{\sum_{\ell=1}^n x_\ell|\zeta_\ell|^2}\right\}.
\end{array}
\tag{40}
$$

The function $\Phi(x)$ of $x \in \mathbf{R}^n_+$ is concave; therefore its minimum on the simplex

$$
S = \{x \in \mathbf{R}^n_+ : \sum_\ell x_\ell = \sum_\ell \lambda_\ell\}
$$

is achieved at a vertex, let it be e. Now let $\widehat{L} \in \mathbf{C}^{d\times n}$ be such that $\widehat{L}^H\widehat{L} = U^H \operatorname{Diag}\{e\}U$. Note that $\widehat{L}$ is a rank 1 matrix (since e is a vertex of S) and that

$$[\|\widehat{L}\|_2^2 =] \quad \operatorname{Tr}(\widehat{L}^H\widehat{L}) = \sum_\ell e_\ell = \sum_\ell \lambda_\ell = \operatorname{Tr}(L^HL) \quad [= \|L\|_2^2].$$

Since the unitary factor in the eigenvalue decomposition of $\widehat{L}^H\widehat{L}$ is U, (40) holds true when L is replaced with $\widehat{L}$ and λ with e, so that

$$\mathbf{E}\left\{\|\widehat{L}\zeta\|_2\|R\zeta\|_2\right\} = \Phi(e) \leq \Phi(\lambda) = \mathbf{E}\left\{\|L\zeta\|_2\|R\zeta\|_2\right\}.$$

Applying the same reasoning to the quantity

$$\mathbf{E}\left\{\|\widehat{L}\zeta\|_2\|R\zeta\|_2\right\}$$

with R playing the role of L, we conclude that there exists a rank 1 matrix $\widehat{R}$ such that

$$\|\widehat{R}\|_2 = \|R\|_2$$

and

$$\mathbf{E}\left\{\|\widehat{L}\zeta\|_2\|\widehat{R}\zeta\|_2\right\} \leq \mathbf{E}\left\{\|\widehat{L}\zeta\|_2\|R\zeta\|_2\right\}.$$

Thus, replacing L and R with the rank 1 matrices $\widehat{L}$, $\widehat{R}$, we do not increase the left hand side in (38) and do not vary the right hand side, so that it indeed suffices to establish (38) in the case when L, R are rank 1 matrices. Note that so far our reasoning did not use the fact that ζ is standard Gaussian.

Now let us look what inequality (38) says in the case of rank 1 matrices L, R. By homogeneity, we can further assume that $\|L\|_2 = \|R\|_2 = 1$. With this normalization, for rank 1 matrices L, R we clearly have $L\zeta = z\ell$ and $R\zeta = wr$ for unit deterministic vectors ℓ, r and a Gaussian random vector $(z, w) \in \mathbf{C}^2 = \mathbf{R}^4$ such that $\mathbf{E}\{|z|^2\} = \mathbf{E}\{|w|^2\} = 1$ (both z and w are just linear combinations, with appropriate deterministic coefficients, of the entries in ζ). Since $\mathbf{E}\{|z|^2\} = \mathbf{E}\{|w|^2\} = 1$, we can express (z, w) in terms of a *standard* Gaussian vector $(\eta, \xi) \in \mathbf{C}^2$ as $z = \eta$, $w = \cos(\theta)\eta + \sin(\theta)\xi$, where $\theta \in [0, \frac{\pi}{2}]$ is such that $\cos(\theta)$ is the absolute value of the correlation $\mathbf{E}\{zw^*\}$ between z and w. With this representation, inequality (38) becomes

$$\phi(\theta) \equiv \int\limits_{\mathbf{C}\times\mathbf{C}} |\eta||\cos(\theta)\eta + \sin(\theta)\xi|dG(\eta, \xi) \geq \frac{\pi}{4} \equiv \phi(\frac{\pi}{2}), \tag{41}$$

where $G(\eta, \xi)$ is the distribution of (η, ξ). We should prove (41) in the range $[0, \frac{\pi}{2}]$ of values of θ; in fact we shall prove this inequality in the larger range $\theta \in [0, \pi]$. Given $\theta \in [0, \pi]$, we set

$$u = \cos(\theta/2)\eta + \sin(\theta/2)\xi, \quad v = -\sin(\theta/2)\eta + \cos(\theta/2)\xi;$$

it is immediately seen that the distribution of (u, v) is exactly G. At the same time,

$$\eta = \cos(\theta/2)u - \sin(\theta/2)v, \quad \cos(\theta)\eta + \sin(\theta)\xi = \cos(\theta/2)u + \sin(\theta/2)v,$$

whence

$$\begin{aligned}\phi(\theta) &= \int\limits_{\mathbf{C}\times\mathbf{C}} |\cos(\theta/2)u - \sin(\theta/2)v||\cos(\theta/2)u + \sin(\theta/2)v| dG(u,v)\\ &= \int\limits_{\mathbf{C}\times\mathbf{C}} |\cos^2(\theta/2)u^2 - \sin^2(\theta/2)v^2| dG(u,v).\end{aligned}$$

We see that

$$\min_{\theta\in[0,\pi]} \phi(\theta) = \min_{0\leq\alpha\leq 1} \psi(\alpha), \quad \psi(\alpha) = \int\limits_{\mathbf{C}\times\mathbf{C}} |\alpha u^2 - (1-\alpha)v^2| dG(u,v).$$

The function $\psi(\alpha)$ clearly is convex and $\psi(1-\alpha) = \psi(\alpha)$ (since the distribution of (u,v) is symmetric in u,v). Consequently, ψ attains its minimum when $\alpha = 1/2$, and ϕ attains its minimum when $\cos^2(\theta/2) = 1/2$, i.e., when $\theta = \pi/2$, which is exactly what is stated in (41).

(ii): Applying exactly the same reasoning as in the proof of (i), we conclude that it suffices to verify (39) in the case when L, R are real rank 1 matrices. In this case, the same argument as above demonstrates that (39) is equivalent to the fact that that if ξ, η are independent real standard Gaussian variables and $G(\xi,\eta)$ is the distribution of (ξ,η), then the function

$$\phi(\theta) = \int\limits_{\mathbf{R}\times\mathbf{R}} |\xi||\cos(\theta)\xi + \sin(\theta)\eta| dG(\xi,\eta) \tag{42}$$

of $\theta \in [0,\pi]$ achieves its minimum when $\theta = \frac{\pi}{2}$. To prove this statement, one can repeat word by word, with evident modifications, the reasoning we have used in the complex case. ■

2.1.5 Completing the proof of Theorem 2.1.(iii)

We are now in a position to complete the proof of Theorem 2.1.(iii). Let us set

$$\begin{array}{rcl} d^{\rm s}_{\mathbf{R}} &=& 2\max\{d_p : p \in I^{\rm r}_{\rm s}\},\\ d^{\rm s}_{\mathbf{C}} &=& 2\max\{d_p : p \in I^{\rm c}_{\rm s}\},\\ \vartheta &=& \max\left[\vartheta_{\mathbf{H}}(d^{\rm s}_{\mathbf{R}}), 4\vartheta_{\mathbf{H}}(d^{\rm s}_{\mathbf{C}}), \frac{4}{\pi}\right]; \end{array} \tag{43}$$

here by definition the maximum over an empty set is 0, and $\vartheta_{\mathbf{H}}(0) = 0$. Note that by (32) one has

$$\vartheta \leq 4\pi\sqrt{d^{\rm s}}$$

(cf. (9), (10)).

Let ζ be a standard Gaussian vector in $\mathbf{C}^n$. Invoking Propositions 2.1 – 2.3, we have (for notation, see Lemma 2.4):

$$\begin{array}{rcl}
\|\lambda(Z^{1/2}A_pZ^{1/2})\|_1 & \leq & \vartheta_{\mathbf{H}}(\mathrm{Rank}(Z^{1/2}A_pZ^{1/2}))\mathbf{E}_\zeta\left\{|\zeta^HZ^{1/2}A_pZ^{1/2}\zeta|\right\}\\
& \leq & \vartheta\mathbf{E}_\zeta\left\{|\zeta^HZ^{1/2}A_pZ^{1/2}\zeta|\right\},\ p\in I_{\mathrm{s}}^{\mathrm{r}}\\
\multicolumn{3}{r}{[\text{by Proposition 2.1 since } A_p=A_p^H \text{ and } \mathrm{Rank}(A_p)=\mathrm{Rank}([L_p^HR_p+R_p^HL_p])\leq 2d_p]}\\
\|\sigma(Z^{1/2}R_p^HL_pZ^{1/2})\|_1 & \leq & 4\vartheta_{\mathbf{H}}(2\,\mathrm{Rank}(Z^{1/2}R_p^HL_pZ^{1/2}))\mathbf{E}_\zeta\left\{|\zeta^HZ^{1/2}R_p^HL_pZ^{1/2}\zeta|\right\}\\
& \leq & \vartheta\mathbf{E}_\zeta\left\{|\zeta^HZ^{1/2}R_p^HL_pZ^{1/2}\zeta|\right\},\ p\in I_{\mathrm{s}}^{\mathrm{c}}\\
\multicolumn{3}{r}{[\text{by Proposition 2.2 since } \mathrm{Rank}(R_p^HL_p)\leq d_p]}\\
\|L_pZ^{1/2}\|_2\|R_pZ^{1/2}\|_2 & \leq & \frac{4}{\pi}\mathbf{E}_\zeta\left\{\|L_pZ^{1/2}\zeta\|_2\|R_pZ^{1/2}\zeta\|_2\right\}\\
& \leq & \vartheta\mathbf{E}_\zeta\left\{\|L_pZ^{1/2}\zeta\|_2\|R_pZ^{1/2}\zeta\|_2\right\}\\
\multicolumn{3}{r}{[\text{by Proposition 2.3.(i)}]}
\end{array}$$

and, of course,

$$\mathbf{E}_\zeta\left\{\zeta^HZ^{1/2}AZ^{1/2}\zeta\right\}=\mathrm{Tr}(Z^{1/2}AZ^{1/2}).$$

In view of these observations, (29) implies that

$$\begin{array}{l}
\gamma\vartheta\Big[\sum\limits_{p\in I_{\mathrm{s}}^{\mathrm{r}}}\mathbf{E}_\zeta\left\{|\zeta^HZ^{1/2}[L_p^HR_p+R_p^HL_p]Z^{1/2}\zeta|\right\}+\sum\limits_{p\in I_{\mathrm{s}}^{\mathrm{c}}}\mathbf{E}_\zeta\left\{2|\zeta^HZ^{1/2}R_p^HL_pZ^{1/2}\zeta|\right\}\\
\qquad+\sum\limits_{p\in I_{\mathrm{f}}^{\mathrm{c}}}\mathbf{E}_\zeta\left\{2\|L_pZ^{1/2}\zeta\|_2\|R_pZ^{1/2}\zeta\|_2\right\}\Big]>\mathbf{E}_\zeta\left\{\zeta^HZ^{1/2}AZ^{1/2}\zeta\right\}
\end{array}$$

(we have substituted the expressions for A_p, see (30)). It follows that there exists a realization $\widehat{\zeta}$ of ζ such that with $\xi=Z^{1/2}\widehat{\zeta}$ one has

$$\gamma\vartheta\left[\sum_{p\in I_{\mathrm{s}}^{\mathrm{r}}}|\xi^H[L_p^HR_p+R_p^HL_p]\xi|+\sum_{p\in I_{\mathrm{s}}^{\mathrm{c}}}2|\xi^HR_p^HL_p\xi|+\sum_{p\in I_{\mathrm{f}}^{\mathrm{c}}}2\|L_p\xi\|_2\|R_p\xi\|_2\right]>\xi^HA\xi. \quad (44)$$

Observe that

- The quantities $\xi^H[L_p^HR_p+R_p^HL_p]\xi$ are real; we therefore can choose $\delta_p=\pm1$, $p\in I_{\mathrm{s}}^{\mathrm{r}}$, in such a way that with $\Delta_p=\delta_pI_{d_p}$ one has
$$\xi^H[L_p^H\Delta_pR_p+R_p^H\Delta_p^HL_p]\xi=|\xi^H[L_p^HR_p+R_p^HL_p]\xi|,\ p\in I_{\mathrm{s}}^{\mathrm{r}};$$
- For $p\in I_{\mathrm{s}}^{\mathrm{c}}$, we can choose $\delta_p\in\mathbf{C}$, $|\delta_p|=1$, in such a way that with $\Delta_p=\delta_pI_{d_p}$ one has
$$\xi^H[L_p^H\Delta_pR_p+R_p^H\Delta_p^HL_p]\xi=2|\xi^HR_p^HL_p\xi|,\ p\in I_{\mathrm{s}}^{\mathrm{c}};$$
- For $p\in I_{\mathrm{f}}^{\mathrm{c}}$, we can choose $\Delta_p\in\mathbf{C}^{d_p\times d_p}$, $\|\Delta_p\|\leq1$, in such a way that
$$\xi^H[L_p^H\Delta_pR_p+R_p^H\Delta_p^HL_p]\xi=2\|L_p\xi\|_2\|R_p\xi\|_2,\ p\in I_{\mathrm{f}}^{\mathrm{c}}.$$

With Δ_p's we have defined, (44) reads

$$\xi^H\underbrace{\Big[A-\gamma\vartheta\sum_{p=1}^{k}[L_p^H\Delta_pR_p+R_p^H\Delta_p^HL_p]\Big]}_{C}\xi<0,$$

so that C is not positive semidefinite; on the other hand, by construction $C\in\mathcal{U}[\vartheta\gamma]$. Thus, the predicate ($\mathcal{I}[\vartheta\gamma]$) is not valid; recalling the definition of ϑ, this completes the proof of Claim 2.1 and thus the proof of Theorem 2.1.(iii). ∎

3 Matrix Cube Theorem, Real case

The Real Matrix Cube problem is as follows:

RMC: *Let m, d_1,...,d_k be positive integers, and $A \in \mathbf{S}^m_+$, $L_p, R_p \in \mathbf{R}^{d_p \times m}$ be given matrices, $L_p \neq 0$. Let also a partition $\{1, 2, ..., k\} = I^{\mathrm{r}}_{\mathrm{s}} \cup I^{\mathrm{r}}_{\mathrm{f}}$ of the index set $\{1, ..., k\}$ into two non-overlapping sets be given. With these data, we associate a parametric family of "matrix boxes"*

$$\mathcal{U}[\gamma] = \left\{ A + \gamma \sum_{p=1}^{k} [L_p^T \Delta_p R_p + R_p^T \Delta_p^T L_p] : \begin{array}{c} \Delta_p \in \mathbf{\Delta}^{\mathbf{R}}_p, \|\Delta_p\| \leq 1, \\ p = 1, ..., k \end{array} \right\} \subset \mathbf{S}^m, \tag{45}$$

where $\gamma \geq 0$ is the parameter and

$$\mathbf{\Delta}^{\mathbf{R}}_p = \begin{cases} \{\delta I_{d_p} : \delta \in \mathbf{R}\}, & p \in I^{\mathrm{r}}_{\mathrm{s}} \text{ [“scalar perturbations”]} \\ \mathbf{R}^{d_p \times d_p}, & p \in I^{\mathrm{r}}_{\mathrm{f}} \text{ [“full size perturbations”]} \end{cases}. \tag{46}$$

Given $\gamma \geq 0$, check whether

$$\mathcal{U}[\gamma] \subset \mathbf{S}^m_+ \tag{$\mathcal{I}_{\mathbf{R}}[\gamma]$}$$

Remark 3.1 *In the sequel, we always assume that $d_p > 1$ for $p \in I^{\mathrm{r}}_{\mathrm{s}}$. Indeed, non-repeated ($d_p = 1$) scalar perturbations always can be regarded as full size perturbations.*

The RMC problem, same as the CMC one, is, in general, NP-hard; similar to the complex case, we intend to build a "computationally tractable" sufficient condition for the validity of ($\mathcal{I}_{\mathbf{R}}[\gamma]$) and to understand how "conservative" is this condition.

Consider, along with predicate ($\mathcal{I}_{\mathbf{R}}[\gamma]$), the predicate

$$\begin{array}{llcl} \multicolumn{4}{l}{\exists Y_p \in \mathbf{S}^m,\ p = 1, ..., k :} \\ (a) & Y_p & \succeq & L_p^T \Delta_p R_p + R_p^T \Delta_p^T L_p \quad \forall (\Delta_p \in \mathbf{\Delta}^{\mathbf{R}}_p, \|\Delta_p\| \leq 1),\ p = 1, ..., k, \\ (b) & A - \gamma \sum\limits_{p=1}^{k} Y_p & \succeq & 0. \end{array} \tag{$\mathcal{II}_{\mathbf{R}}[\gamma]$}$$

The Real case version of Theorem 2.1 is as follows:

Theorem 3.1 [The Real Matrix Cube Theorem] *One has:*

(i) *Predicate ($\mathcal{II}_{\mathbf{R}}[\gamma]$) is stronger than ($\mathcal{I}_{\mathbf{R}}[\gamma]$) – the validity of the former predicate implies the validity of the latter one.*

(ii) ($\mathcal{II}_{\mathbf{R}}[\gamma]$) *is computationally tractable – the validity of the predicate is equivalent to the solvability of the system of LMIs*

$$\begin{array}{lrcl} (s) & Y_p \pm \left[L_p^T R_p + R_p^T L_p\right] & \succeq & 0,\ p \in I^{\mathrm{r}}_{\mathrm{s}}, \\ (f) & \begin{bmatrix} Y_p - \lambda_p L_p^T L_p & R_p^T \\ R_p & \lambda_p I_{d_p} \end{bmatrix} & \succeq & 0,\ p \in I^{\mathrm{r}}_{\mathrm{f}} \\ (*) & A - \gamma \sum\limits_{p=1}^{k} Y_p & \succeq & 0. \end{array} \tag{47}$$

in matrix variables $Y_p \in \mathbf{S}^m$, $p = 1, ..., k$, *and real variables* λ_p, $p \in I_{\rm f}^{\rm r}$.

(iii) *"The gap" between* $(\mathcal{I}_{\mathbf{R}}[\gamma])$ *and* $(\mathcal{II}_{\mathbf{R}}[\gamma])$ *can be bounded solely in terms of the maximal size*

$$d^s = \max\{d_p : p \in I_{\rm s}^{\rm r}\}$$

of the scalar perturbations (here the maximum over an empty set by definition is 0). Specifically, there exists a universal function $\vartheta_{\mathbf{R}}(\nu) \le \pi\sqrt{\nu/2}$, $\nu \ge 1$, *such that*

$$\text{if } (\mathcal{II}_{\mathbf{R}}[\gamma]) \text{ is not valid, then } (\mathcal{I}_{\mathbf{R}}[\vartheta_{\mathbf{R}}(d^{\rm s})\gamma]) \text{ is not valid.} \tag{48}$$

Corollary 3.1 *The efficiently computable supremum* $\widehat{\gamma}$ *of those* $\gamma \ge 0$ *for which the system of LMIs* (47) *is solvable is a lower bound on the supremum* $\gamma^\star$ *of those* $\gamma \ge 0$ *for which* $\mathcal{U}[\gamma] \subset \mathbf{S}^m_+$, *and this lower bound is tight within the factor* $\vartheta_{\mathbf{R}}(d^{\rm s})$*:*

$$\widehat{\gamma} \le \gamma^\star \le \vartheta_{\mathbf{R}}(d^{\rm s})\widehat{\gamma}. \tag{49}$$

Remark 3.2 *From the proof of Theorem 3.1 it follows that* $\vartheta_{\mathbf{R}}(0) = \frac{\pi}{2}$, $\vartheta_{\mathbf{R}}(2) = 2$. *Thus,*

- *when there are no scalar perturbations:* $I_{\rm s}^{\rm r} = \emptyset$ *(cf. Remark 3.1), the factor* ϑ *in the implication*

$$\neg(\mathcal{II}_{\mathbf{R}}[\gamma]) \Rightarrow \neg(\mathcal{I}_{\mathbf{R}}[\vartheta\gamma]) \tag{50}$$

  can be set to $\frac{\pi}{2} = 1.57...$

- *when all scalar perturbations are repeated twice* ($d^{\rm s} = 2$)*, the factor* ϑ *in* (50) *can be set to* 2.

The proof of the Real Matrix Cube Theorem repeats word by word, with evident simplifications, the proof of its complex case counterpart and is therefore omitted. Note that the difference in absolute constant factors in bounds on $\vartheta(\nu)$ in Theorems 2.1 and 3.1 (which is "in favour" of the real case) comes mainly from the absolute constants in (32) which are different for the real and the complex cases. The difference between the absolute constants in Remarks 2.2 and 3.2, which is in favour of the complex case, comes from the difference between (38) and (39). Note also that Remarks 2.3, 2.4 remain valid in the real case.

Matrix Cube theorems and known results on tractable approximations of semi-infinite LMIs. The results we have established compare favourably with known results on the quality of tractable approximations of semi-infinite LMIs (1). Aside of the Matrix Cube theorem of [4] (this result, which is the prototype of all our developments here, was already discussed in Introduction), there is, to the best of our knowledge, a single relevant result, specifically, Theorem 6.2.2 in [2]. This theorem is of the same spirit as Theorem 3.1, specifically, it deals with a bi-affine *real* LMI (1) where *real* perturbations Δ are *diagonal* matrices, and the set $\mathbf{\Delta}$ in (1) is

$$\mathbf{\Delta} = \{\mathrm{Diag}\{(\delta^1, ..., \delta^k)\} : \delta^p \in \mathbf{R}^{d_p}, \|\delta^p\|_2 \le 1, p = 1, ..., k\} \tag{51}$$

(the "structure" $d_1, ..., d_k$ is fixed). The theorem associates with (1) an explicit system $\mathcal{S}_\gamma$ of LMIs in x and additional variables u in such a way that

(a) if x can be extended to a feasible solution of $\mathcal{S}_\gamma$, then x is feasible for the semi-infinite LMI (1), the level of perturbations being γ;

(b) if x *cannot* be extended to a feasible solution to $\mathcal{S}_\gamma$, then x is *not* feasible for (1) *when the perturbation level is increased from γ to $\chi\gamma$.*

The "tightness factor" χ (which plays the same role as the factor ϑ in the Matrix Cube theorems) is shown to be

$$\chi = \min[\sqrt{mk}, \sqrt{\sum_{p=1}^{k} d_p}], \tag{52}$$

(m is the row size of the LMI (1), k is the number of blocks δ^p,and $d_p = \dim \delta^p$). When comparing this result with those given by Theorem 3.1, it makes sense to restrict ourselves with the case when $d_p = 1$, $p = 1, ..., k$ – this is the only case where the statements under considerations speak about the same perturbation set $\boldsymbol{\Delta}$. Note that in the case in question (1) becomes the semi-infinite LMI

$$A[x] + \sum_{p=1}^{k} \delta^p A_p[x] \succeq 0 \quad \forall(\delta = (\delta^1, ..., \delta^k) \in \mathbf{R}^k : \|\delta\|_\infty \leq \gamma), \tag{53}$$

where $A[x]$, $A_p[x]$ are symmetric matrices affinely depending on x. On a closest inspection, it turns out that *in the case of $d_p = 1$, $p = 1, ..., k$, both tractable approximations of* (1) *– the one built in [2] and the one given in Theorem 3.1 – are identical to each other* and are given by the system of LMIs

$$Y_p \succeq \pm A_p[x],\ p = 1, ..., k,\ A[x] - \gamma \sum_{p=1}^{k} Y_p \succeq 0 \tag{54}$$

in the original variables x and additional matrix variables Y_1,...,Y_p. Although both Theorem 6.2.2 from [2] and our Theorem 3.1 speak about the same pair of entities (53), (54), the tightness factors provided by these two statements are different: in the case of $d_1 = ... = d_k = 1$, (52) results in the tightness factor $\chi = \sqrt{k}$, while Theorem 3.1 says that the factor is at most $\vartheta \leq \pi\sqrt{\max\limits_{1\leq p\leq k} \max\limits_{x} \mathrm{Rank}(A_p[x])/2}$. Formally speaking, these two upper bounds are in "general position" – no one of them dominates the other one. However, in typical applications (e.g., those considered in [4] or to be considered below) the second bound is by far better than the first one.

We are about to illustrate the use of the Matrix Cube Theorems by two application examples. The first example (Section 4) is a complex-case version of the $\frac{\pi}{2}$-Theorem of Yu. Nesterov [11]. The second example (Section 5) deals with an important Control entity – the structured singular value.

4 Maximizing Hermitian quadratic form over the complex unit cube

Let $S \in \mathbf{H}^m$, $S \succ 0$. Consider the problem of maximizing the quadratic form $x^H S x$ over the complex unit cube:

$$\omega_\star(S) = \max\left\{z^H S z : \|z\|_\infty \equiv \max_r |z_r| \leq 1\right\}. \tag{55}$$

It is well-known that the real case version of the problem (S is real symmetric, and the complex unit cube is replaced with the real one) is NP-hard; the same can be shown to be true for the complex case (55). It is also known that in the real case the standard semidefinite relaxation bound

$$\widehat{\omega}(S) = \min\left\{\sum_{r=1}^m \lambda_r : \lambda \in \mathbf{R}^m_+, \mathrm{Diag}\{\lambda\} \succeq S\right\} \tag{56}$$

is an upper bound on $\omega_\star(S)$ tight within the factor $\frac{\pi}{2}$ ("$\frac{\pi}{2}$-Theorem" of Yu. Nesterov, [11]). It is immediate to see that $\widehat{\omega}(S)$ is an upper bound on $\omega_\star(S)$ in the complex case as well. Indeed, if $\lambda \in \mathbf{R}^m_+$ is such that $\mathrm{Diag}\{\lambda\} \succeq S$, and $\|z\|_\infty \leq 1$, then

$$z^H S z \leq z^H \mathrm{Diag}\{\lambda\} z = \sum_r \lambda_r |z_r|^2 \leq \sum_r \lambda_r.$$

We are about to demonstrate that in the complex case $\widehat{\omega}(S)$ coincides with $\omega_\star(S)$ within the factor $\frac{4}{\pi}$:

$$\omega_\star(S) \leq \widehat{\omega}(S) \leq \frac{4}{\pi}\omega_\star(S). \tag{57}$$

The proof follows the lines of an alternative proof of the $\frac{\pi}{2}$-Theorem given in [4]. Observe that $\omega_\star(S)$ is the minimum of those $\omega \in \mathbf{R}$ for which the ellipsoid $\{z \in \mathbf{C}^m : z^H S z \leq \omega\}$ contains the complex unit cube $\{z \in \mathbf{C}^m : \|z\|_\infty \leq 1\}$. This inclusion is equivalent to the fact that the polar of the ellipsoid (which is the ellipsoid $\{\zeta \in \mathbf{C}^m : \zeta^H S^{-1}\zeta \leq \omega^{-1}\}$) is contained in the polar of the unit cube (which is the set $\{\zeta \in \mathbf{C}^m : \|\zeta\|_1 \equiv \sum\limits_r |z_r| \leq 1\}$). In turn, the inclusion $\{\zeta \in \mathbf{C}^m : \zeta^H S^{-1}\zeta \leq \omega^{-1}\} \subset \{\zeta \in \mathbf{C}^m : \|\zeta\|_1 \equiv \sum\limits_r |z_r| \leq 1\}$ is, by homogeneity, equivalent to the fact $\omega\zeta^H S^{-1}\zeta \geq \|\zeta\|_1^2$ for all $\zeta \in \mathbf{C}^m$. Combining our observations, we arrive at the equality

$$\omega_\star^{-1}(S) = \max\left\{\gamma \in \mathbf{R}_+ : \zeta^H S^{-1}\zeta \geq \gamma\|\zeta\|_1^2 \quad \forall \zeta\right\}.$$

Further, by evident reasons one has

$$\|\zeta\|_1^2 = \max\left\{\zeta^H B \zeta : B \in \mathbf{H}^m, |B_{rs}| \leq 1, 1 \leq r \leq s \leq m\right\}.$$

Indeed, when $B = B^H$, $|B_{rs}| \leq 1$, the quantity $|\zeta^H B \zeta|$ clearly is $\leq \|\zeta\|_1^2$ and is equal to $\|\zeta\|_1^2$ when $H = \widehat{\zeta}\widehat{\zeta}^H$, where $\widehat{\zeta}_r = \zeta_r/|\zeta_r|$ (when $\zeta_r = 0$, one can set $\widehat{\zeta}_r = 0$ as well). Combining our observations, we arrive at the relation

$$\begin{array}{rcl}\omega_\star^{-1}(S) & = & \max\left\{\gamma \in \mathbf{R}_+ : \zeta^H S^{-1}\zeta \geq \gamma\zeta^H B\zeta \quad \forall (B \in \mathbf{H}^m : |B_{rs}| \leq 1)\right\} \\ & = & \max\left\{\gamma \in \mathbf{R}_+ : S^{-1} - \gamma B \succeq 0 \quad \forall (B \in \mathbf{H}^m : |B_{rs}| \leq 1)\right\}.\end{array} \tag{58}$$

Denoting by e_r the standard basic orths in $\mathbf{C}^m$ and specifying the data of a Complex Matrix Cube problem as

$$\begin{array}{rcl} d_{rs} & = & 1,\, 1 \le r \le s \le m, \\ R_{rs} & = & e_r^H,\, 1 \le r \le s \le m, \\ L_{rs} & = & \begin{cases} \frac{1}{2} e_r^H, & r = s \\ e_s^H, & r < s \end{cases},\, 1 \le r \le s \le m, \\ A & = & S^{-1}, \\ I_{\rm f}^{\rm c} & = & \{(r,s) : 1 \le r \le s \le m\}, \\ I_{\rm s}^{\rm r} = I_{\rm s}^{\rm c} & = & \emptyset \end{array} \tag{59}$$

(here it is convenient to index the perturbations by pairs of integers rather than by single integer), we see that (58) says exactly that $\omega_\star^{-1}(S)$ is the largest $\gamma = \gamma^\star$ for which the complex matrix box given by the data (59) is contained in $\mathbf{H}_+^m$.

Applying the Complex case Matrix Cube Theorem, we arrive at an explicit Generalized Eigenvalue problem such that its optimal value, let it be $\widehat{\gamma}$, is a lower bound, tight within the factor $\frac{4}{\pi}$, for $\gamma^\star = \omega_\star^{-1}(S)$ (note that we are in the case when there are no scalar perturbations). Consequently, $\widehat{\gamma}^{-1}$ is an upper bound, tight within the same factor $\frac{4}{\pi}$, for $\omega_\star(S)$. Exactly in the same way as in the real case (see [4], Section 4), it can be further verified that $\widehat{\gamma}^{-1}$ is nothing but the semidefinite relaxation bound $\widehat{\omega}(S)$. ■

5 Lyapunov stability radius and Structured singular value

5.1 Preliminaries

When a problem of the form CMC comes from applications, its "structure" m, k, $d_1, ..., d_k$, $I_{\rm s}^{\rm r}$, $I_{\rm s}^{\rm c}$, $I_{\rm f}^{\rm c}$ is usually fixed, while "the data" A, $\{L_p, R_p\}_{p=1}^k$ may depend on a vector of design variables x of a certain "master problem" of the generic form

$$\gamma^\star = \max_{\gamma, x} \{\gamma \ge 0 : x \in \mathcal{X} \ \&\ (\mathcal{I}^x[\gamma]) \text{ is valid}\} \tag{$\mathcal{P}$}$$

where $(\mathcal{I}^x[\gamma])$ is the predicate $(\mathcal{I}[\gamma])$ associated with the data $A = A[x]$, $L_p = L_p[x]$, $R_p = R_p[x]$ (cf. the Lyapunov Stability Analysis example presented in Introduction). In these cases, the following observation allows to build a tractable approximation of the master problem $(\mathcal{P})$:

Proposition 5.1 *Assume that $A[x]$, $R_p[x]$ are affine in x and L_p are independent of x, and let the set $\mathcal{X}$ in $(\mathcal{P})$ be semidefinite-representable:*

$$x \in \mathcal{X} \Leftrightarrow \exists u : \mathcal{C}[x,u] \succeq 0, \tag{60}$$

where $\mathcal{C}[x,u]$ is a symmetric matrix affinely depending on (x,u). Under these assumptions, *the problem*

$$\widehat{\gamma} = \max\left\{\gamma \geq 0 : \begin{array}{ll} \mathcal{C}[x,u] \succeq 0; & \\ Y_p \pm \left[L_p^H R_p[x] + R_p^H[x] L_p\right] \succeq 0, & p \in I_{\rm s}^{\rm r} \\ \left[\begin{array}{cc} Y_p - V_p & L_p^H R_p[x] \\ R_p^H[x] L_p & V_p \end{array}\right] \succeq 0, & p \in I_{\rm s}^{\rm c} \\ \left[\begin{array}{cc} Y_p - \lambda_p L_p^H L_p & R_p^H[x] \\ R_p[x] & \lambda_p I_{d_p} \end{array}\right] \succeq 0, & p \in I_{\rm f}^{\rm c} \\ A[x] - \gamma \sum\limits_{p=1}^{k} Y_p \succeq 0 & \end{array}\right\} \tag{$\mathcal{A}$}$$

in matrix variables $Y_p \in \mathbf{H}^m$*,* $p = 1, ..., k$*,* $V_p \in \mathbf{H}^m$*,* $p \in I_{\rm s}^{\rm c}$*,* $\lambda_p \in \mathbf{R}$*,* $p \in I_{\rm f}^{\rm c}$*,* x, u *is an explicit Generalized Eigenvalue problem which is a conservative approximation of* $(\mathcal{P})$*: whenever* (γ, x, u) *can be extended to a feasible solution of* $(\mathcal{A})$*, then* (γ, x, u) *is feasible for* $(\mathcal{P})$*. The quality of this approximation can be quantified as follows: if* $(\mathcal{P})$ *is feasible, then so is* $(\mathcal{A})$*, and*

$$\widehat{\gamma} \leq \gamma^{\star} \leq \vartheta_{\mathbf{C}}(d^{\rm s})\widehat{\gamma}, \tag{61}$$

(cf. Theorem 2.1). Note that $\widehat{\gamma}$ *is the optimal value in an explicit Generalized Eigenvalue problem and is therefore efficiently computable.*

This statement is readily given by Theorem 2.1 and admits a straightforward real case analogy implied by Theorem 3.1.

We are about to consider an instructive application example for Proposition 5.1.

5.2 Estimating Lyapunov stability radius for an uncertain discrete time dynamical system

Consider a discrete time dynamical system with states $z_t \in \mathbf{C}^m$ obeying the dynamics

$$z_{t+1} = A_t z_t,\ t = 0, 1, ... \tag{62}$$

We assume that the system is uncertain, in the sense that the matrices A_t are *not* known in advance; all we know is that they vary in a given *uncertainty set*:

$$\forall t : A_t \in \mathcal{A}_{\gamma} = \{A = A_* + \gamma \sum_{p=1}^{k} C_p^H \Delta_p D_p : \Delta_p \in \mathbf{\Delta}_p\}, \tag{63}$$

where γ is the perturbation level and $\mathbf{\Delta}_p$ are given by (7).

We are interested to certify the stability of (62) – the fact that all trajectories $\{z_t\}$ of all "realizations" $\{A_t \in \mathcal{A}_{\gamma}\}_{t=0}^{\infty}$ of the system converge to 0 as $t \to \infty$. The standard *sufficient* condition for stability is that all matrices $A \in \mathcal{A}_t$ share a common discrete-time *Lyapunov stability certificate* Y, i.e., there exists $Y \succ 0$ such that $A^H Y A \prec Y$ for all $A \in \mathcal{A}_{\gamma}$. Setting $X = Y^{-1}$ and applying the SCL, it is easily seen that the existence of a common Lyapunov stability certificate is equivalent to the solvability of the following semi-infinite system of LMIs in matrix variable X:

$$\left[\begin{array}{cc} X & XA^H \\ AX & X \end{array}\right] \succ 0\ \forall A \in \mathcal{A}_{\gamma};\ X \succeq I.$$

Recalling the description of $\mathcal{A}_\gamma$, we can rewrite this system equivalently as

$$\underbrace{\left[\begin{array}{cc} X & XA_*^H \\ A_*X & X \end{array}\right]}_{A[X]} + \gamma \sum_{p=1}^{k} \underbrace{\left[\begin{array}{cc} & XD_p^H\Delta_p^H C_p \\ C_p^H \Delta_p D_p X & \end{array}\right]}_{\substack{L_p^H \Delta_p R_p[X] + R_p^H[X]\Delta_p^H L_p \\ \left[\begin{array}{c} L_p = [0_{d_p\times m}, C_p], \\ R_p[X] = [D_pX, 0_{d_p\times m}] \end{array}\right]}} \succ 0 \quad \forall (\Delta_p \in \boldsymbol{\Delta}_p,\, p=1,...,k) \tag{64}$$

The *Lyapunov stability radius* γ^* of uncertain system (62) – (63) is the supremum of those uncertainty levels γ for which the system admits a Lyapunov stability certificate, or, which is the same, for which the semi-infinite system of LMIs (64) is solvable. Assuming that the "nominal" matrix A_* defines a stable system (i.e., the spectral radius $\rho(A_*)$ of A_* is less than 1), it is immediately seen that

$$\gamma_* = \sup_{\gamma>0,X} \left\{ \gamma : \begin{array}{l} X \succeq I \\ A[X] + \gamma \sum\limits_p [L_p^H \Delta_p R_p[X] + R_p^H[X]\Delta_p^H L_p] \succeq 0 \,\forall \left(\begin{array}{c} \Delta_p \in \boldsymbol{\Delta}_p, \\ p = 1,...,k \end{array} \right) \end{array} \right\} \tag{65}$$

We see that the problem of computing the Lyapunov stability radius is of the generic form $(\mathcal{P})$ (with X playing the role of x and $\mathcal{X} = \{X : X \succeq I\}$). Applying Proposition 5.1, we arrive at the following conclusion.

Corollary 5.1 *The Lyapunov stability radius γ^* of* (62) – (63) (which by itself is, in general, NP-hard to compute) *admits an efficiently computable lower bound $\widehat{\gamma}$ which conisides with γ^* up to a factor not exceeding $O(1)\sqrt{d^{\rm s}}$, where $d^{\rm s}$ is the maximum of row sizes of scalar perturbation blocks Δ_p. When there are no scalar perturbation blocks, one has $\frac{\gamma^*}{\widehat{\gamma}} \leq \frac{4}{\pi}$.*

The results similar to Corollary 5.1 hold true for the Lyapunov stability radius of continuous time uncertain system (see Introduction); besides this, we could consider the case of real systems and perturbations (in both the discrete- and the continuous-time settings). The particular setup we dealt with is motivated by the desire to link our considerations with an important Control entity – the *complex structured singular value*.

5.2.1 Bounding the complex structured singular value

The *complex structured singular value* of a matrix is an important Control entity (for an overview of the corresponding *μ-theory*, see, e.g., [12]) which is defined as follows.

• A (complex) *block structure* on $\mathbf{C}^m$ is an ordered collection of positive integers $d_1, ..., d_k$, $\sum\limits_{p=1}^{k} d_k = m$, along with a partitioning of the index set $\{1, ..., k\}$ into two non-overlapping sets $I_{\rm s}^{\rm c}$, $I_{\rm f}^{\rm c}$. Same as in the description of the CMC problem, such a structure defines the sets $\boldsymbol{\Delta}_p \subset \mathbf{C}^{d_p \times d_p}$ of "scalar complex" and "full complex" perturbation blocks, $p = 1, ..., k$. We set

$$\boldsymbol{\Delta} = \{\Delta = \mathrm{Diag}\{\Delta_1, ..., \Delta_k\} : \Delta_p \in \boldsymbol{\Delta}_p,\, p = 1, ..., k\} \subset \mathbf{C}^{m\times m}.$$

• Given a block structure, the corresponding *structured singular value* $\mu_{\mathbf{\Delta}}(M)$ of a matrix $M \in \mathbf{C}^{m\times m}$ is defined as

$$\mu_{\mathbf{\Delta}}(M) = \max\left\{\rho(\Delta M) : \Delta \in \mathbf{\Delta}, \|\Delta\| \leq 1\right\}, \tag{66}$$

where $\rho(S)$ is the spectral radius of a square matrix S. Recalling that the spectral radius of a matrix A is < 1 if and only if A admits a discrete time Lyapunov stability certificate (i.e., there exists $X \succ 0$ such that $\left[\begin{array}{cc} X & XA^H \\ AX & X \end{array}\right] \succ 0$), an equivalent definition of $\mu_{\mathbf{\Delta}}(M)$ is as follows:

> (!) *The quantity $\frac{1}{\mu_{\mathbf{\Delta}}(M)}$ is the supremum of those $\gamma \geq 0$ for which every one of the matrices $\gamma M\Delta$, $\Delta \in \mathbf{\Delta}$, admits discrete time Lyapunov stability certificate.*

In general, it is NP-hard to compute the quantity $\mu_{\mathbf{\Delta}}(\cdot)$, this is why an important role in μ-theory is played by computable bounds on $\mu_{\mathbf{\Delta}}$. As far as upper bounds are concerned, the standard one (and, to the best of our knowledge, the only one) is the *scaling* upper bound $\widehat{\mu}_{\mathbf{\Delta}}(M)$ defined as follows. Let $\mathbf{D}$ be the set of all Hermitian $m \times m$ matrices $D \succeq I_m$ which commute with all matrices from $\mathbf{\Delta}$; in other words, $D \in \mathbf{D}$ if and only if $D = \mathrm{Diag}\{D_1, ..., D_k\}$, where $D_p \in \mathbf{H}^{d_p}$ are $\succeq I_{d_p}$ and, besides this, $D_p = \lambda_p I_{d_p}$ for $p \in I_{\mathrm{f}}^{\mathrm{c}}$. The bound $\widehat{\mu}_{\mathbf{\Delta}}(M)$ is defined as

$$\widehat{\mu}_{\mathbf{\Delta}}(M) = \frac{1}{\widehat{\gamma}}, \tag{67}$$

where

$$\widehat{\gamma} = \sup_{\gamma\geq 0, D\in\mathbf{D}} \left\{\gamma : \left[\begin{array}{cc} D & \gamma DM^H \\ \gamma MD & D \end{array}\right] \succeq 0\right\} \tag{68}$$

(note that by homogeneity reasons, the optimal value in the latter problem remains unchanged when the normalization constraint $D \succeq I$ in the definition of $\mathbf{D}$ is relaxed to $D \succ 0$). The fact that $\widehat{\mu}_{\mathbf{\Delta}}(M)$ is an upper bound on $\mu_{\mathbf{\Delta}}(M)$ is immediate. Indeed, taking into account the definitions of $\mu_{\mathbf{\Delta}}$ and $\widehat{\mu}_{\mathbf{\Delta}}$, we observe that to prove that $\widehat{\mu}_{\mathbf{\Delta}}(M) \geq \mu_{\mathbf{\Delta}}(M)$ is the same as to verify that if γ and $D \in \mathbf{D}$ are such that $\left[\begin{array}{cc} D & \gamma DM^H \\ \gamma MD & D \end{array}\right] \succ 0$ (or, which is the same, $\gamma^2 M^H D^{-1} M \prec D^{-1}$), then $\rho(t\Delta M) < 1$ for all $\Delta \in \mathbf{\Delta}$ and $0 \leq t < \gamma$. To verify the latter claim, note that

$$\gamma^2 M^H D^{-1} M \prec D^{-1} \Rightarrow t^2 (M\Delta)^H D^{-1} (M\Delta) \preceq (t/\gamma)^2 \Delta^H D^{-1} \Delta \underbrace{\preceq}_{(*)} (t/\gamma)^2 D^{-1} \prec D^{-1},$$

where $(*)$ is readily given by the fact that Δ commutes with $D \in \mathbf{D}$ and $\|\Delta\| \leq 1$. We see that the matrix $tM\Delta$ admits a discrete time Lyapunov stability certificate, or, which is exactly the same, $\rho(t\Delta M) = \rho(tM\Delta) < 1$, as required.

As defined above (and as arising in the Control literature), the scaling upper bound $\widehat{\mu}_{\mathbf{\Delta}}(M)$ looks as a very useful "ad hoc" invention. Could we build a computable upper bound on $\mu_{\mathbf{\Delta}}(M)$ in a more systematic way? The answer is affirmative. Indeed, consider the set of matrices

$$\mathcal{A}_\gamma = \gamma\{\Delta M : \Delta \in \mathbf{\Delta}\}; \tag{69}$$

note that this set is of the form (63) with $A_* = 0_{m\times m}$. By (!), $\frac{1}{\mu_{\mathbf{\Delta}}(M)}$ is the supremum of those γ for which every $A \in \mathcal{A}_\gamma$ admits discrete time Lyapunov stability certificate. This property, of course, is weaker than the existence a *common* Lyapunov stability certificate for all matrices from $\mathcal{A}_\gamma$; it follows that

$$\frac{1}{\mu_{\mathbf{\Delta}}(M)} \geq \gamma^* \geq \widehat{\gamma},$$

where γ^* is the Lyapunov stability radius of uncertain dynamical system with uncertainty set (69), and $\widehat{\gamma}$ is the computable lower bound for this radius mentioned in Corollary 5.1. *We have arrived at a computable upper bound on* $\mu_{\mathbf{\Delta}}(M)$, specifically, the quantity $\widehat{\omega}_{\mathbf{\Delta}}(M) = [\widehat{\gamma}]^{-1}$. An explicit description of $\widehat{\omega}_{\mathbf{\Delta}}(M)$ is given by the optimization problem $(\mathcal{A})$, the data of the problem coming from the description (69) of the underlying uncertainty set. Skipping the purely "mechanical" derivation, here is the resulting description of $\widehat{\omega}_{\mathbf{\Delta}}(M)$:

$$\frac{1}{\widehat{\omega}_{\mathbf{\Delta}}(M)} = \max_{\substack{\gamma\in\mathbf{R}, X\in\mathbf{H}^m,\\ Y_p,V_p\in\mathbf{H}^{2m},\lambda_p\in\mathbf{R}}} \left\{\gamma : \begin{array}{ll} X \succeq I_m & (a)\\ \begin{bmatrix} Y_p - V_p & L_p^H R_p[X] \\ R_p^H[X] L_p & V_p\end{bmatrix} \succeq 0, \quad p\in I_{\rm s}^{\rm c} & (b)\\ \begin{bmatrix} Y_p - \lambda_p L_p^H L_p & R_p^H[X] \\ R_p[X] & \lambda_p I_{d_p}\end{bmatrix} \succeq 0, \quad p\in I_{\rm f}^{\rm c} & (c)\\ A[X] - \gamma \sum_{p=1}^k Y_p \succeq 0 & (d)\end{array}\right\}$$
$$\left[\begin{array}{rcl} P_p &=& \left[0_{d_p\times d_1+\ldots+d_{p-1}}, I_{d_p}, 0_{d_p\times d_{p+1}+\ldots+d_k}\right] \in \mathbf{C}^{d_p\times m},\\ R_p[X] &=& \left[P_p M X, 0_{d_p\times m}\right], \quad L_p = \left[0_{d_p\times m}, P_p\right], \quad A[X] = \begin{bmatrix} X & \\ & X\end{bmatrix}\end{array}\right] \tag{70}$$

Now a natural question arises: there are two efficiently computable upper bounds on the "computationally intractable" structured singular value $\mu_{\mathbf{\Delta}}(M)$ – the usual scaling bound $\widehat{\mu}_{\mathbf{\Delta}}(M)$ (67) – (68) and the bound $\widehat{\omega}_{\mathbf{\Delta}}(M)$ given by (70). What is the relation between these bounds? Here is the answer:

Proposition 5.2 *One always has*

$$\widehat{\omega}_{\mathbf{\Delta}}(M) = \widehat{\mu}_{\mathbf{\Delta}}(M). \tag{71}$$

Proof. The proof decomposes into two parts.

1. Let us first prove that $\widehat{\omega}_{\mathbf{\Delta}}(M) \leq \widehat{\mu}_{\mathbf{\Delta}}(M)$. Recalling the origins of $\widehat{\mu}_{\mathbf{\Delta}}(M)$ and $\widehat{\omega}_{\mathbf{\Delta}}(M)$, in order to verify this inequality we should prove that if $(\gamma > 0, D)$ is feasible for (68), then γ can be extended to a feasible solution of problem (70). Let $(\gamma > 0, D)$ be feasible for (68), so that $D = \mathrm{Diag}\{D_1, ..., D_k\}$ with positive definite Hermitian $d_p \times d_p$ blocks D_p

which are scalar matrices $\lambda_p I_{d_p}$ for $p \in I_{\rm f}^{\rm c}$. By homogeneity of the LMI in (68) w.r.t. D we may assume that $\gamma D \succeq I_m$. Let us set

$$\begin{array}{rcl} X &=& \gamma D, \\ Y_p &=& L_p^H D_p L_p + R_p^H[\gamma D] D_p^{-1} R_p[\gamma D],\ p = 1, ..., k. \end{array} \tag{72}$$

It suffices to demonstrate that the matrices we have defined can be extended by properly chosen V_p, $p \in I_{\rm s}^{\rm c}$, and λ_p, $p \in I_{\rm f}^{\rm c}$, to a feasible solution of (70). Here is the demonstration:

1^0. (70.a) immediately follows from the definition of X due to our normalization $\gamma D \succeq I_m$ of D.

2^0. To prove (70.b), we first need

Lemma 5.1 *Let $L, R \in \mathbf{C}^{d\times n}$ and $W \in \mathbf{H}^d$, $W \succ 0$. Then the matrix*

$$Y = L^H W L + R^H W^{-1} R$$

satisfies the relation

$$Y \succeq [\delta L^H R + \delta^* R^H L] \quad \forall (\delta \in \mathbf{C}, |\delta| \leq 1). \tag{73}$$

Proof. Let $\xi \in \mathbf{C}^n$. Then

$$\begin{array}{rcl} \xi^H Y \xi &=& \xi^H L^H W L \xi + \xi^H R^H W^{-1} R \xi = \|W^{1/2} L \xi\|_2^2 + \|W^{-1/2} R \xi\|_2^2 \\ &\geq& 2\|W^{1/2} L\xi\|_2 \|W^{-1/2} R\xi\|_2 \geq 2|[W^{1/2} L\xi]^H [W^{-1/2} R\xi]| \\ &=& 2|\xi^H L^H R \xi| \geq |\xi^H [\delta L^H R + \delta^* R^H L]\xi| \quad \forall (\delta, |\delta| \leq 1). \end{array} \qquad \blacksquare$$

Combining Lemmas 5.1 and 2.1, we conclude that for every one of the matrices Y_p, $p \in I_{\rm s}^{\rm c}$, one can find a matrix V_p in such a way that relations (70.b) are satisfied.

3^0. In fact, we already have in our disposal the λ_p's required in (70.c) (recall that for $p \in I_{\rm f}^{\rm c}$ one has $D_p = \lambda_p I_{d_p}$ with $\lambda_p > 0$). When $p \in I_{\rm f}^{\rm c}$, (72) says that $Y_p = \lambda_p L_p^H L_p + \lambda_p^{-1} R_p^H[X] R_p[X]$, which, by the SCL, implies the validity of (70.c).

4^0. It remains to verify (70.d), which is immediate. Observe, first, that by construction of the matrices P_p (see (70)) one has

$$\Delta = \mathrm{Diag}\{\Delta_1, ..., \Delta_k\} \in \boldsymbol{\Delta} \Rightarrow \Delta = \sum_{p=1}^k P_p^H \Delta_p P_p, \tag{74}$$

whence

$$\begin{array}{l} \forall \Delta = \mathrm{Diag}\{\Delta_1, ..., \Delta_k\} \in \boldsymbol{\Delta} : \\ \left[\begin{array}{cc} X & \gamma X M^H \Delta^H \\ \gamma \Delta M X & X \end{array}\right] \;=\; A[X] + \gamma \sum_{p=1}^k \left[L_p^H \Delta_p R_p[X] + R_p^H[X] \Delta_p^H L_p\right] \end{array} \tag{75}$$

with $A[X]$, L_p, $R_p[X]$ given by (70). Since both D and D^{-1} are block-diagonal with the sizes of the diagonal blocks $d_1, ..., d_k$, (74) implies that

$$D = \sum_{p=1}^k P_p^H D_p P_p, \quad D^{-1} = \sum_{p=1}^k P_p^H D_p^{-1} P_p. \tag{76}$$

Consequently,

$$A[X]-\gamma\sum_{p=1}^{k}Y_p = A[\gamma D]-\gamma\sum_{p=1}^{k}\left[L_p^H D_p L_p + R_p^H[\gamma D]D_p^{-1}R_p[\gamma D]\right]$$

$$= \left[\begin{array}{c|c} \gamma D-\gamma\sum_{p=1}^{k}[\gamma D]M^H P_p^H D_p^{-1}P_p M[\gamma D] & \\ \hline & \gamma D-\gamma\sum_{p=1}^{k}P_p^H D_p P_p \end{array}\right] \quad \text{[see (75) and (72)]}$$

$$= \left[\begin{array}{c|c} \gamma D-\gamma^3 DM^H D^{-1}MD & \\ \hline & \gamma D-\gamma D \end{array}\right] \quad \text{[see (76)]}$$

$$= \left[\begin{array}{c|c} \gamma[D-\gamma^2 DM^H D^{-1}MD] & \\ \hline & \end{array}\right].$$

To see that the resulting matrix is $\succeq 0$ (which means the validity of (70.d)), note that $D\succeq\gamma^2 DM^H D^{-1}MD$ since (γ,D) is feasible for (68).

2. Now let us prove that $\widehat{\mu}_{\mathbf{\Delta}}(M)\le\widehat{\omega}_{\mathbf{\Delta}}(M)$. This is exactly the same as to prove that, given a feasible solution $(\gamma>0,X,\{Y_p\},\{V_p\},\{\lambda_p\})$ of (70) and $\gamma'\in(0,\gamma)$, we can build $D\in\mathbf{D}$ such that (γ',D) is feasible for (68).

For $\epsilon>0$, let us set

$$\begin{array}{rcll} \lambda_{p,\epsilon} &=& \lambda_p+\epsilon, & p\in I_{\rm f}^{\rm c},\\ Y_{p,\epsilon} &=& \lambda_{p,\epsilon}L_p^H L_p+\lambda_{p,\epsilon}^{-1}R_p^H[X]R_p[X], & p\in I_{\rm f}^{\rm c},\\ W_{p,\epsilon} &=& R_p[X][\underbrace{V_p+\epsilon I}_{V_{p,\epsilon}}]^{-1}R_p^H[X]+\epsilon I_{d_p}, & p\in I_{\rm s}^{\rm c},\\ Y_{p,\epsilon} &=& L_p^H W_{p,\epsilon}L_p+R_p^H[X]W_{p,\epsilon}^{-1}R_p[X], & p\in I_{\rm s}^{\rm c}. \end{array} \tag{77}$$

In view of (70.c), for $p\in I_{\rm f}^{\rm c}$ one has

$$\left[\begin{array}{cc} Y_p-\lambda_p L_p^H L_p & R_p^H[X]\\ R_p[X] & \lambda_{p,\epsilon}I_{d_p}\end{array}\right]\succeq 0\Rightarrow Y_p\succeq\lambda_p L_p^H L_p+\lambda_{p,\epsilon}^{-1}R_p^H[X]R_p[H]\Rightarrow Y_{p,\epsilon}\preceq Y_p+\epsilon L_p^H L_p. \tag{78}$$

In view of (70.b), for $p\in I_{\rm s}^{\rm c}$ one has

$$\begin{array}{l} \left[\begin{array}{cc} Y_p-V_p & L_p^H R_p[X]\\ R_p^H[X]L_p & V_{p,\epsilon}\end{array}\right]\succeq 0\Rightarrow Y_p\succeq V_p+L_p^H\left[R_p[X]V_{p,\epsilon}^{-1}R_p^H[X]\right]L_p\\ \qquad\Rightarrow V_p+L_p^H W_{p,\epsilon}L_p\preceq Y_p+\epsilon L_p^H L_p. \end{array} \tag{79}$$

Moreover,

$$\begin{array}{l} \left[\begin{array}{cc} V_{p,\epsilon} & I_{2m}\\ I_{2m} & V_{p,\epsilon}^{-1}\end{array}\right]\succeq 0\Rightarrow \left[\begin{array}{cc} I_{2m} & 0_{2m\times d_p}\\ 0_{d_p\times 2m} & R_p[X]\end{array}\right]\left[\begin{array}{cc} V_{p,\epsilon} & I_{2m}\\ I_{2m} & V_{p,\epsilon}^{-1}\end{array}\right]\left[\begin{array}{cc} I_{2m} & 0_{2m\times d_p}\\ 0_{d_p\times 2m} & R_p[X]\end{array}\right]^H\succeq 0\\ \qquad\Rightarrow\left[\begin{array}{cc} V_{p,\epsilon} & R_p^H[X]\\ R_p[X] & R_p[X]V_{p,\epsilon}^{-1}R_p^H[X]\end{array}\right]\succeq 0\Rightarrow\left[\begin{array}{cc} V_{p,\epsilon} & R_p^H[X]\\ R_p[X] & W_{p,\epsilon}\end{array}\right]\succeq 0\\ \qquad\Rightarrow V_{p,\epsilon}\succeq R_p^H[X]W_{p,\epsilon}^{-1}R_p[X]. \end{array} \tag{80}$$

Combining the latter matrix inequality with the concluding relation in (79), we get for $p \in I_s^c$:

$$Y_{p,\epsilon} = L_p^H W_{p,\epsilon} L_p + R_p^H[X] W_{p,\epsilon}^{-1} R_p[X] \preceq Y_p + \epsilon(L_p^H L_p + I). \tag{81}$$

Taking into account (78), (81), the validity of (70.d), and the fact that $A[X] \succ 0$ due to (70.a), we arrive at the following intermediate conclusion: there exist positive reals $\bar{\lambda}_p$, $p \in I_f^c$, and positive definite $d_p \times d_p$ matrices $\bar{W}_p$, $p \in I_s^c$, such that

$$\gamma' \left[\sum_{p \in I_f^c} \left[\bar{\lambda}_p L_p^H L_p + \bar{\lambda}_p^{-1} R_p^H[X] R_p[X] \right] + \sum_{p \in I_s^c} \left[L_p^H \bar{W}_p L_p + R_p^H[X] \bar{W}_p^{-1} R_p[X] \right] \right] \prec A[X]. \tag{82}$$

Setting $D_p = \bar{\lambda}_p I_{d_p}$, $p \in I_f^c$, $D_p = \bar{W}_p$, $p \in I_s^c$, $D = \mathrm{Diag}\{D_1, ..., D_k\}$, we see that D, up to multiplication by a positive scalar, belongs to $\mathbf{D}$. Recalling the origin of L_p, $R_p[X]$ and $A[X]$ (see (75)), (82) reads

$$\begin{bmatrix} X - \gamma' X M^H D^{-1} M X & \\ & X - \gamma' D \end{bmatrix} \succ 0, \tag{83}$$

whence $\gamma' M^H D^{-1} M \prec X^{-1} \prec (\gamma')^{-1} D^{-1}$, which by the SCL yields $\begin{bmatrix} D & \gamma' D M^H \\ \gamma' M D & D \end{bmatrix} \succ 0$. Multiplying, if necessary, D by positive real to satisfy the requirement $D \succeq I$, we get a feasible solution (γ', D) of (68). ■

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