# Lecture 3

# Robust Conic Quadratic and Semidefinite Optimization

In this lecture, we extend the RO methodology onto *non-linear* convex optimization problems, specifically, *conic* ones.

## 3.1 Uncertain Conic Optimization: Preliminaries

### 3.1.1 Conic Programs

A *conic* optimization (CO) problem (also called *conic program*) is of the form

$$\min_x \left\{ c^T x + d : Ax - b \in \mathbf{K} \right\}, \tag{3.1.1}$$

where $x \in \mathbb{R}^n$ is the decision vector, $\mathbf{K} \subset \mathbb{R}^m$ is a closed pointed convex cone with a nonempty interior, and $x \mapsto Ax - b$ is a given affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. Conic formulation is one of the universal forms of a Convex Programming problem; among the many advantages of this specific form is its "unifying power." An extremely wide variety of convex programs is covered by just three types of cones:

1. Direct products of nonnegative rays, i.e., $\mathbf{K}$ is a non-negative orthant $\mathbb{R}^m_+$. These cones give rise to Linear Optimization problems

$$\min_x \left\{ c^T x + d : a_i^T x - b_i \geq 0,\ 1 \leq i \leq m \right\}.$$

2. Direct products of *Lorentz* (or *Second-order*, or Ice-cream) cones $\mathbf{L}^k = \{x \in \mathbb{R}^k : x_k \geq \sqrt{\sum_{j=1}^{k-1} x_j^2}\}$. These cones give rise to Conic Quadratic Optimization (called also Second Order Conic Optimization). The Mathematical Programming form of a CO problem is

$$\min_x \left\{ c^T x + d : \|A_i x - b_i\|_2 \leq c_i^T x - d_i,\ 1 \leq i \leq m \right\};$$

here $i$-th scalar constraint (called *Conic Quadratic Inequality*) (CQI) expresses the fact that the vector $[A_i x; c_i^T x] - [b_i; d_i]$ that depends affinely on $x$ belongs to the Lorentz cone $\mathbf{L}_i$ of appropriate dimension, and the system of all constraints says that the affine mapping

$$x \mapsto \left[ [A_1 x; c_1^T x]; \ldots; [A_m x; c_m^T x] \right] - [[b_1; d_1]; \ldots, ; [b_m; d_m]]$$

maps $x$ into the direct product of the Lorentz cones $\mathbf{L}_1 \times ... \times \mathbf{L}_m$.

3. Direct products of *semidefinite* cones $\mathbf{S}^k_+$.

   $\mathbf{S}^k_+$ is the cone of positive semidefinite $k \times k$ matrices; it "lives" in the space $\mathbf{S}^k$ of symmetric $k \times k$ matrices. We treat $\mathbf{S}^k$ as Euclidean space equipped with the *Frobenius* inner product $\langle A, B \rangle = \mathrm{Tr}(AB) = \sum\limits_{i,j=1}^{k} A_{ij}B_{ij}$.

   The family of semidefinite cones gives rise to *Semidefinite Optimization* (SDO) — optimization programs of the form

$$\min_x \left\{ c^T x + d : \mathcal{A}_i x - B_i \succeq 0,\ 1 \le i \le m \right\},$$

   where

$$x \mapsto \mathcal{A}_i x - B_i \equiv \sum_{j=1}^{n} x_j A^{ij} - B_i$$

   is an affine mapping from $\mathbb{R}^n$ to $\mathbf{S}^{k_i}$ (so that $A^{ij}$ and $B_i$ are symmetric $k_i \times k_i$ matrices), and $A \succeq 0$ means that $A$ is a symmetric positive semidefinite matrix. The constraint of the form "a symmetric matrix affinely depending on the decision vector should be positive semidefinite" is called an LMI — Linear Matrix Inequality. Thus, a Semidefinite Optimization problem (called also *semidefinite program*) is the problem of minimizing a linear objective under finitely many LMI constraints. One can rewrite an SDO program in the Mathematical Programming form, e.g., as

$$\min_x \left\{ c^T x + d : \lambda_{\min}(\mathcal{A}_i x - B_i) \ge 0,\ 1 \le i \le m \right\},$$

   where $\lambda_{\min}(A)$ stands for the minimal eigenvalue of a symmetric matrix $A$, but this reformulation usually is of no use.

Keeping in mind our future needs related to *Globalized Robust Counterparts*, it makes sense to modify slightly the format of a conic program, specifically, to pass to programs of the form

$$\min_x \left\{ c^T x + d : A_i x - b_i \in \mathbf{Q}_i,\ 1 \le i \le m \right\}, \tag{3.1.2}$$

where $\mathbf{Q}_i \subset \mathbb{R}^{k_i}$ are nonempty closed convex sets given by finite lists of conic inclusions:

$$\mathbf{Q}_i = \{u \in \mathbb{R}^{k_i} : Q_{i\ell} u - q_{i\ell} \in \mathbf{K}_{i\ell},\ \ell = 1, ..., L_i\}, \tag{3.1.3}$$

with closed convex pointed cones $\mathbf{K}_{i\ell}$. We will restrict ourselves to the cases where $\mathbf{K}_{i\ell}$ are nonnegative orthants, or Lorentz, or Semidefinite cones. Clearly, a problem in the form (3.1.2) is equivalent to the conic problem

$$\min_x \left\{ c^T x + d : Q_{i\ell} A_i x - [Q_{i\ell} b_i + q_{i\ell}] \in \mathbf{K}_{i\ell}\ \forall (i, \ell \le L_i) \right\}$$

We treat the collection $(c, d, \{A_i, b_i\}_{i=1}^m)$ as *natural data* of problem (3.1.2). The collection of sets $\mathbf{Q}_i$, $i = 1, ..., m$, is interpreted as the *structure* of problem (3.1.2), and thus the quantities $Q_{i\ell}, q_{i\ell}$ specifying these sets are considered as certain data.

### 3.1.2 Uncertain Conic Problems and their Robust Counterparts

*Uncertain conic problem* (3.1.2) is a problem with fixed structure and uncertain natural data *affinely* parameterized by a *perturbation* vector $\zeta \in \mathbb{R}^L$

$$(c, d, \{A_i, b_i\}_{i=1}^m) = (c^0, d^0, \{A_i^0, b_i^0\}_{i=1}^m) + \sum_{\ell=1}^{L} \zeta_\ell (c^\ell, d^\ell, \{A_i^\ell, b_i^\ell\}_{i=1}^m). \tag{3.1.4}$$

running through a given perturbation set $\mathcal{Z} \subset \mathbb{R}^L$.

#### Robust Counterpart of an uncertain conic problem

The notions of a robust feasible solution and the *Robust Counterpart* (RC) of uncertain problem (3.1.2) are defined exactly as in the case of an uncertain LO problem (see Definition 1.4):

**Definition 3.1** *Let an uncertain problem* (3.1.2), (3.1.4) *be given and let* $\mathcal{Z} \subset \mathbb{R}^L$ *be a given perturbation set.*

(i) *A candidate solution* $x \in \mathbb{R}^n$ *is robust feasible, if it remains feasible for all realizations of the perturbation vector from the perturbation set:*

$$\begin{array}{c} x \textit{ is robust feasible} \\ \Updownarrow \\ [A_i^0 + \sum\limits_{\ell=1}^{L} \zeta_\ell A_i^\ell]x - [b_i^0 + \sum\limits_{\ell=1}^{L} \zeta_\ell b_i^\ell] \in \mathbf{Q}_i \ \forall \left(i, 1 \leq i \leq m, \zeta \in \mathcal{Z}\right). \end{array}$$

(ii) *The Robust Counterpart of* (3.1.2), (3.1.4) *is the problem*

$$\min_{x,t} \left\{ t : \left. \begin{array}{l} [c^0 + \sum\limits_{\ell=1}^{L} \zeta_\ell c^\ell]^T x + [d^0 + \sum\limits_{\ell=1}^{L} \zeta_\ell d^\ell] - t \in \mathbf{Q}_0 \equiv \mathbb{R}_-, \\ [A_i^0 + \sum\limits_{\ell=1}^{L} \zeta_\ell A_i^\ell]x - [b_i^0 + \sum\limits_{\ell=1}^{L} \zeta_\ell b_i^\ell] \in \mathbf{Q}_i, \ 1 \leq i \leq m \end{array} \right\} \forall \zeta \in \mathcal{Z} \right\} \tag{3.1.5}$$

*of minimizing the guaranteed value of the objective over the robust feasible solutions.*

As in the LO case, it is immediately seen that the RC remains intact when the perturbation set $\mathcal{Z}$ is replaced with its closed convex hull; so, from now on we assume the perturbation set to be closed and convex. Note also that the case when the entries of the uncertain data $[A; b]$ are affected by perturbations in a *non-affine* fashion in principle could be reduced to the case of affine perturbations (see section 1.5); however, we do not know meaningful cases beyond uncertain LO where such a reduction leads to a tractable RC.

### 3.1.3 Robust Counterpart of Uncertain Conic Problem: Tractability

In contrast to uncertain LO, where the RC turn out to be computationally tractable whenever the perturbation set is so, uncertain conic problems with computationally tractable RCs are a "rare commodity." The ultimate reason for this phenomenon is rather simple: the RC (3.1.5) of an uncertain conic problem (3.1.2), (3.1.4) is a convex problem with linear objective and constraints of the generic form

$$P(y, \zeta) = \pi(y) + \Phi(y)\zeta = \phi(\zeta) + \Phi(\zeta)y \in \mathbf{Q}, \tag{3.1.6}$$

where $\pi(y), \Phi(y)$ are affine in the vector $y$ of the decision variables, $\phi(\zeta), \Phi(\zeta)$ are affine in the perturbation vector $\zeta$, and $\mathbf{Q}$ is a "simple" closed convex set. For such a problem, its computational tractability is, essentially, equivalent to the possibility to check efficiently whether a given candidate solution $y$ is or is not feasible. The latter question, in turn, is whether the image of the perturbation set $\mathcal{Z}$ under an affine mapping $\zeta \mapsto \pi(y) + \Phi(y)\zeta$ is or is not contained in a given convex set $\mathbf{Q}$. This question is easy when $\mathbf{Q}$ is a polyhedral set given by an explicit list of scalar linear inequalities $a_i^T u \le b_i$, $i = 1, ..., I$ (in particular, when $\mathbf{Q}$ is a nonpositive ray, that is what we deal with in LO), in which case the required verification consists in checking whether the maxima of $I$ affine functions $a_i^T(\pi(y) + \Phi(y)\zeta) - b_i$ of $\zeta$ over $\zeta \in \mathcal{Z}$ are or are not nonnegative. Since the maximization of an affine (and thus concave!) function over a computationally tractable convex set $\mathcal{Z}$ is easy, so is the required verification. When $\mathbf{Q}$ is given by *nonlinear* convex inequalities $a_i(u) \le 0$, $i = 1, ..., I$, the verification in question requires checking whether the *maxima* of *convex* functions $a_i(\pi(y) + \Phi(y)\zeta)$ over $\zeta \in \mathcal{Z}$ are or are not nonpositive. A problem of maximizing a convex function $f(\zeta)$ over a convex set $\mathcal{Z}$ can be computationally intractable already in the case of $\mathcal{Z}$ as simple as the unit box and $f$ as simple as a convex quadratic form $\zeta^T Q \zeta$. Indeed, it is known that the problem

$$\max_{\zeta} \left\{ \zeta^T B \zeta : \|\zeta\|_\infty \le 1 \right\}$$

with positive semidefinite matrix $B$ is NP-hard; in fact, it is already NP-hard to approximate the optimal value in this problem within a relative accuracy of 4%, even when probabilistic algorithms are allowed [55]. This example immediately implies that the RC of a generic uncertain conic quadratic problem with a perturbation set as simple as a box is computationally intractable.

Indeed, consider a simple-looking uncertain conic quadratic inequality

$$\|0 \cdot y + Q\zeta\|_2 \le 1$$

($Q$ is a given square matrix) along with its RC, the perturbation set being the unit box:

$$\|0 \cdot y + Q\zeta\|_2 \le 1 \ \forall(\zeta : \|\zeta\|_\infty \le 1). \qquad \text{(RC)}$$

The feasible set of the RC is either the entire space of $y$-variables, or is empty, which depends on whether or not one has

$$\max_{\|\zeta\|_\infty \le 1} \zeta^T B \zeta \le 1. \qquad [B = Q^T Q]$$

Varying $Q$, we can get, as $B$, an arbitrary positive semidefinite matrix of a given size. Now, assuming that we can process (RC) efficiently, we can check efficiently whether the feasible set of (RC) is or is not empty, that is, we can compare efficiently the maximum of a positive semidefinite quadratic form over the unit box with the value 1. If we can do it, we can compute the maximum of a general-type positive semidefinite quadratic form $\zeta^T B \zeta$ over the unit box within relative accuracy $\epsilon$ in time polynomial in the dimension of $\zeta$ and $\ln(1/\epsilon)$ (by comparing $\max_{\|\zeta\|_\infty \le 1} \lambda \zeta^T B \zeta$ with 1 and applying bisection in $\lambda > 0$). Thus, the NP-hard problem of computing $\max_{\|\zeta\|_\infty \le 1} \zeta^T B \zeta$, $B \succ 0$, within relative accuracy $\epsilon = 0.04$ reduces to checking feasibility of the RC of a CQI with a box perturbation set, meaning that it is NP-hard to process the RC in question.

The unpleasant phenomenon we have just outlined leaves us with only two options:

A. To identify meaningful particular cases where the RC of an uncertain conic problem is computationally tractable; and

B. To develop *tractable approximations* of the RC in the remaining cases.

Note that the RC, same as in the LO case, is a "constraint-wise" construction, so that investigating tractability of the RC of an uncertain conic problem reduces to the same question for the RCs of the conic constraints constituting the problem. Due to this observation, from now on we focus on tractability of the RC

$$\forall(\zeta \in \mathcal{Z}) : A(\zeta)x + b(\zeta) \in \mathbf{Q}$$

of a *single* uncertain conic inequality.

### 3.1.4 Safe Tractable Approximations of RCs of Uncertain Conic Inequalities

In sections 3.2, 3.4 we will present a number of special cases where the RC of an uncertain CQI/LMI is computationally tractable; these cases have to do with rather specific perturbation sets. The question is, what to do when the RC is *not* computationally tractable. A natural course of action in this case is to look for a *safe tractable approximation* of the RC, defined as follows:

**Definition 3.2** *Consider the RC*

$$\underbrace{A(\zeta)x + b(\zeta)}_{\equiv \alpha(x)\zeta + \beta(x)} \in \mathbf{Q}\ \forall \zeta \in \mathcal{Z} \tag{3.1.7}$$

*of an uncertain constraint*

$$A(\zeta)x + b(\zeta) \in \mathbf{Q}. \tag{3.1.8}$$

*($A(\zeta) \in \mathbb{R}^{k\times n}, b(\zeta) \in \mathbb{R}^k$ are affine in $\zeta$, so that $\alpha(x)$, $\beta(x)$ are affine in the decision vector $x$). We say that a system $\mathcal{S}$ of convex constraints in variables $x$ and, perhaps, additional variables $u$ is a safe approximation of the RC* (3.1.7)*, if the projection of the feasible set of $\mathcal{S}$ on the space of $x$ variables is contained in the feasible set of the RC:*

$$\forall x : (\exists u :\ (x, u) \text{ satisfies } \mathcal{S}) \Rightarrow x \text{ satisfies } (3.1.7).$$

*This approximation is called tractable, provided that $\mathcal{S}$ is so, (e.g., $\mathcal{S}$ is an explicit system of CQIs/LMIs or, more generally, the constraints in $\mathcal{S}$ are efficiently computable).*

The rationale behind the definition is as follows: assume we are given an uncertain conic problem (3.1.2) with vector of design variables $x$ and a certain objective $c^Tx$ (as we remember, the latter assumption is w.l.o.g.) and we have at our disposal a safe tractable approximation $\mathcal{S}_i$ of $i$-th constraint of the problem, $i = 1, ..., m$. Then the problem

$$\min_{x,u^1,...,u^m} \left\{c^Tx : (x, u^i) \text{ satisfies } \mathcal{S}_i,\ 1 \le i \le m\right\}$$

is a computationally tractable *safe approximation* of the RC, meaning that the $x$-component of every feasible solution to the approximation is feasible for the RC, and thus an optimal solution to the approximation is a *feasible* suboptimal solution to the RC.

In principle, there are many ways to build a safe tractable approximation of an uncertain conic problem. For example, assuming $\mathcal{Z}$ bounded, which usually is the case, we could find a simplex $\Delta = \mathrm{Conv}\{\zeta^1, ..., \zeta^{L+1}\}$ in the space $\mathbb{R}^L$ of perturbation vectors that is large enough to contain the actual perturbation set $\mathcal{Z}$. The RC of our uncertain problem, the perturbation set being $\Delta$, is computationally tractable (see section 3.2.1) and is a safe approximation of the RC associated with the actual perturbation set $\mathcal{Z}$ due to $\Delta \supset \mathcal{Z}$. The essence of the matter is, of course, how conservative an approximation is: how much it "adds" to the built-in conservatism of the worst-case-oriented RC. In order to answer the latter question, we should quantify the "conservatism" of an approximation. There is no evident way to do it. One possible way could be to look by how much the optimal value of the approximation is larger than the optimal value of the true RC, but here we run into a severe difficulty. It may well happen that the feasible set of an approximation is empty, while the true feasible set of the RC is not so. Whenever this is the case, the optimal value of the approximation is "infinitely worse" than the true optimal value. It follows that comparison of optimal values makes sense only when the approximation scheme in question guarantees that the approximation inherits the feasibility properties of the true problem. On a closer inspection, such a requirement is, in general, not less restrictive than the requirement for the approximation to be precise.

The way to quantify the conservatism of an approximation to be used in this book is as follows. Assume that $0 \in \mathcal{Z}$ (this assumption is in full accordance with the interpretation of vectors $\zeta \in \mathcal{Z}$ as data perturbations, in which case $\zeta = 0$ corresponds to the nominal data). With this assumption, we can embed our closed convex perturbation set $\mathcal{Z}$ into a single-parametric family of perturbation sets

$$\mathcal{Z}_\rho = \rho\mathcal{Z},\ 0 < \rho \leq \infty, \tag{3.1.9}$$

thus giving rise to a single-parametric family

$$\underbrace{A(\zeta)x + b(\zeta)}_{\equiv \alpha(x)\zeta + \beta(x)} \in \mathbf{Q}\ \forall \zeta \in \mathcal{Z}_\rho \tag{RC$_\rho$}$$

of RCs of the uncertain conic constraint (3.1.8). One can think about $\rho$ as *perturbation level*; the original perturbation set $\mathcal{Z}$ and the associated RC (3.1.7) correspond to the perturbation level 1. Observe that the feasible set $X_\rho$ of (RC$_\rho$) shrinks as $\rho$ grows. This allows us to quantify the conservatism of a safe approximation to (RC) by "positioning" the feasible set of $\mathcal{S}$ with respect to the scale of "true" feasible sets $X_\rho$, specifically, as follows:

**Definition 3.3** *Assume that we are given an approximation scheme that puts into correspondence to* (3.1.9), (RC$_\rho$) *a finite system $\mathcal{S}_\rho$ of efficiently computable convex constraints on variables $x$ and, perhaps, additional variables $u$, depending on $\rho > 0$ as on a parameter, in such a way that for every $\rho$ the system $\mathcal{S}_\rho$ is a safe tractable approximation of (RC$_\rho$), and let $\widehat{X}_\rho$ be the projection of the feasible set of $\mathcal{S}_\rho$ onto the space of $x$ variables.*

*We say that the conservatism (or "tightness factor") of the approximation scheme in question does not exceed $\vartheta \geq 1$ if, for every $\rho > 0$, we have*

$$X_{\vartheta\rho} \subset \widehat{X}_\rho \subset X_\rho.$$

Note that the fact that $\mathcal{S}_\rho$ is a safe approximation of (RC$_\rho$) tight within factor $\vartheta$ is equivalent to the following pair of statements:

1. [safety] *Whenever a vector $x$ and $\rho > 0$ are such that $x$ can be extended to a feasible solution of $\mathcal{S}_\rho$, $x$ is feasible for* (RC$_\rho$);

2. [tightness] *Whenever a vector $x$ and $\rho > 0$ are such that $x$ can<u>not</u> be extended to a feasible solution of $\mathcal{S}_\rho$, $x$ is <u>not</u> feasible for* (RC$_{\vartheta\rho}$).

Clearly, a tightness factor equal to 1 means that the approximation is precise: $\widehat{X}_\rho = X_\rho$ for all $\rho$. In many applications, especially in those where the level of perturbations is known only "up to an order of magnitude," a safe approximation of the RC with a moderate tightness factor is almost as useful, from a practical viewpoint, as the RC itself.

An important observation is that *with a bounded perturbation set $\mathcal{Z} = \mathcal{Z}_1 \subset \mathbb{R}^L$ that is symmetric w.r.t. the origin, we can always point out a safe computationally tractable approximation scheme for* (3.1.9), (RC$_\rho$) *with tightness factor $\leq L$.*

Indeed, w.l.o.g. we may assume that int$\mathcal{Z} \neq \emptyset$, so that $\mathcal{Z}$ is a closed and bounded convex set symmetric w.r.t. the origin. It is known that for such a set, there always exist two similar ellipsoids, centered at the origin, with the similarity ratio at most $\sqrt{L}$, such that the smaller ellipsoid is contained in $\mathcal{Z}$, and the larger one contains $\mathcal{Z}$. In particular, one can choose, as the smaller ellipsoid, the largest volume ellipsoid contained in $\mathcal{Z}$; alternatively, one can choose, as the larger ellipsoid, the smallest volume ellipsoid containing $\mathcal{Z}$. Choosing coordinates in which the smaller ellipsoid is the unit Euclidean ball $B$, we conclude that $B \subset \mathcal{Z} \subset \sqrt{L}B$. Now observe that $B$, and therefore $\mathcal{Z}$, contains the convex hull $\underline{\mathcal{Z}} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_1 \leq 1\}$ of the $2L$ vectors $\pm e_\ell$, $\ell = 1, ..., L$, where $e_\ell$ are the basic orths of the axes in question. Since $\underline{\mathcal{Z}}$ clearly contains $L^{-1/2}B$, the convex hull $\widehat{\mathcal{Z}}$ of the vectors $\pm Le_\ell$, $\ell = 1, ..., L$, contains $\mathcal{Z}$ and is contained in $L\mathcal{Z}$. Taking, as $\mathcal{S}_\rho$, the RC of our uncertain constraint, the perturbation set being $\rho\widehat{\mathcal{Z}}$, we clearly get an $L$-tight safe approximation of (3.1.9), (RC$_\rho$), and this approximation is merely the system of constraints

$$A(\rho Le_\ell)x + b(\rho Le_\ell) \in \mathbf{Q},\ A(-\rho Le_\ell)x + b(-\rho Le_\ell) \in \mathbf{Q},\ \ell = 1, ..., L,$$

that is, our approximation scheme is computationally tractable.

## 3.2 Uncertain Conic Quadratic Problems with Tractable RCs

In this section we focus on uncertain conic quadratic problems (that is, the sets $\mathbf{Q}_i$ in (3.1.2) are given by explicit lists of conic quadratic inequalities) for which the RCs are computationally tractable.

### 3.2.1 A Generic Solvable Case: Scenario Uncertainty

We start with a simple case where the RC of an uncertain conic problem (not necessarily a conic quadratic one) is computationally tractable — the case of *scenario uncertainty*.

**Definition 3.4** *We say that a perturbation set $\mathcal{Z}$ is scenario generated, if $\mathcal{Z}$ is given as the convex hull of a given finite set of scenarios $\zeta^{(\nu)}$:*

$$\mathcal{Z} = \text{Conv}\{\zeta^{(1)}, ..., \zeta^{(N)}\}. \tag{3.2.1}$$

**Theorem 3.1** *The RC* (3.1.5) *of uncertain problem* (3.1.2), (3.1.4) *with scenario perturbation set* (3.2.1) *is equivalent to the explicit conic problem*

$$\min_{x,t} \left\{ t : \left. \begin{array}{c} [c^0 + \sum\limits_{\ell=1}^{L} \zeta_\ell^{(\nu)} c^\ell]^T x + [d^0 + \sum\limits_{\ell=1}^{L} \zeta_\ell^{(\nu)} d^\ell] - t \leq 0 \\ [A_i^0 + \sum\limits_{\ell=1}^{L} \zeta_\ell^{(\nu)} A_i^\ell]^T x - [b^0 + \sum\limits_{\ell=1}^{L} \zeta_\ell^{(\nu)} b^\ell] \in \mathbf{Q}_i, \\ \hfill 1 \leq i \leq m \end{array} \right\}, 1 \leq \nu \leq N \right\} \tag{3.2.2}$$

*with a structure similar to the one of the instances of the original uncertain problem.*

**Proof.** This is evident due to the convexity of $\mathbf{Q}_i$ and the affinity of the left hand sides of the constraints in (3.1.5) in $\zeta$. $\square$

The situation considered in Theorem 3.1 is "symmetric" to the one considered in lecture 1, where we spoke about problems (3.1.2) with the simplest possible sets $\mathbf{Q}_i$ — just nonnegative rays, and the RC turns out to be computationally tractable whenever the perturbation set is so. Theorem 3.1 deals with another extreme case of the tradeoff between the geometry of the right hand side sets $\mathbf{Q}_i$ and that of the perturbation set. Here the latter is as simple as it could be — just the convex hull of an explicitly listed finite set, which makes the RC computationally tractable for rather general (just computationally tractable) sets $\mathbf{Q}_i$. Unfortunately, the second extreme is not too interesting: in the large scale case, a "scenario approximation" of a reasonable quality for typical perturbation sets, like boxes, requires an astronomically large number of scenarios, thus preventing listing them explicitly and making problem (3.2.2) computationally intractable. This is in sharp contrast with the first extreme, where the simple sets were $\mathbf{Q}_i$ — Linear Optimization is definitely interesting and has a lot of applications.

In what follows, we consider a number of less trivial cases where the RC of an uncertain conic quadratic problem is computationally tractable. As always with RC, which is a constraint-wise construction, we may focus on computational tractability of the RC of a *single* uncertain CQI

$$\|\underbrace{A(\zeta)y + b(\zeta)}_{\equiv\alpha(y)\zeta+\beta(y)}\|_2 \leq \underbrace{c^T(\zeta)y + d(\zeta)}_{\equiv\sigma^T(y)\zeta+\delta(y)}, \tag{3.2.3}$$

where $A(\zeta) \in \mathbb{R}^{k\times n}, b(\zeta) \in \mathbb{R}^k, c(\zeta) \in \mathbb{R}^n, d(\zeta) \in \mathbb{R}$ are affine in $\zeta$, so that $\alpha(y)$, $\beta(y), \sigma(y), \delta(y)$ are affine in the decision vector $y$.

### 3.2.2 Solvable Case I: Simple Interval Uncertainty

Consider uncertain conic quadratic constraint (3.2.3) and assume that:

1. The uncertainty is *side-wise*: the perturbation set $\mathcal{Z} = \mathcal{Z}^{\text{left}} \times \mathcal{Z}^{\text{right}}$ is the direct product of two sets (so that the perturbation vector $\zeta \in \mathcal{Z}$ is split into blocks $\eta \in \mathcal{Z}^{\text{left}}$ and $\chi \in \mathcal{Z}^{\text{right}}$), with the left hand side data $A(\zeta), b(\zeta)$ depending solely on $\eta$ and the right hand side data $c(\zeta), d(\zeta)$ depending solely on $\chi$, so that (3.2.3) reads

$$\|\underbrace{A(\eta)y + b(\eta)}_{\equiv\alpha(y)\eta+\beta(y)}\|_2 \leq \underbrace{c^T(\chi)y + d(\chi)}_{\equiv\sigma^T(y)\chi+\delta(y)}, \tag{3.2.4}$$

   and the RC of this uncertain constraint reads

$$\|A(\eta)y + b(\eta)\|_2 \leq c^T(\chi)y + d(\chi) \quad \forall(\eta \in \mathcal{Z}^{\text{left}}, \chi \in \mathcal{Z}^{\text{right}}); \tag{3.2.5}$$

2. The right hand side perturbation set is as described in Theorem 1.1, that is,

$$\mathcal{Z}^{\text{right}} = \{\chi : \exists u : P\chi + Qu + p \in \mathbf{K}\},$$

   where either $\mathbf{K}$ is a closed convex pointed cone, and the representation is strictly feasible, or $\mathbf{K}$ is a polyhedral cone given by an explicit finite list of linear inequalities;

3. The left hand side uncertainty is a simple interval one:

$$\begin{array}{rcl}\mathcal{Z}^{\rm left} & = & \big\{\eta=[\delta A,\delta b]:|(\delta A)_{ij}|\leq\delta_{ij},1\leq i\leq k,1\leq j\leq n,\\ & & |(\delta b)_i|\leq\delta_i,\,1\leq i\leq k\big\},\\ {[A(\zeta),b(\zeta)]} & = & [A^{\rm n},b^{\rm n}]+[\delta A,\delta b].\end{array}$$

In other words, every entry in the left hand side data $[A, b]$ of (3.2.3), independently of all other entries, runs through a given segment centered at the nominal value of the entry.

**Proposition 3.1** *Under assumptions 1 – 3 on the perturbation set $\mathcal{Z}$, the RC of the uncertain CQI* (3.2.3) *is equivalent to the following explicit system of conic quadratic and linear constraints in variables $y, z, \tau, v$:*

$$\begin{array}{ll}(a) & \begin{array}{l}\tau+p^Tv\leq\delta(y),\;P^Tv=\sigma(y),\\ Q^Tv=0,\;v\in\mathbf{K}_*\end{array}\\ (b) & \begin{array}{l}z_i\geq|(A^{\rm n}y+b^{\rm n})_i|+\delta_i+\sum\limits_{j=1}^n|\delta_{ij}y_j|,\;i=1,...,k\\ \|z\|_2\leq\tau\end{array}\end{array}\tag{3.2.6}$$

*where $\mathbf{K}_*$ is the cone dual to $\mathbf{K}$.*

**Proof.** Due to the side-wise structure of the uncertainty, a given $y$ is robust feasible if and only if there exists $\tau$ such that

$$\begin{array}{ll}(a) & \tau\leq\min\limits_{\chi\in\mathcal{Z}^{\rm right}}\big\{\sigma^T(y)\chi+\delta(y)\big\}\\ & =\min\limits_{\chi,u}\big\{\sigma^T(y)\chi:P\chi+Qu+p\in\mathbf{K}\big\}+\delta(y),\\ (b) & \tau\geq\max\limits_{\eta\in\mathcal{Z}^{\rm left}}\|A(\eta)y+b(\eta)\|_2\\ & =\max\limits_{\delta A,\delta b}\left\{\|[A^{\rm n}y+b^{\rm n}]+[\delta Ay+\delta b]\|_2:|\delta A|_{ij}\leq\delta_{ij},|\delta b_i|\leq\delta_i\right\}.\end{array}$$

By Conic Duality, a given $\tau$ satisfies $(a)$ if and only if $\tau$ can be extended, by properly chosen $v$, to a solution of (3.2.6.$a$); by evident reasons, $\tau$ satisfies $(b)$ if and only if there exists $z$ satisfying (3.2.6.$b$). □

### 3.2.3 Solvable Case II: Unstructured Norm-Bounded Uncertainty

Consider the case where the uncertainty in (3.2.3) is still side-wise ($\mathcal{Z}=\mathcal{Z}^{\rm left}\times\mathcal{Z}^{\rm right}$) with the right hand side uncertainty set $\mathcal{Z}^{\rm right}$ as in section 3.2.2, while the left hand side uncertainty is *unstructured norm-bounded*, meaning that

$$\mathcal{Z}^{\rm left}=\big\{\eta\in\mathbb{R}^{p\times q}:\|\eta\|_{2,2}\leq 1\big\}\tag{3.2.7}$$

and either

$$A(\eta)y+b(\eta)=A^{\rm n}y+b^{\rm n}+L^T(y)\eta R\tag{3.2.8}$$

with $L(y)$ affine in $y$ and $R\neq 0$, or

$$A(\eta)y+b(\eta)=A^{\rm n}y+b^{\rm n}+L^T\eta R(y)\tag{3.2.9}$$

with $R(y)$ affine in $y$ and $L\neq 0$. Here

$$\|\eta\|_{2,2}=\max_u\{\|\eta u\|_2:u\in\mathbb{R}^q,\|u\|_2\leq 1\}$$

is the usual matrix norm of a $p\times q$ matrix $\eta$ (the maximal singular value),

**Example 3.1**

(i) Imagine that some $p \times q$ submatrix $P$ of the left hand side data $[A, b]$ of (3.2.4) is uncertain and differs from its nominal value $P^{\rm n}$ by an additive perturbation $\Delta P = M^T \Delta N$ with $\Delta$ having matrix norm at most 1, and all entries in $[A, b]$ outside of $P$ are certain. Denoting by $I$ the set of indices of the rows in $P$ and by $J$ the set of indices of the columns in $P$, let $U$ be the natural projector of $\mathbb{R}^{n+1}$ on the coordinate subspace in $\mathbb{R}^{n+1}$ given by $J$, and $V$ be the natural projector of $\mathbb{R}^k$ on the subspace of $\mathbb{R}^k$ given by $I$ (e.g., with $I = \{1, 2\}$ and $J = \{1, 5\}$, $Uu = [u_1; u_5] \in \mathbb{R}^2$ and $Vu = [u_1; u_2] \in \mathbb{R}^2$). Then the outlined perturbations of $[A, b]$ can be represented as

$$[A(\eta), b(\eta)] = [A^{\rm n}, b^{\rm n}] + \underbrace{V^T M^T}_{L^T} \eta \underbrace{(NU)}_{R},\ \|\eta\|_{2,2} \leq 1,$$

whence, setting $Y(y) = [y; 1]$,

$$A(\eta)y + b(\eta) = [A^{\rm n} y + b^{\rm n}] + L^T \eta \underbrace{[RY(y)]}_{R(y)},$$

and we are in the situation (3.2.7), (3.2.9).

(ii) [Simple ellipsoidal uncertainty] Assume that the left hand side perturbation set $\mathcal{Z}^{\rm left}$ is a $p$-dimensional ellipsoid; w.l.o.g. we may assume that this ellipsoid is just the unit Euclidean ball $B = \{\eta \in \mathbb{R}^p : \|\eta\|_2 \leq 1\}$. Note that for vectors $\eta \in \mathbb{R}^p = \mathbb{R}^{p \times 1}$ their usual Euclidean norm $\|\eta\|_2$ and their matrix norm $\|\eta\|_{2,2}$ are the same. We now have

$$A(\eta)y + b(\eta) = [A^0 y + b^0] + \sum_{\ell=1}^{p} \eta_\ell [A^\ell y + b^\ell] = [A^{\rm n} y + b^{\rm n}] + L^T(y)\eta R,$$

where $A^{\rm n} = A^0$, $b^{\rm n} = b^0$, $R = 1$ and $L(y)$ is the matrix with the rows $[A^\ell y + b^\ell]^T$, $\ell = 1, ..., p$. Thus, we are in the situation (3.2.7), (3.2.8).

**Theorem 3.2** *The RC of the uncertain CQI* (3.2.4) *with unstructured norm-bounded uncertainty is equivalent to the following explicit system of LMIs in variables $y, \tau, u, \lambda$:*
(i) *In the case of left hand side perturbations* (3.2.7), (3.2.8)*:*

$$\begin{array}{ll} (a) & \tau + p^T v \leq \delta(y),\ P^T v = \sigma(y),\ Q^T v = 0,\ v \in \mathbf{K}_* \\ (b) & \left[\begin{array}{c|c|c} \tau I_k & L^T(y) & A^{\rm n} y + b^{\rm n} \\ \hline L(y) & \lambda I_p & \\ \hline [A^{\rm n} y + b^{\rm n}]^T & & \tau - \lambda R^T R \end{array}\right] \succeq 0. \end{array} \quad (3.2.10)$$

(ii) *In the case of left hand side perturbations* (3.2.7), (3.2.9)*:*

$$\begin{array}{ll} (a) & \tau + p^T v \leq \delta(y),\ P^T v = \sigma(y),\ Q^T v = 0,\ v \in \mathbf{K}_* \\ (b) & \left[\begin{array}{c|c|c} \tau I_k - \lambda L^T L & & A^{\rm n} y + b^{\rm n} \\ \hline & \lambda I_q & R(y) \\ \hline [A^{\rm n} y + b^{\rm n}]^T & R^T(y) & \tau \end{array}\right] \succeq 0. \end{array} \quad (3.2.11)$$

*Here $\mathbf{K}_*$ is the cone dual to $\mathbf{K}$.*

**Proof.** Same as in the proof of Proposition 3.1, $y$ is robust feasible for (3.2.4) if and only if there exists $\tau$ such that

$$\begin{array}{ll}(a) & \tau \leq \min\limits_{\chi\in\mathcal{Z}^{\text{right}}} \left\{\sigma^T(y)\chi + \delta(y)\right\} \\ & \quad = \min\limits_{\chi,u} \left\{\sigma^T(y)\chi : P\chi + Qu + p \in \mathbf{K}\right\}, \\ (b) & \tau \geq \max\limits_{\eta\in\mathcal{Z}^{\text{left}}} \|A(\eta)y + b(\eta)\|_2,\end{array} \tag{3.2.12}$$

and a given $\tau$ satisfies $(a)$ if and only if it can be extended, by a properly chosen $v$, to a solution of (3.2.10.$a$)⇔(3.2.11.$a$). It remains to understand when $\tau$ satisfies $(b)$. This requires two basic facts.

**Lemma 3.1** [Semidefinite representation of the Lorentz cone] *A vector* $[y;t] \in \mathbb{R}^k \times \mathbb{R}$ *belongs to the Lorentz cone* $\mathbf{L}^{k+1} = \{[y;t] \in \mathbb{R}^{k+1} : t \geq \|y\|_2\}$ *if and only if the "arrow matrix"*

$$\text{Arrow}(y,t) = \left[\begin{array}{c|c} t & y^T \\ \hline y & tI_k \end{array}\right]$$

*is positive semidefinite.*

Proof of Lemma 3.1: We use the following fundamental fact:

**Lemma 3.2 [Schur Complement Lemma]** *A symmetric block matrix*

$$A = \left[\begin{array}{c|c} P & Q^T \\ \hline Q & R \end{array}\right]$$

*with* $R \succ 0$ *is positive (semi)definite if and only if the matrix*

$$P - Q^T R^{-1} Q$$

*is positive (semi)definite.*

Schur Complement Lemma ⇒ Lemma 3.1: When $t = 0$, we have $[y;t] \in \mathbf{L}^{k+1}$ iff $y = 0$, and $\text{Arrow}(y,t) \succeq 0$ iff $y = 0$, as claimed in Lemma 3.1. Now let $t > 0$. Then the matrix $tI_k$ is positive definite, so that by the Schur Complement Lemma we have $\text{Arrow}(y,t) \succeq 0$ if and only if $t \geq t^{-1}y^Ty$, or, which is the same, iff $[y;t] \in \mathbf{L}^{k+1}$. When $t < 0$, we have $[y;t] \notin \mathbf{L}^{k+1}$ and $\text{Arrow}(y,t) \not\succeq 0$. □

Proof of the Schur Complement Lemma: Matrix $A = A^T$ is $\succeq 0$ iff $u^TPu + 2u^TQ^Tv + v^TRv \geq 0$ for all $u, v$, or, which is the same, iff

$$\forall u : 0 \leq \min_v \left\{u^TPu + 2u^TQ^Tv + v^TRv\right\} = u^TPu - u^TQ^TR^{-1}Qu$$

(indeed, since $R \succ 0$, the minimum in $v$ in the last expression is achieved when $v = R^{-1}Qu$). The concluding relation $\forall u : u^T[P - Q^TR^{-1}Q]u \geq 0$ is valid iff $P - Q^TR^{-1}Q \succeq 0$. Thus, $A \succeq 0$ iff $P - Q^TR^{-1}Q \succeq 0$. The same reasoning implies that $A \succ 0$ iff $P - Q^TR^{-1}Q \succ 0$. □

We further need the following fundamental result:

**Lemma 3.3** [$\mathcal{S}$**-Lemma**]

(i) [homogeneous version] *Let* $A, B$ *be symmetric matrices of the same size such that* $\bar{x}^T A\bar{x} > 0$ *for some* $\bar{x}$. *Then the implication*

$$x^T Ax \geq 0 \Rightarrow x^T Bx \geq 0$$

*holds true if and only if*

$$\exists \lambda \geq 0 : B \succeq \lambda A.$$

(ii) [inhomogeneous version] *Let* $A, B$ *be symmetric matrices of the same size, and let the quadratic form* $x^T Ax + 2a^T x + \alpha$ *be strictly positive at some point. Then the implication*

$$x^T Ax + 2a^T x + \alpha \geq 0 \Rightarrow x^T Bx + 2b^T x + \beta \geq 0$$

*holds true if and only if*

$$\exists \lambda \geq 0 : \left[\begin{array}{c|c} B - \lambda A & b^T - \lambda a^T \\ \hline b - \lambda a & \beta - \lambda\alpha \end{array}\right] \succeq 0.$$

For proof of this fundamental Lemma, see, e.g., [9, section 4.3.5].

Coming back to the proof of Theorem 3.2, we can now understand when a given pair $\tau, y$ satisfies (3.2.12.$b$). Let us start with the case (3.2.8). We have

$$\begin{array}{ll}
& (y,\tau) \text{ satisfies } (3.2.12.b) \\
\Leftrightarrow & [\overbrace{[A^{\rm n}y + b^{\rm n}]}^{\widehat{y}} + L^T(y)\eta R; \tau] \in \mathbf{L}^{k+1} \quad \forall (\eta : \|\eta\|_{2,2} \leq 1) \\
& \hfill [\text{by } (3.2.8)] \\
\Leftrightarrow & \left[\begin{array}{c|c} \tau & \widehat{y}^T + R^T\eta^T L(y) \\ \hline \widehat{y} + L^T(y)\eta R & \tau I_k \end{array}\right] \succeq 0 \quad \forall (\eta : \|\eta\|_{2,2} \leq 1) \\
& \hfill [\text{by Lemma 3.1}] \\
\Leftrightarrow & \tau s^2 + 2sr^T[\widehat{y} + L^T(y)\eta R] + \tau r^T r \geq 0 \quad \forall [s;r] \; \forall (\eta : \|\eta\|_{2,2} \leq 1) \\
\Leftrightarrow & \tau s^2 + 2s\widehat{y}^T r + 2 \min\limits_{\eta : \|\eta\|_{2,2} \leq 1} \left[s(\eta^T L(y) r)^T R\right] + \tau r^T r \geq 0 \quad \forall [s;r] \\
\Leftrightarrow & \tau s^2 + 2s\widehat{y}^T r - 2\|L(y)r\|_2 \|sR\|_2 + \tau r^T r \geq 0 \quad \forall [s;r] \\
\Leftrightarrow & \tau r^T r + 2(L(y)r)^T \xi + 2sr^T\widehat{y} + \tau s^2 \geq 0 \quad \forall (s, r, \xi : \xi^T\xi \leq s^2 R^T R) \\
\Leftrightarrow & \exists \lambda \geq 0 : \left[\begin{array}{c|c|c} \tau I_k & L^T(y) & \widehat{y} \\ \hline L(y) & \lambda I_p & \\ \hline \widehat{y}^T & & \tau - \lambda R^T R \end{array}\right] \succeq 0 \\
& \hfill [\text{by the homogeneous } \mathcal{S}\text{-Lemma; note that } R \neq 0].
\end{array}$$

The requirement $\lambda \geq 0$ in the latter relation is implied by the LMI in the relation and is therefore redundant. Thus, in the case of (3.2.8) relation (3.2.12.$b$) is equivalent to the possibility to extend $(y, \tau)$ to a solution of (3.2.10.$b$).

Now let (3.2.9) be the case. We have

$$
\begin{array}{rl}
& (y,\tau) \text{ satisfies } (3.2.12.b)\\
\Leftrightarrow & [\overbrace{[A^{\rm n}y+b^{\rm n}]}^{\widehat{y}}+L^T\eta R(y);\tau]\in \mathbf{L}^{k+1}\quad \forall(\eta:\|\eta\|_{2,2}\leq 1)\ [\text{by } (3.2.9)]\\
\Leftrightarrow & \left[\begin{array}{c|c}\tau & \widehat{y}^T+R^T(y)\eta^T L\\ \hline \widehat{y}+L^T\eta R(y) & \tau I_k\end{array}\right]\succeq 0\quad \forall(\eta:\|\eta\|_{2,2}\leq 1)\\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{by Lemma 3.1}]\\
\Leftrightarrow & \tau s^2+2sr^T[\widehat{y}+L^T\eta R(y)]+\tau r^Tr\geq 0\quad \forall[s;r]\ \forall(\eta:\|\eta\|_{2,2}\leq 1)\\
\Leftrightarrow & \tau s^2+2s\widehat{y}^Tr+2\min\limits_{\eta:\|\eta\|_{2,2}\leq 1}\left[s(\eta^TLr)^TR(y)\right]+\tau r^Tr\geq 0\quad \forall[s;r]\\
\Leftrightarrow & \tau s^2+2s\widehat{y}^Tr-2\|Lr\|_2\|sR(y)\|_2+\tau r^Tr\geq 0\quad \forall[s;r]\\
\Leftrightarrow & \tau r^Tr+2sR^T(y)\xi+2sr^T\widehat{y}+\tau s^2\geq 0\quad \forall(s,r,\xi:\xi^T\xi\leq r^TL^TLr)\\
\Leftrightarrow & \exists\lambda\geq 0: \left[\begin{array}{c|c|c}\tau I_k-\lambda L^TL & & \widehat{y}\\ \hline & \lambda I_q & R(y)\\ \hline \widehat{y}^T & R^T(y) & \tau\end{array}\right]\succeq 0\\
& \qquad [\text{by the homogeneous } \mathcal{S}\text{-Lemma; note that } L\neq 0].
\end{array}
$$

As above, the restriction $\lambda \geq 0$ is redundant. We see that in the case of (3.2.9) relation (3.2.12.$b$) is equivalent to the possibility to extend $(y, \tau)$ to a solution of (3.2.11.$b$). □

### 3.2.4 Solvable Case III: Convex Quadratic Inequality with Unstructured Norm-Bounded Uncertainty

A special case of an uncertain conic quadratic constraint (3.2.3) is a convex quadratic constraint

$$
\begin{array}{ll}
(a) & y^TA^T(\zeta)A(\zeta)y\leq 2y^Tb(\zeta)+c(\zeta)\\
 & \Updownarrow\\
(b) & \|[2A(\zeta)y;1-2y^Tb(\zeta)-c(\zeta)]\|_2\leq 1+2y^Tb(\zeta)+c(\zeta).
\end{array}
\tag{3.2.13}
$$

Here $A(\zeta)$ is $k\times n$.

Assume that the uncertainty affecting this constraint is an unstructured norm-bounded one, meaning that

$$
\begin{array}{ll}
(a) & \mathcal{Z}=\{\zeta\in\mathbb{R}^{p\times q}:\|\zeta\|_{2,2}\leq 1\},\\
(b) & \left[\begin{array}{c}A(\zeta)y\\ y^Tb(\zeta)\\ c(\zeta)\end{array}\right]=\left[\begin{array}{c}A^{\rm n}y\\ y^Tb^{\rm n}\\ c^{\rm n}\end{array}\right]+L^T(y)\zeta R(y),
\end{array}
\tag{3.2.14}
$$

where $L(y)$, $R(y)$ are matrices of appropriate sizes affinely depending on $y$ and such that at least one of the matrices is constant. We are about to prove that the RC of (3.2.13), (3.2.14) is computationally tractable. Note that the just defined unstructured norm-bounded uncertainty in the data of convex quadratic constraint (3.2.13.$a$) implies similar uncertainty in the left hand side data of the equivalent uncertain CQI (3.2.13.$a$). Recall that Theorem 3.2 ensures that the RC of a general-type uncertain CQI with side-wise uncertainty and unstructured norm-bounded perturbations in the left hand side data is tractable. The result to follow removes the requirement of "side-wiseness" of the uncertainty at the cost of restricting the structure of the CQI in question — now it should come from an uncertain convex quadratic constraint. Note also that the case we are about to consider covers in particular the one when the data $(A(\zeta), b(\zeta), c(\zeta))$ of (3.2.13.$a$) are affinely parameterized by $\zeta$ varying in an ellipsoid (cf. Example 3.1.(ii)).

**Proposition 3.2** *Let us set $L(y) = [L_A(y), L_b(y), L_c(y)]$, where $L_b(y)$, $L_c(y)$ are the last two columns in $L(y)$, and let*

$$\begin{array}{c} \widehat{L}^T(y) = \left[L_b^T(y) + \frac{1}{2}L_c^T(y); L_A^T(y)\right],\ \widehat{R}(y) = [R(y), 0_{q\times k}], \\ \mathcal{A}(y) = \left[\begin{array}{c|c} 2y^T b^{\rm n} + c^{\rm n} & [A^{\rm n}y]^T \\ \hline A^{\rm n}y & I_k \end{array}\right], \end{array} \tag{3.2.15}$$

*so that $\mathcal{A}(y)$, $\widehat{L}(y)$ and $\widehat{R}(y)$ are affine in $y$ and at least one of the latter two matrices is constant.*

*The RC of* (3.2.13), (3.2.14) *is equivalent to the explicit LMI $\mathcal{S}$ in variables $y$, $\lambda$ as follows:*

(i) *In the case when $\widehat{L}(y)$ is independent of $y$ and is nonzero, $\mathcal{S}$ is*

$$\left[\begin{array}{c|c} \mathcal{A}(y) - \lambda \widehat{L}^T\widehat{L} & \widehat{R}^T(y) \\ \hline \widehat{R}(y) & \lambda I_q \end{array}\right] \succeq 0; \tag{3.2.16}$$

(ii) *In the case when $\widehat{R}(Y)$ is independent of $y$ and is nonzero, $\mathcal{S}$ is*

$$\left[\begin{array}{c|c} \mathcal{A}(y) - \lambda \widehat{R}^T\widehat{R} & \widehat{L}^T(y) \\ \hline \widehat{L}(y) & \lambda I_p \end{array}\right] \succeq 0; \tag{3.2.17}$$

(iii) *In all remaining cases (that is, when either $\widehat{L}(y) \equiv 0$, or $\widehat{R}(y) \equiv 0$, or both), $\mathcal{S}$ is*

$$\mathcal{A}(y) \succeq 0. \tag{3.2.18}$$

**Proof.** We have

$$\begin{array}{rl} & y^T A^T(\zeta)A(\zeta)y \le 2y^T b(\zeta) + c(\zeta) \quad \forall \zeta \in \mathcal{Z} \\ \Leftrightarrow & \left[\begin{array}{c|c} 2y^T b(\zeta) + c(\zeta) & [A(\zeta)y]^T \\ \hline A[\zeta]y & I_k \end{array}\right] \succeq 0 \quad \forall \zeta \in \mathcal{Z} \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{[Schur Complement Lemma]} \\ \Leftrightarrow & \overbrace{\left[\begin{array}{c|c} 2y^T b^{\rm n} + c^{\rm n} & [A^{\rm n}y]^T \\ \hline A^{\rm n}y & I \end{array}\right]}^{\mathcal{A}(y)} \\ & + \overbrace{\left[\begin{array}{c|c} 2L_b^T(y)\zeta R(y) + L_c^T(y)\zeta R(y) & R^T(y)\zeta^T L_A(y) \\ \hline L_A^T(y)\zeta R(y) & \end{array}\right]}^{\mathcal{B}(y,\zeta)} \succeq 0 \quad \forall (\zeta : \|\zeta\|_{2,2} \le 1) \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{[by (3.2.14)]} \\ \Leftrightarrow & \mathcal{A}(y) + \widehat{L}^T(y)\zeta\widehat{R}(y) + \widehat{R}^T(y)\zeta^T\widehat{L}(y) \succeq 0 \quad \forall (\zeta : \|\zeta\|_{2,2} \le 1) \qquad \text{[by (3.2.15)].} \end{array}$$

Now the reasoning can be completed exactly as in the proof of Theorem 3.2. Consider, e.g., the case of (i). We have

$$
\begin{array}{rl}
& y^TA^T(\zeta)A(\zeta)y \le 2y^Tb(\zeta)+c(\zeta) \;\; \forall \zeta\in\mathcal{Z}\\
\Leftrightarrow & \mathcal{A}(y)+\widehat{L}^T\zeta\widehat{R}(y)+\widehat{R}^T(y)\zeta^T\widehat{L}\succeq 0\;\forall(\zeta:\|\zeta\|_{2,2}\le 1)\text{ [already proved]}\\
\Leftrightarrow & \xi^T\mathcal{A}(y)\xi+2(\widehat{L}\xi)^T\zeta\widehat{R}(y)\xi\ge 0\;\;\forall\xi\;\;\forall(\zeta:\|\zeta\|_{2,2}\le 1)\\
\Leftrightarrow & \xi^T\mathcal{A}(y)\xi-2\|\widehat{L}\xi\|_2\|\widehat{R}(y)\xi\|_2\ge 0\;\;\forall\xi\\
\Leftrightarrow & \xi^T\mathcal{A}(y)\xi+2\eta^T\widehat{R}(y)\xi\ge 0\;\;\forall(\xi,\eta:\eta^T\eta\le\xi^T\widehat{L}^T\widehat{L}\xi)\\
\Leftrightarrow & \exists\lambda\ge 0:\left[\begin{array}{c|c}\mathcal{A}(y)-\lambda\widehat{L}^T\widehat{L} & \widehat{R}^T(y)\\ \hline \widehat{R}(y) & \lambda I_q\end{array}\right]\succeq 0\text{ [}\mathcal{S}\text{-Lemma]}\\
\Leftrightarrow & \exists\lambda:\left[\begin{array}{c|c}\mathcal{A}(y)-\lambda\widehat{L}^T\widehat{L} & \widehat{R}^T(y)\\ \hline \widehat{R}(y) & \lambda I_q\end{array}\right]\succeq 0,
\end{array}
$$

and we arrive at (3.2.16). □

### 3.2.5 Solvable Case IV: CQI with Simple Ellipsoidal Uncertainty

The last solvable case we intend to present is of uncertain CQI (3.2.3) with an ellipsoid as the perturbation set. Now, unlike the results of Theorem 3.2 and Proposition 3.2, we neither assume the uncertainty side-wise, nor impose specific structural restrictions on the CQI in question. However, whereas in all tractability results stated so far we ended up with a "well-structured" tractable reformulation of the RC (mainly in the form of an explicit system of LMIs), now the reformulation will be less elegant: we shall prove that the feasible set of the RC admits an efficiently computable *separation oracle* — an efficient computational routine that, given on input a candidate decision vector $y$, reports whether this vector is robust feasible, and if it is not the case, returns a *separator* — a linear form $e^Tz$ on the space of decision vectors such that

$$e^Ty > \sup_{z\in Y} e^Tz,$$

where $Y$ is the set of all robust feasible solutions. Good news is that equipped with such a routine, one can optimize efficiently a linear form over the intersection of $Y$ with any convex compact set $Z$ that is itself given by an efficiently computable separation oracle. On the negative side, the family of "theoretically efficient" optimization algorithms available in this situation is much more restricted than the family of algorithms available in the situations we encountered so far. Specifically, in these past situations, we could process the RC by high-performance Interior Point polynomial time methods, while in our present case we are forced to use slower black-box-oriented methods, like the Ellipsoid algorithm. As a result, the design dimensions that can be handled in a realistic time can drop considerably.

We are about to describe an efficient separation oracle for the feasible set

$$Y=\{y:\|\alpha(y)\zeta+\beta(y)\|_2\le\sigma^T(y)\zeta+\delta(y)\;\;\forall(\zeta:\zeta^T\zeta\le 1)\}\tag{3.2.19}$$

of the uncertain CQI (3.2.3) with the unit ball in the role of the perturbation set; recall that $\alpha(y)$, $\beta(y)$, $\sigma(y)$, $\delta(y)$ are affine in $y$.

Observe that $y \in Y$ if and only if the following two conditions hold true:

$$\begin{array}{|ll|c|}
\hline
 & 0 \le \sigma^T(y)\zeta + \delta(y) \quad \forall(\zeta : \|\zeta\|_2 \le 1) & \\
\Leftrightarrow & \|\sigma(y)\|_2 \le \delta(y) & (a) \\
\hline
 & (\sigma^T(y)\zeta + \delta(y))^2 - [\alpha(y)\zeta + \beta(y)]^T[\alpha(y)\zeta + \beta(y)] \ge 0 & \\
 & \hfill \forall(\zeta : \zeta^T\zeta \le 1) & \\
\Leftrightarrow & \exists \lambda \ge 0 : & \\
 & A_y(\lambda) \equiv \left[\begin{array}{c|c} \begin{array}{c}\lambda I_L + \sigma(y)\sigma^T(y) \\ -\alpha^T(y)\alpha(y)\end{array} & \begin{array}{c}\delta(y)\sigma^T(y) \\ -\beta^T(y)\alpha(y)\end{array} \\ \hline \begin{array}{c}\delta(y)\sigma(y) \\ -\alpha^T(y)\beta(y)\end{array} & \begin{array}{c}\delta^2(y) - \beta^T(y)\beta(y) \\ -\lambda\end{array}\end{array}\right] \succeq 0 & (b) \\
\hline
\end{array} \qquad (3.2.20)$$

where the second $\Leftrightarrow$ is due to the inhomogeneous $\mathcal{S}$-Lemma. Observe that *given $y$, it is easy to verify the validity of* (3.2.20). Indeed,

1. Verification of (3.2.20.$a$) is trivial.

2. To verify (3.2.20.$b$), we can use bisection in $\lambda$ as follows.
   First note that any $\lambda \ge 0$ satisfying the matrix inequality (MI) in (3.2.20.$b$) clearly should be $\le \lambda_+ \equiv \delta^2(y) - \beta^T(y)\beta(y)$. If $\lambda_+ < 0$, then (3.2.20.$b$) definitely does not take place, and we can terminate our verification. When $\lambda_+ \ge 0$, we can build a shrinking sequence of localizers $\Delta_t = [\underline{\lambda}_t, \overline{\lambda}_t]$ for the set $\Lambda_*$ of solutions to our MI, namely, as follows:

   • We set $\underline{\lambda}_0 = 0$, $\overline{\lambda}_0 = \lambda_+$, thus ensuring that $\Lambda_* \subset \Delta_0$.

   • Assume that after $t-1$ steps we have in our disposal a segment $\Delta_{t-1}$, $\Delta_{t-1} \subset \Delta_{t-2} \subset ... \subset \Delta_0$, such that $\Lambda_* \subset \Delta_{t-1}$. Let $\lambda_t$ be the midpoint of $\Delta_{t-1}$. At step $t$, we check whether the matrix $A_y(\lambda_t)$ is $\succeq 0$; to this end we can use any one from the well-known Linear Algebra routines capable to check in $O(k^3)$ operations positive semidefiniteness of a $k \times k$ matrix $A$, and if it is not the case, to produce a "certificate" for the fact that $A \not\succeq 0$ — a vector $z$ such that $z^TAz < 0$. If $A_y(\lambda_t) \succeq 0$, we are done, otherwise we get a vector $z_t$ such that the affine function $f_t(\lambda) \equiv z_t^T A_y(\lambda) z_t$ is negative when $\lambda = \lambda_t$. Setting $\Delta_t = \{\lambda \in \Delta_{t-1} : f_t(\lambda) \ge 0\}$, we clearly get a new localizer for $\Lambda_*$ that is at least twice shorter than $\Delta_{t-1}$; if this localizer is nonempty, we pass to step $t+1$, otherwise we terminate with the claim that (3.2.20.$b$) is not valid.

   Since the sizes of subsequent localizers shrink at each step by a factor of at least 2, the outlined procedure rapidly converges: for all practical purposes[1] we may assume that the procedure terminates after a small number of steps with either a $\lambda$ that makes the MI in (3.2.20) valid, or with an empty localizer, meaning that (3.2.20.$b$) is invalid.

So far we built an efficient procedure that checks whether or not $y$ is robust feasible (i.e., whether or not $y \in Y$). To complete the construction of a separation oracle for $Y$, it remains to build a separator of $y$ and $Y$ when $y \notin Y$. Our "separation strategy" is as follows. Recall that $y \in Y$ if and only if all vectors $v_y(\zeta) = [\alpha(y)\zeta + \beta(y); \sigma^T(y)\zeta + \delta(y)]$ with $\|\zeta\|_2 \le 1$ belong to the Lorentz cone $\mathbf{L}^{k+1}$, where $k = \dim \beta(y)$. Thus, $y \notin Y$ if there exists $\bar{\zeta}$ such that $\|\bar{\zeta}\|_2 \le 1$ and $v_y(\bar{\zeta}) \notin \mathbf{L}^{k+1}$. Given such a $\bar{\zeta}$, we can immediately build a separator of $y$ and $Y$ as follows:

[1]We could make our reasoning precise, but it would require going into tedious technical details that we prefer to skip.

1. Since $v_y(\bar\zeta) \notin \mathbf{L}^{k+1}$, we can easily separate $v_y(\bar\zeta)$ and $\mathbf{L}^{k+1}$. Specifically, setting $v_y(\bar\zeta) = [a; b]$, we have $b < \|a\|_2$, so that setting $e = [a/\|a\|_2; -1]$, we have $e^T v_y(\bar\zeta) = \|a\|_2 - b > 0$, while $e^T u \le 0$ for all $u \in \mathbf{L}^{k+1}$.

2. After a separator $e$ of $v_y(\bar\zeta)$ and $\mathbf{L}^{k+1}$ is built, we look at the function $\phi(z) = e^T v_z(\bar\zeta)$. This is an affine function of $z$ such that

$$\sup_{z\in Y} \phi(z) \le \sup_{u\in\mathbf{L}^{k+1}} e^T u < e^T v_y(\bar\zeta) = \phi(y)$$

where the first $\le$ is given by the fact that $v_z(\bar\zeta) \in \mathbf{L}^{k+1}$ when $z \in Y$. Thus, the homogeneous part of $\phi(\cdot)$, (which is a linear form readily given by $e$), separates $y$ and $Y$.

In summary, all we need is an efficient routine that, in the case when $y \notin Y$, i.e.,

$$\widehat{\mathcal{Z}}_y \equiv \{\bar\zeta : \|\bar\zeta\|_2 \le 1,\, v_y(\bar\zeta) \notin \mathbf{L}^{k+1}\} \ne \emptyset,$$

finds a point $\bar\zeta \in \widehat{\mathcal{Z}}_y$ ("an infeasibility certificate"). Here is such a routine. First, recall that our algorithm for verifying robust feasibility of $y$ reports that $y \notin Y$ in two situations:

• $\|\sigma(y)\|_2 > \delta(y)$. In this case we can without any difficulty find a $\bar\zeta$, $\|\bar\zeta\|_2 \le 1$, such that $\sigma^T(y)\bar\zeta + \delta(y) < 0$. In other words, the vector $v_y(\bar\zeta)$ has a negative last coordinate and therefore it definitely does not belong to $\mathbf{L}^{k+1}$. Such a $\bar\zeta$ is an infeasibility certificate.

• We have discovered that (a) $\lambda_+ < 0$, or (b) got $\Delta_t = \emptyset$ at a certain step $t$ of our bisection process. In this case building an infeasibility certificate is more tricky.

**Step 1: Separating the positive semidefinite cone and the "matrix ray" $\{A_y(\lambda) : \lambda \ge 0\}$.** Observe that with $z_0$ defined as the last basic orth in $\mathbb{R}^{L+1}$, we have $f_0(\lambda) \equiv z_0^T A_y(\lambda) z_0 < 0$ when $\lambda > \lambda_+$. Recalling what our bisection process is, we conclude that in both cases (a), (b) we have at our disposal a collection $z_0, ..., z_t$ of $(L+1)$-dimensional vectors such that with $f_s(\lambda) = z_s^T A_y(\lambda) z_s$ we have $f(\lambda) \equiv \min\,[f_0(\lambda), f_1(\lambda), ..., f_t(\lambda)] < 0$ for all $\lambda \ge 0$. By construction, $f(\lambda)$ is a piecewise linear concave function on the nonnegative ray; looking at what happens at the maximizer of $f$ over $\lambda \ge 0$, we conclude that an appropriate convex combination of just two of the "linear pieces" $f_0(\lambda), ..., f_t(\lambda)$ of $f$ is negative everywhere on the nonnegative ray. That is, with properly chosen and easy-to-find $\alpha \in [0,1]$ and $\tau_1, \tau_2 \le t$ we have

$$\phi(\lambda) \equiv \alpha f_{\tau_1}(\lambda) + (1-\alpha) f_{\tau_2}(\lambda) < 0 \quad \forall \lambda \ge 0.$$

Recalling the origin of $f_\tau(\lambda)$ and setting $z^1 = \sqrt{\alpha} z_{\tau_1}$, $z^2 = \sqrt{1-\alpha} z_{\tau_2}$, $Z = z^1[z^1]^T + z^2[z^2]^T$, we have

$$0 > \phi(\lambda) = [z^1]^T A_y(\lambda) z^1 + [z^2]^T A_y(\lambda) z^2 = \mathrm{Tr}(A_y(\lambda) Z) \quad \forall \lambda \ge 0. \tag{3.2.21}$$

This inequality has a simple interpretation: the function $\Phi(X) = \mathrm{Tr}(XZ)$ is a linear form on $\mathbf{S}^{L+1}$ that is nonnegative on the positive semidefinite cone (since $Z \succeq 0$ by construction) and is negative everywhere on the "matrix ray" $\{A_y(\lambda) : \lambda \ge 0\}$, thus certifying that this ray does not intersect the positive semidefinite cone (the latter is exactly the same as the fact that (3.2.20.$b$) is false).

**Step 2: from $Z$ to $\bar\zeta$.** Relation (3.2.21) says that an affine function $\phi(\lambda)$ is negative everywhere on the nonnegative ray, meaning that the slope of the function is nonpositive, and the value at the origin is negative. Taking into account (3.2.20), we get

$$Z_{L+1,L+1} \geq \sum_{i=1}^{L} Z_{ii}, \ \mathrm{Tr}(Z \underbrace{\left[\begin{array}{c|c} \sigma(y)\sigma^T(y) & \delta(y)\sigma^T(y) \\ -\alpha^T(y)\alpha(y) & -\beta^T(y)\alpha(y) \\ \hline \delta(y)\sigma(y) & \delta^2(y) - \beta^T(y)\beta(y) \\ -\alpha^T(y)\beta(y) & \end{array}\right]}_{A_y(0)}) < 0. \tag{3.2.22}$$

Besides this, we remember that $Z$ is given as $z^1[z^1]^T + z^2[z^2]^T$. We claim that

(!) *We can efficiently find a representation* $Z = ee^T + ff^T$ *such that* $e, f \in \mathbf{L}^{L+1}$.

Taking for the time being (!) for granted, let us build an infeasibility certificate. Indeed, from the second relation in (3.2.22) it follows that either $\mathrm{Tr}(A_y(0)ee^T) < 0$, or $\mathrm{Tr}(A_y(0)ff^T) < 0$, or both. Let us check which one of these inequalities indeed holds true; w.l.o.g., let it be the first one. From this inequality, in particular, $e \neq 0$, and since $e \in \mathbf{L}^{L+1}$, we have $e_{L+1} > 0$. Setting $\bar{e} = e/e_{L+1} = [\bar{\zeta}; 1]$, we have $\mathrm{Tr}(A_y(0)\bar{e}\bar{e}^T) = \bar{e}^T A_y(0)\bar{e} < 0$, that is,

$$\begin{aligned} &\delta^2(y) - \beta^T(y)\beta(y) + 2\delta(y)\sigma^T(y)\bar{\zeta} - 2\beta^T(y)\alpha(y)\bar{\zeta} + \bar{\zeta}^T\sigma(y)\sigma^T(y)\bar{\zeta} \\ &-\bar{\zeta}^T\alpha^T(y)\alpha(y)\bar{\zeta} < 0, \end{aligned}$$

or, which is the same,

$$(\delta(y) + \sigma^T(y)\bar{\zeta})^2 < (\alpha(y)\bar{\zeta} + \beta(y))^T(\alpha(y)\bar{\zeta} + \beta(y)).$$

We see that the vector $v_y(\bar{\zeta}) = [\alpha(y)\bar{\zeta} + \beta(y); \sigma^T(y)\bar{\zeta} + \delta(y)]$ does not belong to $\mathbf{L}^{L+1}$, while $\bar{e} = [\bar{\zeta}; 1] \in \mathbf{L}^{L+1}$, that is, $\|\bar{\zeta}\|_2 \leq 1$. We have built a required infeasibility certificate.

It remains to justify (!). Replacing, if necessary, $z^1$ with $-z^1$ and $z^2$ with $-z^2$, we can assume that $Z = z^1[z^1]^T + z^2[z^2]^T$ with $z^1 = [p; s]$, $z^2 = [q; r]$, where $s, r \geq 0$. It may happen that $z^1, z^2 \in \mathbf{L}^{L+1}$ — then we are done. Assume now that not both $z^1$, $z^2$ belong to $\mathbf{L}^{L+1}$, say, $z^1 \notin \mathbf{L}^{L+1}$, that is, $0 \leq s < \|p\|_2$. Observe that $Z_{L+1,L+1} = s^2 + r^2$ and $\sum\limits_{i=1}^{L} Z_{ii} = p^Tp + q^Tq$; therefore the first relation in (3.2.22) implies that $s^2 + r^2 \geq p^Tp + q^Tq$. Since $0 \leq s < \|p\|_2$ and $r \geq 0$, we conclude that $r > \|q\|_2$. Thus, $s < \|p\|_2$, $r > \|q\|_2$, whence there exists (and can be easily found) $\alpha \in (0, 1)$ such that for the vector $e = \sqrt{\alpha}z^1 + \sqrt{1-\alpha}z^2 = [u; t]$ we have $e_{L+1} = \sqrt{e_1^2 + ... + e_L^2}$. Setting $f = -\sqrt{1-\alpha}z^1 + \sqrt{\alpha}z^2$, we have $ee^T + ff^T = z^1[z^1]^T + z^2[z^2]^T = Z$. We now have

$$0 \leq Z_{L+1,L+1} - \sum_{i=1}^{L} Z_{ii} = e_{L+1}^2 + f_{L+1}^2 - \sum_{i=1}^{L}[e_i^2 + f_i^2] = f_{L+1}^2 - \sum_{i=1}^{L} f_i^2;$$

thus, replacing, if necessary, $f$ with $-f$, we see that $e, f \in \mathbf{L}^{L+1}$ and $Z = ee^T + ff^T$, as required in (!).

### Semidefinite Representation of the RC of an Uncertain CQI with Simple Ellipsoidal Uncertainty

This book was nearly finished when the topic considered in this section was significantly advanced by R. Hildebrand [56, 57] who discovered an explicit SDP representation of the cone of "Lorentz-positive" $n \times m$ matrices (real $m \times n$ matrices that map the Lorentz cone $\mathbf{L}^m$ into the Lorentz cone $\mathbf{L}^n$). Existence of such a representation was a long-standing open question. As a byproduct of answering this question, the construction of Hildebrand offers an explicit SDP reformulation of the RC of an uncertain conic quadratic inequality with ellipsoidal uncertainty.

**The RC of an uncertain conic quadratic inequality with ellipsoidal uncertainty and Lorentz-positive matrices.** Consider the RC of an uncertain conic quadratic inequality with simple ellipsoidal uncertainty; w.l.o.g., we assume that the uncertainty set $\mathcal{Z}$ is the unit Euclidean ball in some $\mathbb{R}^{m-1}$, so that the RC is the semi-infinite constraint of the form

$$B[x]\zeta + b[x] \in \mathbf{L}^n \ \forall(\zeta \in \mathbb{R}^{m-1} : \zeta^T\zeta \leq 1), \tag{3.2.23}$$

with $B[x]$, $b[x]$ affinely depending on $x$. This constraint is clearly exactly the same as the constraint

$$B[x]\xi + \tau b[x] \in \mathbf{L}^n \ \forall([\xi;\tau] \in \mathbf{L}^m).$$

We see that *$x$ is feasible for the RC in question if and only if the $n \times m$ matrix $M[x] = [B[x], b[x]]$ affinely depending on $x$ is Lorentz-positive, that is, maps the cone $\mathbf{L}^m$ into the cone $\mathbf{L}^n$.* It follows that *in order to get an explicit SDP representation of the RC, is suffices to know an explicit SDP representation of the set $P_{n,m}$ of $n \times m$ matrices mapping $\mathbf{L}^m$ into $\mathbf{L}^n$.*

**SDP representation of $P_{n,m}$** as discovered by R. Hildebrand (who used tools going far beyond those used in this book) is as follows.

A. Given $m, n$, we define a linear mapping $A \mapsto \mathcal{W}(A)$ from the space $\mathbb{R}^{n\times m}$ of real $n \times m$ matrices into the space $\mathbf{S}^N$ of symmetric $N \times N$ matrices with $N = (n-1)(m-1)$, namely, as follows.

Let $W_n[u] = \begin{bmatrix} u_n + u_1 & u_2 & \cdots & u_{n-1} \\ u_2 & u_n - u_1 & & \\ \vdots & & \ddots & \\ u_{n-1} & & & u_n - u_1 \end{bmatrix}$, so that $W_n$ is a symmetric $(n-1) \times (n-1)$ matrix depending on a vector $u$ of $n$ real variables. Now consider the *Kronecker product* $W[u,v] = W_n[u] \bigotimes W_m[v]$. [2] $W$ is a symmetric $N \times N$ matrix with entries that are bilinear functions of $u$ and $v$ variables, so that an entry is of the form "weighted sum of pair products of the $u$ and the $v$-variables." Now, given an $n \times m$ matrix $A$, let us replace pair products $u_iv_k$ in the representation of the entries in $W[u,v]$ with the entries $A_{ik}$ of $A$. As a result of this formal substitution, $W$ will become a symmetric $(n-1) \times (m-1)$ matrix $\mathcal{W}(A)$ that depends linearly on $A$.

B. We define a linear subspace $\mathcal{L}_{m,n}$ in the space $\mathbf{S}^N$ as the linear span of the Kronecker products $S \bigotimes T$ of all skew-symmetric real $(n-1)\times(n-1)$ matrices $S$ and skew-symmetric real $(m-1)\times(m-1)$ matrices $T$. Note that the Kronecker product of two skew-symmetric matrices is a symmetric matrix, so that the definition makes sense. Of course, we can easily build a basis in $\mathcal{L}_{m,n}$ — it is comprised of pairwise Kronecker products of the basic $(n-1)$-dimensional and $(m-1)$-dimensional skew-symmetric matrices.

The Hildebrand SDP representation of $P_{n,m}$ is given by the following:

**Theorem 3.3** [Hildebrand [57, Theorem 5.6]] *Let* $\min[m,n] \geq 3$. *Then an $n \times m$ matrix $A$ maps $\mathbf{L}^m$ into $\mathbf{L}^n$ if and only if $A$ can be extended to a feasible solution to the explicit system of LMIs*

$$\mathcal{W}(A) + X \succeq 0, \ X \in \mathcal{L}_{m,n}$$

*in variables $A, X$.*

[2] Recall that the Kronecker product $A \bigotimes B$ of a $p \times q$ matrix $A$ and an $r \times s$ matrix $B$ is the $pr \times qs$ matrix with rows indexed by pairs $(i,k)$, $1 \leq i \leq p$, $1 \leq k \leq r$, and columns indexed by pairs $(j,\ell)$, $1 \leq j \leq q$, $1 \leq \ell \leq s$, and the $((i,k),(j,\ell))$-entry equal to $A_{ij}B_{k\ell}$. Equivalently, $A \bigotimes B$ is a $p \times q$ block matrix with $r \times s$ blocks, the $(i,j)$-th block being $A_{ij}B$.

As a corollary,

*When* $m-1 := \dim\zeta \geq 2$ *and* $n := \dim b[x] \geq 3$, *the explicit* $(n-1)(m-1)\times(n-1)(m-1)$ *LMI*

$$\mathcal{W}([B[x], b[x]]) + X \succeq 0 \tag{3.2.24}$$

*in variables* $x$ *and* $X \in \mathcal{L}_{m,n}$ *is an equivalent SDP representation of the semi-infinite conic quadratic inequality* (3.2.23) *with ellipsoidal uncertainty set.*

The lower bounds on the dimensions of $\zeta$ and $b[x]$ in the corollary do not restrict generality — we can always ensure their validity by adding zero columns to $B[x]$ and/or adding zero rows to $[B[x], b[x]]$.

### 3.2.6 Illustration: Robust Linear Estimation

Consider the situation as follows: we are given noisy observations

$$w = (I_p + \Delta)z + \xi \tag{3.2.25}$$

of a signal $z$ that, in turn, is the result of passing an unknown input signal $v$ through a given linear filter: $z = Av$ with known $p\times q$ matrix $A$. The measurements contain errors of two kinds:

• bias $\Delta z$ linearly depending on $z$, where the only information on the bias matrix $\Delta$ is given by a bound $\|\Delta\|_{2,2} \leq \rho$ on its norm;

• random noise $\xi$ with zero mean and known covariance matrix $\Sigma = \mathbf{E}\{\xi\xi^T\}$.

The goal is to estimate a given linear functional $f^T v$ of the input signal. We restrict ourselves with estimators that are linear in $w$:

$$\widehat{f} = x^T w,$$

where $x$ is a fixed weight vector. For a linear estimator, the mean squares error is

$$\begin{array}{rcl}\mathrm{EstErr} & = & \sqrt{\mathbf{E}\{(x^T[(I+\Delta)Av+\xi] - f^T v)^2\}} \\ & = & \sqrt{([A^T(I+\Delta^T)x - f]^T v)^2 + x^T\Sigma x}.\end{array}$$

Now assume that our a priori knowledge of the true signal is that $v^T Q v \leq R^2$, where $Q \succ 0$ and $R > 0$. In this situation it makes sense to look for the *minimax optimal* weight vector $x$ that minimizes the worst, over $v$ and $\Delta$ compatible with our a priori information, mean squares estimation error. In other words, we choose $x$ as the optimal solution to the following optimization problem

$$\min_x \max_{\substack{v: v^T Q v \leq R^2 \\ \Delta: \|\Delta\|_{2,2} \leq \rho}} \left(\left([\underbrace{A^T(I+\Delta^T)}_{S} x - f]^T v\right)^2 + x^T\Sigma x\right)^{1/2}. \tag{$P$}$$

Now,

$$\begin{array}{l}\max\limits_{v: v^T Q v \leq R^2} [Sx - f]^T v = \max\limits_{u: u^T u \leq 1} [Sx - f]^T (RQ^{-1/2}u) \\ = R\|Q^{-1/2}Sx - \underbrace{Q^{-1/2}f}_{\widehat{f}}\|_2,\end{array}$$

so that $(P)$ reduces to the problem

$$\min_x \sqrt{x^T\Sigma x + R^2 \max_{\|\Delta\|_{2,2}\leq\rho} \|\underbrace{Q^{-1/2}A^T(I+\Delta^T)}_{B} x - \widehat{f}\|_2^2},$$

which is exactly the RC of the uncertain conic quadratic program

$$\min_{x,t,r,s}\left\{t:\ \begin{array}{c}\sqrt{r^2+s^2}\le t,\ \|\Sigma^{1/2}x\|_2\le r,\\ \|Bx-\widehat{f}\|_2\le R^{-1}s\end{array}\right\}, \tag{3.2.26}$$

where the only uncertain element of the data is the matrix $B = Q^{-1/2}A^T(I+\Delta^T)$ running through the uncertainty set

$$\mathcal{U} = \{B = \underbrace{Q^{-1/2}A^T}_{B_{\rm n}} + \rho Q^{-1/2}A^T\zeta, \zeta\in\mathcal{Z} = \{\zeta\in\mathbb{R}^{p\times p}: \|\zeta\|_{2,2}\le 1\}\}. \tag{3.2.27}$$

The uncertainty here is the unstructured norm-bounded one; the RC of (3.2.26), (3.2.27) is readily given by Theorem 3.2 and Example 3.1.(i). Specifically, the RC is the optimization program

$$\min_{x,t,r,s,\lambda}\left\{t:\ \begin{array}{c}\sqrt{r^2+s^2}\le t,\ \|\Sigma^{1/2}x\|_2\le r,\\ \left[\begin{array}{c|c|c} R^{-1}sI_q-\lambda\rho^2B_{\rm n}B_{\rm n}^T & & B_{\rm n}x-\widehat{f}\\ \hline & \lambda I_p & x\\ \hline [B_{\rm n}x-\widehat{f}]^T & x^T & R^{-1}s\end{array}\right]\succeq 0\end{array}\right\}, \tag{3.2.28}$$

which can further be recast as an SDP.

Next we present a numerical illustration.

**Example 3.2** Consider the problem as follows:

> A thin homogeneous iron plate occupies the 2-D square $D=\{(x,y): 0\le x,y\le 1\}$. At time $t=0$ it was heated to temperature $T(0,x,y)$ such that $\int_D T^2(0,x,y)dxdy\le T_0^2$ with a given $T_0$, and then was left to cool; the temperature along the perimeter of the plate is kept at the level $0^o$ all the time. At a given time $2\tau$ we measure the temperature $T(2\tau,x,y)$ along the 2-D grid
>
> $$\Gamma=\{(u_\mu,u_\nu):1\le\mu,\nu\le N\},\ u_k = frack-1/2N$$
>
> The vector $w$ of measurements is obtained from the vector
>
> $$z=\{T(2\tau,u_\mu,u_\nu):1\le\mu,\nu\le N\}$$
>
> according to (3.2.25), where $\|\Delta\|_{2,2}\le\rho$ and $\xi_{\mu\nu}$ are independent Gaussian random variables with zero mean and standard deviation $\sigma$. Given the measurements, we need to estimate the temperature $T(\tau,1/2,1/2)$ at the center of the plate at time $\tau$.

It is known from physics that the evolution in time of the temperature $T(t,x,y)$ of a homogeneous plate occupying a 2-D domain $\Omega$, with no sources of heat in the domain and heat exchange solely via the boundary, is governed by the *heat equation*

$$\frac{\partial}{\partial t}T = -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)T$$

(In fact, in the right hand side there should be a factor $\gamma$ representing material's properties, but by an appropriate choice of the time unit, this factor can be made equal to 1.) For the case of $\Omega=D$ and zero boundary conditions, the solution to this equation is as follows:

$$T(t,x,y)=\sum_{k,\ell=1}^{\infty} a_{k\ell}\exp\{-(k^2+\ell^2)\pi^2 t\}\sin(\pi kx)\sin(\pi\ell y), \tag{3.2.29}$$

where the coefficients $a_{k\ell}$ can be obtained by expanding the initial temperature into a series in the orthogonal basis $\phi_{k\ell}(x,y) = \sin(\pi kx)\sin(\pi \ell y)$ in $L_2(D)$:

$$a_{k\ell} = 4\int\limits_D T(0,x,y)\phi_{k\ell}(x,y)dxdy.$$

In other words, the Fourier coefficients of $T(t,\cdot,\cdot)$ in an appropriate orthogonal spatial basis decrease exponentially as $t$ grows, with the "decay time" (the smallest time in which every one of the coefficients is multiplied by factor $\leq 0.1$) equal to

$$\Delta = \frac{\ln(10)}{2\pi^2}.$$

Setting $v_{k\ell} = a_{k\ell}\exp\{-(k^2+\ell^2)\pi^2\tau\}$, the problem in question becomes to estimate

$$T(\tau,1/2,1/2) = \sum_{k,\ell} v_{k\ell}\phi_{k\ell}(1/2,1/2)$$

given observations

$$w = (I+\Delta)z + \xi,\ z = \{T(2\tau,u_\mu,u_\nu) : 1 \leq \mu,\nu \leq N\},$$
$$\xi = \{\xi_{\mu\nu} \sim \mathcal{N}(0,\sigma^2) : 1 \leq \mu,\nu \leq N\}$$

($\xi_{\mu\nu}$ are independent).

**Finite-dimensional approximation.** Observe that

$$a_{k\ell} = \exp\{\pi^2(k^2+\ell^2)\tau\}v_{k\ell}$$

and that

$$\sum_{k,\ell} v_{k\ell}^2\exp\{2\pi^2(k^2+\ell^2)\tau\} = \sum_{k,\ell} a_{k\ell}^2 = 4\int_D T^2(0,x,y)dxdy \leq 4T_0^2. \tag{3.2.30}$$

It follows that

$$|v_{k\ell}| \leq 2T_0\exp\{-\pi^2(k^2+\ell^2)\tau\}.$$

Now, given a tolerance $\epsilon > 0$, we can easily find $L$ such that

$$\sum_{k,\ell:k^2+\ell^2>L^2} \exp\{-\pi^2(k^2+\ell^2)\tau\} \leq \frac{\epsilon}{2T_0},$$

meaning that when replacing by zeros the actual (unknown!) $v_{k\ell}$ with $k^2+\ell^2 > L^2$, we change temperature at time $\tau$ (and at time $2\tau$ as well) at every point by at most $\epsilon$. Choosing $\epsilon$ really small (say, $\epsilon = 1.$e-16), we may assume for all practical purposes that $v_{k\ell} = 0$ when $k^2+\ell^2 > L^2$, which makes our problem a finite-dimensional one, specifically, as follows:

Given the parameters $L$, $N$, $\rho$, $\sigma$, $T_0$ and observations

$$w = (I+\Delta)z + \xi, \tag{3.2.31}$$

where $\|\Delta\|_{2,2} \leq \rho$, $\xi_{\mu\nu} \sim \mathcal{N}(0,\sigma^2)$ are independent, $z = Av$ is defined by the relations

$$z_{\mu\nu} = \sum_{k^2+\ell^2\leq L^2} \exp\{-\pi^2(k^2+\ell^2)\tau\}v_{k\ell}\phi_{k\ell}(u_\mu,u_\nu),\ 1 \leq \mu,\nu \leq N,$$

and $v = \{v_{k\ell}\}_{k^2+\ell^2 \le L^2}$ is known to satisfy the inequality

$$v^T Q v \equiv \sum_{k^2+\ell^2 \le L^2} v_{k\ell}^2 \exp\{2\pi^2(k^2+\ell^2)\tau\} \le 4T_0^2,$$

estimate the quantity

$$\sum_{k^2+\ell^2 \le L^2} v_{k\ell}\phi_{k\ell}(1/2,1/2),$$

where $\phi_{k\ell}(x,y) = \sin(\pi k x)\sin(\pi \ell y)$ and $u_\mu = \frac{\mu-1/2}{N}$.

The latter problem fits the framework of robust estimation we have built, and we can recover $T = T(\tau, 1/2, 1/2)$ by a linear estimator

$$\widehat{T} = \sum_{\mu,\nu} x_{\mu\nu} w_{\mu\nu}$$

with weights $x_{\mu\nu}$ given by an optimal solution to the associated problem (3.2.28).

Assume, for example, that $\tau$ is half of the decay time of our system:

$$\tau = \frac{1}{2}\frac{\ln(10)}{2\pi^2} \approx 0.0583,$$

and let

$$T_0 = 1000, N = 4.$$

With $\epsilon$ = 1.e-15, we get $L = 8$ (this corresponds to just 41-dimensional space for $v$'s). Now consider four options for $\rho$ and $\sigma$:

(a) $\rho$ = 1.e-9, $\sigma$ = 1.e-9
(b) $\rho = 0$, $\sigma$ = 1.e-3
(c) $\rho$ = 1.e-3, $\sigma$ = 1.e-3
(d) $\rho$ = 1.e-1, $\sigma$ = 1.e-1

*In the case of (a)*, the optimal value in (3.2.28) is 0.0064, meaning that the expected squared error of the minimax optimal estimator never exceeds $(0.0064)^2$. The minimax optimal weights are

$$\begin{bmatrix} 6625.3 & -2823.0 & -2.8230 & 6625.3 \\ -2823.0 & 1202.9 & 1202.9 & -2823.0 \\ -2823.0 & 1202.9 & 1202.9 & -2823.0 \\ 6625.3 & -2823.0 & -2823.0 & 6625.3 \end{bmatrix} \qquad \text{(A)}$$

(we represent the weights as a 2-D array, according to the natural structure of the observations).

*In the case of (b)*, the optimal value in (3.2.28) is 0.232, and the minimax optimal weights are

$$\begin{bmatrix} -55.6430 & -55.6320 & -55.6320 & -55.6430 \\ -55.6320 & 56.5601 & 56.5601 & -55.6320 \\ -55.6320 & 56.5601 & 56.5601 & -55.6320 \\ -55.6430 & -55.6320 & -55.6320 & -55.6430 \end{bmatrix}. \qquad \text{(B)}$$

*In the case of (c)*, the optimal value in (3.2.28) is 8.92, and the minimax optimal weights are

$$\begin{bmatrix} -0.4377 & -0.2740 & -0.2740 & -0.4377 \\ -0.2740 & 1.2283 & 1.2283 & -0.2740 \\ -0.2740 & 1.2283 & 1.2283 & -0.2740 \\ -0.4377 & -0.2740 & -0.2740 & -0.4377 \end{bmatrix}. \qquad \text{(C)}$$

*In the case of (d)*, the optimal value in (3.2.28) is 63.9, and the minimax optimal weights are

$$\begin{bmatrix} 0.1157 & 0.2795 & 0.2795 & 0.1157 \\ 0.2795 & 0.6748 & 0.6748 & 0.2795 \\ 0.2795 & 0.6748 & 0.6748 & 0.2795 \\ 0.1157 & 0.2795 & 0.2795 & 0.1157 \end{bmatrix}. \tag{D}$$

Now, in reality we can hardly know exactly the bounds $\rho$, $\sigma$ on the measurement errors. What happens when we under- or over-estimate these quantities? To get an orientation, let us use every one of the weights given by (A), (B), (C), (D) in every one of the situations (a), (b), (c), (d). This is what happens with the errors (obtained as the average of observed errors over 100 random simulations using the "nearly worst-case" signal $v$ and "nearly worst-case" perturbation matrix $\Delta$):

| | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| (A) | 0.001 | 18.0 | 6262.9 | 6.26e5 |
| (B) | 0.063 | 0.232 | 89.3 | 8942.7 |
| (C) | 8.85 | 8.85 | 8.85 | 108.8 |
| (D) | 8.94 | 8.94 | 8.94 | 63.3 |

We clearly see that, first, in our situation taking into account measurement errors, even pretty small ones, is a must (this is so in all *ill-posed* estimation problems — those where the condition number of $B_{\rm n}$ is large). Second, we see that underestimating the magnitude of measurement errors seems to be much more dangerous than overestimating them.

## 3.3 Approximating RCs of Uncertain Conic Quadratic Problems

In this section we focus on *tight* tractable approximations of uncertain CQIs — those with tightness factor independent (or nearly so) of the "size" of the description of the perturbation set. Known approximations of this type deal with side-wise uncertainty and two types of the left hand side perturbations: the first is the case of *structured norm-bounded perturbations* to be considered in section 3.3.1, while the second is the case of $\cap$-*ellipsoidal* left hand side perturbation sets to be considered in section 3.3.2.

### 3.3.1 Structured Norm-Bounded Uncertainty

Consider the case where the uncertainty in CQI (3.2.3) is side-wise with the right hand side uncertainty as in section 3.2.2, and with *structured norm-bounded* left hand side uncertainty, meaning that

1. The left hand side perturbation set is

$$\mathcal{Z}_\rho^{\rm left} = \rho \mathcal{Z}_1^{\rm left} = \left\{ \eta = (\eta^1, ..., \eta^N) : \begin{array}{l} \eta^\nu \in \mathbb{R}^{p_\nu \times q_\nu} \,\forall \nu \le N \\ \|\eta^\nu\|_{2,2} \le \rho \,\forall \nu \le N \\ \eta^\nu = \theta_\nu I_{p_\nu}, \theta_\nu \in \mathbb{R},\, \nu \in \mathcal{I}_{\rm S} \end{array} \right\} \tag{3.3.1}$$

Here $\mathcal{I}_{\rm S}$ is a given subset of the index set $\{1, ..., N\}$ such that $p_\nu = q_\nu$ for $\nu \in \mathcal{I}_{\rm S}$.

Thus, the left hand side perturbations $\eta \in \mathcal{Z}_1^{\text{left}}$ are block-diagonal matrices with $p_\nu \times q_\nu$ diagonal blocks $\eta^\nu, \nu = 1, ..., N$. All of these blocks are of matrix norm not exceeding 1, and, in addition, prescribed blocks should be proportional to the unit matrices of appropriate sizes. The latter blocks are called *scalar*, and the remaining — *full* perturbation blocks.

2. We have

$$A(\eta)y + b(\eta) = A^{\text{n}}y + b^{\text{n}} + \sum_{\nu=1}^{N} L_\nu^T(y)\eta^\nu R_\nu(y), \tag{3.3.2}$$

where all matrices $\mathbf{L}_\nu(y) \not\equiv 0$, $R_\nu(y) \not\equiv 0$ are affine in $y$ and for every $\nu$, either $L_\nu(y)$, or $R_\nu(y)$, or both are independent of $y$.

**Remark 3.1** *W.l.o.g., we assume from now on that all scalar perturbation blocks are of the size* $1 \times 1$*:* $p_\nu = q_\nu = 1$ *for all* $\nu \in \mathcal{I}_{\text{S}}$.

To see that this assumption indeed does not restrict generality, note that if $\nu \in \mathcal{I}_{\text{S}}$, then in order for (3.3.2) to make sense, $R_\nu(y)$ should be a $p_\nu \times 1$ vector, and $L_\nu(y)$ should be a $p_\nu \times k$ matrix, where $k$ is the dimension of $b(\eta)$. Setting $\bar{R}_\nu(y) \equiv 1$, $\bar{L}_\nu(y) = R_\nu^T(y)L_\nu(y)$, observe that $\bar{L}_\nu(y)$ is affine in $y$, and the contribution $\theta_\nu L_\nu^T(y)R_\nu(y)$ of the $\nu$-th scalar perturbation block to $A(\eta)y + b(\eta)$ is exactly the same as if this block were of size $1 \times 1$, and the matrices $L_\nu(y)$, $R_\nu(y)$ were replaced with $\bar{L}_\nu(y)$, $\bar{R}_\nu(y)$, respectively.

Note that Remark 3.1 is equivalent to the assumption that *there are no scalar perturbation blocks at all* — indeed, $1 \times 1$ scalar perturbation blocks can be thought of as full ones as well. [3].

Recall that we have already considered the particular case $N = 1$ of the uncertainty structure. Indeed, with a single perturbation block, that, as we just have seen, we can treat as a full one, we find ourselves in the situation of side-wise uncertainty with unstructured norm-bounded left hand side perturbation (section 3.2.3). In this situation the RC of the uncertain CQI in question is computationally tractable. The latter is not necessarily the case for general ($N > 1$) structured norm-bounded left hand side perturbations. To see that the general structured norm-bounded perturbations are difficult to handle, note that they cover, in particular, the case of *interval uncertainty*, where $\mathcal{Z}_1^{\text{left}}$ is the box $\{\eta \in \mathbb{R}^L : \|\eta\|_\infty \leq 1\}$ and $A(\eta)$, $b(\eta)$ are arbitrary affine functions of $\eta$.

Indeed, the interval uncertainty

$$\begin{aligned} A(\eta)y + b(\eta) &= [A^{\text{n}}y + b^{\text{n}}] + \sum_{\nu=1}^{N} \eta_\nu [A^\nu y + b^\nu] \\ &= [A^{\text{n}}y + b^{\text{n}}] \quad + \sum_{\nu=1}^{N} \underbrace{[A^\nu y + b^\nu]}_{L_\nu^T(y)} \cdot \eta_\nu \cdot \underbrace{1}_{R_\nu(y)}, \end{aligned} \tag{3.3.3}$$

is nothing but the structured norm-bounded perturbation with $1 \times 1$ perturbation blocks.

From the beginning of section 3.1.3 we know that the RC of uncertain CQI with side-wise uncertainty and interval uncertainty in the left hand side in general is computationally intractable, meaning that structural norm-bounded uncertainty can be indeed difficult.

[3] A reader could ask, why do we need the scalar perturbation blocks, given that finally we can get rid of them without loosing generality. The answer is, that we intend to use the same notion of structured norm-bounded uncertainty in the case of uncertain LMIs, where Remark 3.1 does not work.

**Approximating the RC of Uncertain Least Squares Inequality**

We start with deriving a safe tractable approximation of the RC of an *uncertain Least Squares constraint*

$$\|A(\eta)y + b(\eta)\|_2 \leq \tau, \tag{3.3.4}$$

with structured norm-bounded perturbation (3.3.1), (3.3.2).

**Step 1: reformulating the RC of (3.3.4), (3.3.1), (3.3.2) as a semi-infinite LMI.** Given a $k$-dimensional vector $u$ ($k$ is the dimension of $b(\eta)$) and a real $\tau$, let us set

$$\text{Arrow}(u,t) = \left[\begin{array}{c|c} \tau & u^T \\ \hline u & \tau I_k \end{array}\right].$$

Recall that by Lemma 3.1 $\|u\|_2 \leq \tau$ if and only if $\text{Arrow}(u,\tau) \succeq 0$. It follows that the RC of (3.3.4), (3.3.1), (3.3.2), which is the semi-infinite Least Squares inequality

$$\|A(\eta)y + b(\eta)\|_2 \leq \tau \ \forall \eta \in \mathcal{Z}_\rho^{\text{left}},$$

can be rewritten as

$$\text{Arrow}(A(\eta)y + b(\eta), \tau) \succeq 0 \ \forall \eta \in \mathcal{Z}_\rho^{\text{left}}. \tag{3.3.5}$$

Introducing $k \times (k+1)$ matrix $\mathcal{L} = [0_{k\times 1}, I_k]$ and $1 \times (k+1)$ matrix $\mathcal{R} = [1, 0, ..., 0]$, we clearly have

$$\begin{array}{l} \text{Arrow}(A(\eta)y + b(\eta), \tau) = \text{Arrow}(A^{\text{n}}y + b^{\text{n}}, \tau) \\ \quad + \sum\limits_{\nu=1}^N \left[\mathcal{L}^T L_\nu^T(y)\eta^\nu R_\nu(y)\mathcal{R} + \mathcal{R}^T R_\nu^T(y)[\eta^\nu]^T L_\nu(y)\mathcal{L}\right]. \end{array} \tag{3.3.6}$$

Now, since for every $\nu$, either $L_\nu(y)$, or $R_\nu(y)$, or both, are independent of $y$, renaming, if necessary $[\eta^\nu]^T$ as $\eta^\nu$, and swapping $L_\nu(y)\mathcal{L}$ and $R_\nu(y)\mathcal{R}$, we may assume w.l.o.g. that in the relation (3.3.6) all factors $L_\nu(y)$ are independent of $y$, so that the relation reads

$$\begin{array}{l} \text{Arrow}(A(\eta)y + b(\eta), \tau) = \text{Arrow}(A^{\text{n}}y + b^{\text{n}}, \tau) \\ \quad + \sum\limits_{\nu=1}^N \big[\underbrace{\mathcal{L}^T L_\nu^T}_{\widehat{L}_\nu^T} \eta^\nu \underbrace{R_\nu(y)\mathcal{R}}_{\widehat{R}_\nu(y)} + \widehat{R}_\nu^T(y)[\eta^\nu]^T \widehat{L}_\nu\big] \end{array}$$

where $\widehat{R}_\nu(y)$ are affine in $y$ and $\widehat{L}_\nu \neq 0$. Observe also that all the symmetric matrices

$$B_\nu(y,\eta^\nu) = \widehat{L}_\nu^T \eta^\nu \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y)[\eta^\nu]^T \widehat{L}_\nu$$

are differences of two matrices of the form $\text{Arrow}(u,\tau)$ and $\text{Arrow}(u',\tau)$, so that these are matrices of rank at most 2. The intermediate summary of our observations is as follows:

(#): *The RC of* (3.3.4), (3.3.1), (3.3.2) *is equivalent to the semi-infinite LMI*

$$\begin{array}{c} \underbrace{\text{Arrow}(A^{\text{n}}y + b^{\text{n}}, \tau)}_{B_0(y,\tau)} + \sum\limits_{\nu=1}^N B_\nu(y,\eta^\nu) \succeq 0 \ \forall \left(\eta : \begin{array}{c} \eta^\nu \in \mathbb{R}^{p_\nu \times q_\nu}, \\ \|\eta^\nu\|_{2,2} \leq \rho \, \forall \nu \leq N \end{array}\right) \\ \left[\begin{array}{c} B_\nu(y,\eta^\nu) = \widehat{L}_\nu^T \eta^\nu \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y)[\eta^\nu]^T \widehat{L}_\nu, \ \nu = 1, ..., N \\ p_\nu = q_\nu = 1 \, \forall \nu \in \mathcal{I}_{\text{S}} \end{array}\right] \end{array} \tag{3.3.7}$$

*Here $\widehat{R}(y)$ are affine in $y$, and for all $y$, all $\nu \geq 1$ and all $\eta^\nu$ the ranks of the matrices $B_\nu(y,\eta^\nu)$ do not exceed 2.*

**Step 2. Approximating (3.3.7).** Observe that an evident *sufficient* condition for the validity of (3.3.7) for a given $y$ is the existence of symmetric matrices $Y_\nu$, $\nu = 1, ..., N$, such that

$$Y_\nu \succeq B_\nu(y, \eta^\nu) \,\forall\, (\eta^\nu \in \mathcal{Z}_\nu = \{\eta^\nu : \|\eta^\nu\|_{2,2} \le 1; \nu \in \mathcal{I}_\mathrm{S} \Rightarrow \eta^\nu \in \mathbb{R}I_{p_\nu}\}) \tag{3.3.8}$$

and

$$B_0(y, \tau) - \rho \sum_{\nu=1}^{N} Y_\nu \succeq 0. \tag{3.3.9}$$

We are about to demonstrate that the semi-infinite LMIs (3.3.8) in variables $Y_\nu, y, \tau$ can be represented by explicit finite systems of LMIs, so that the system $\mathcal{S}^0$ of semi-infinite constraints (3.3.8), (3.3.9) on variables $Y_1, ..., Y_N, y, \tau$ is equivalent to an explicit finite system $\mathcal{S}$ of LMIs. Since $\mathcal{S}^0$, due to its origin, is a safe approximation of (3.3.7), so will be $\mathcal{S}$, (which, in addition, is tractable). Now let us implement our strategy.

$1^0$. Let us start with $\nu \in \mathcal{I}_\mathrm{S}$. Here (3.3.8) clearly is equivalent to just two LMIs

$$Y_\nu \succeq B_\nu(y) \equiv \widehat{L}_\nu^T \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y) \widehat{L}_\nu \;\&\; Y_\nu \succeq -B_\nu(y). \tag{3.3.10}$$

$2^0$. Now consider relation (3.3.8) for the case $\nu \notin \mathcal{I}_\mathrm{S}$. Here we have

$$\begin{array}{cc}
 & (Y_\nu, y) \text{ satisfies } (3.3.8) \\
\Leftrightarrow & u^T Y_\nu u \ge u^T B_\nu(y, \eta^\nu) u \;\forall u \forall (\eta^\nu : \|\eta^\nu\|_{2,2} \le 1) \\
\Leftrightarrow & u^T Y_\nu u \ge u^T \widehat{L}_\nu^T \eta^\nu \widehat{R}_\nu(y) u + u^T \widehat{R}_\nu^T(y) [\eta^\nu]^T \widehat{L}_\nu u \;\forall u \forall (\eta^\nu : \|\eta^\nu\|_{2,2} \le 1) \\
\Leftrightarrow & u^T Y_\nu u \ge 2 u^T \widehat{L}_\nu^T \eta^\nu \widehat{R}_\nu(y) u \;\forall u \forall (\eta^\nu : \|\eta^\nu\|_{2,2} \le 1) \\
\Leftrightarrow & u^T Y_\nu u \ge 2 \|\widehat{L}_\nu u\|_2 \|\widehat{R}(y) u\|_2 \;\forall u \\
\Leftrightarrow & u^T Y_\nu u - 2\xi^T \widehat{R}_\nu(y) u \;\forall (u, \xi : \xi^T \xi \le u^T \widehat{L}_\nu^T \widehat{L}_\nu u)
\end{array}$$

Invoking the $\mathcal{S}$-Lemma, the concluding condition in the latter chain is equivalent to

$$\exists \lambda_\nu \ge 0 : \left[\begin{array}{c|c} Y_\nu - \lambda_\nu \widehat{L}_\nu^T \widehat{L}_\nu & -\widehat{R}_\nu^T(y) \\ \hline -\widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu} \end{array}\right] \succeq 0, \tag{3.3.11}$$

where $k_\nu$ is the number of rows in $\widehat{R}_\nu(y)$.

We have proved the first part of the following statement:

**Theorem 3.4** *The explicit system of LMIs*

$$\begin{array}{c}
Y_\nu \succeq \pm(\widehat{L}_\nu^T \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y) \widehat{L}_\nu),\; \nu \in \mathcal{I}_\mathrm{S} \\
\left[\begin{array}{c|c} Y_\nu - \lambda_\nu \widehat{L}_\nu^T \widehat{L}_\nu & \widehat{R}_\nu^T(y) \\ \hline \widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu} \end{array}\right] \succeq 0,\; \nu \notin \mathcal{I}_\mathrm{S} \\
\mathrm{Arrow}(A^\mathrm{n} y + b^\mathrm{n}, \tau) - \rho \sum\limits_{\nu=1}^{N} Y_\nu \succeq 0
\end{array} \tag{3.3.12}$$

*(for notation, see* (3.3.7)*) in variables* $Y_1, ..., Y_N, \lambda_\nu, y, \tau$ *is a safe tractable approximation of the RC of the uncertain Least Squares inequality* (3.3.4), (3.3.1), (3.3.2)*. The tightness factor of this approximation never exceeds* $\pi/2$*, and equals to 1 when* $N = 1$.

**Proof.** By construction, (3.3.12) indeed is a safe tractable approximation of the RC of (3.3.4), (3.3.1), (3.3.2) (note that a matrix of the form $\left[\begin{array}{c|c} A & B \\ \hline B^T & A \end{array}\right]$ is $\succeq 0$ if and only if the matrix $\left[\begin{array}{c|c} A & -B \\ \hline -B^T & A \end{array}\right]$ is so). By Remark and Theorem 3.2, our approximation is exact when $N = 1$. The fact that the tightness factor never exceeds $\pi/2$ is an immediate corollary of the real case Matrix Cube Theorem (Theorem A.7), and we use the corresponding notation in the rest of the proof. Observe that a given pair $(y, \tau)$ is robust feasible for (3.3.4), (3.3.1), (3.3.2) if and only if the matrices $B_0 = B_0(y, \tau)$, $B_i = B_{\nu_i}(y, 1)$, $i = 1, ..., p$, $L_j = \widehat{L}_{\mu_j}, R_j = \widehat{R}_{\mu_j}(y)$, $j = 1, ..., q$, satisfy $\mathcal{A}(\rho)$; here $\mathcal{I}_{\rm S} = \{\nu_1 < ... < \nu_p\}$ and $\{1, ..., L\}\backslash\mathcal{I}_{\rm S} = \{\mu_1 < ... < \mu_q\}$. At the same time, the validity of the corresponding predicate $\mathcal{B}(\rho)$ is equivalent to the possibility to extend $y$ to a solution of (3.3.12) due to the origin of the latter system. Since all matrices $B_i$, $i = 1, ..., p$, are of rank at most 2 by (#), the Matrix Cube Theorem implies that if $(y, \tau)$ cannot be extended to a feasible solution to (3.3.12), then $(y, \tau)$ is not robust feasible for (3.3.4), (3.3.1), (3.3.2) when the uncertainty level is increased by the factor $\vartheta(2) = \frac{\pi}{2}$. □

**Illustration: Antenna Design revisited.** Consider the Antenna Design example (Example 1.1) and assume that instead of measuring the closeness of a synthesized diagram to the target one in the uniform norm, as was the case in section 1.1.3, we want to use the Euclidean norm, specifically, the weighted 2-norm

$$\|f(\cdot)\|_{2,w} = \left(\sum_{i=1}^m f^2(\theta_i)\mu_i\right)^{1/2} \qquad [\theta_i = \frac{i\pi}{2m}, 1 \le i \le m = 240,\ \mu_i = \frac{\cos(\theta_i)}{\sum_{s=1}^m \cos(\theta_s)}]$$

To motivate the choice of weights, recall that the functions $f(\cdot)$ we are interested in are restrictions of diagrams (and their differences) on the equidistant $L$-point grid of altitude angles. The diagrams in question are, physically speaking, functions of a 3D direction from the upper half-space (a point on the unit 2D hemisphere) which depend solely on the altitude angle and are independent of the longitude angle. A "physically meaningful" $L_2$-norm here corresponds to uniform distribution on the hemisphere; after discretization of the altitude angle, this $L_2$ norm becomes our $\|\cdot\|_2$.

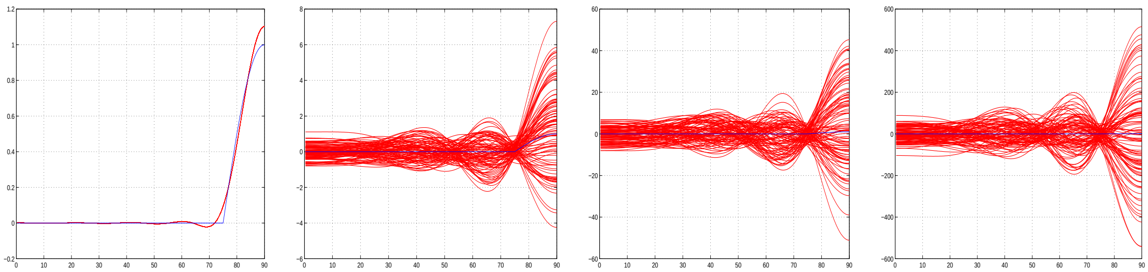
with this measure of discrepancy between a synthesized and the target diagram, the problem of interest becomes the uncertain problem

$$\left\{\min_{y,\tau}\{\tau : \|WD[I + \mathrm{Diag}\{\eta\}]y - b\|_2 \le \tau\} : \eta \in \rho\mathcal{Z}\right\},\ \mathcal{Z} = \{\eta \in \mathbb{R}^{L=10} : \|\eta\|_\infty \le 1\}, \quad (3.3.13)$$

where

- $D = [D_{ij} = D_j(\theta_i)]_{\substack{1\le i\le m=240,\\ 1\le j\le L=10}}]$ is the matrix comprised of the diagrams of $L = 10$ antenna elements (central circle and surrounding rings), see section 1.1.3,
- $W = \mathrm{Diag}\{\sqrt{\mu_1}, ..., \sqrt{\mu_m}\}$, so that $\|Wz\|_2 = \|z\|_{2,w}$, and $b = W[D_*(\theta_1); ...; D_*(\theta_m)]$ comes from the target diagram $D_*(\cdot)$, and
- $\eta$ is comprised of actuation errors, and $\rho$ is the uncertainty level.

*Nominal design.* Solving the nominal problem (corresponding to $\rho = 0$), we end up with the nominal optimal design which "in the dream" – with no actuation errors – is really nice (figure



$\rho = 0$ $\rho = 0.0001$ $\rho = 0.001$ $\rho = 0.01$

Figure 3.1: "Dream and reality," nominal optimal design: samples of 100 actual diagrams (red) for different uncertainty levels. Blue: the target diagram

| | Dream | Reality | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\rho = 0$ | $\rho = 0.0001$ | | | $\rho = 0.001$ | | | $\rho = 0.01$ | | |
| | value | min | mean | max | min | mean | max | min | mean | max |
| $\|\cdot\|_{2,w}$-distance to target | 0.011 | 0.077 | 0.424 | 0.957 | 1.177 | 4.687 | 9.711 | 8.709 | 45.15 | 109.5 |
| energy concentration | 99.4% | 0.23% | 20.1% | 77.7% | 0.70% | 19.5% | 61.5% | 0.53% | 18.9% | 61.5% |

Table 3.1: Quality of nominal antenna design: dream and reality. Data over 100 samples of actuation errors per each uncertainty level $\rho$.

3.1, case of $\rho = 0$): the $\|\cdot\|_{2,w}$-distance of the nominal diagram to the target is as small as 0.0112, and the energy concentration for this diagram is as large as 99.4%. Unfortunately, the data in figure 3.1 and table 3.1 show that "in reality," with the uncertainty level as small as $\rho = 0.01\%$, the nominal design is a complete disaster.

*Robust design.* Let us build a robust design. The set $\mathcal{Z}$ is a unit box, that is, we are in the case of interval uncertainty, or, which the same, structured norm-bounded uncertainty with $L = 10$ scalar perturbation blocks $\eta_\ell$. Denoting $\ell$-th column of $WD$ by $[WD]_\ell$, we have

$$WD[I + \mathrm{Diag}\{\eta\}]y - b = [WDy - b] + \sum_{\ell=1}^{L} \eta_\ell \left[y_\ell [WD]_\ell\right],$$

that is, taking into account (3.3.3), we have in the notation of (3.3.2):

$$A^{\mathrm{n}}y + b^{\mathrm{n}} = WDy - b,\ L_\nu(y) = y_\nu [WD]_\nu^T,\ \eta^\nu \equiv \eta_\nu,\ R_\nu(y) \equiv 1,\ 1 \le \nu \le N \equiv L = 10,$$

so that in the notation of Theorem 3.4 we have

$$\widehat{R}_\nu(y) = [0, y_\nu [WD]_\nu^T],\ \widehat{L}_\nu = [1, 0_{1\times n}], k_\nu = 1, ., 1 \le \nu \le N \equiv L.$$

Since we are in our right to treat all perturbation blocks as scalar, a tight within the factor $\pi/2$ safe tractable approximation, given by Theorem 3.4, of the RC of our uncertain Least Squares problem reads

$$\min_{\tau,y,Y_1,\dots,Y_L}\left\{\tau : \mathrm{Arrow}(WDy-b,\tau)-\rho\sum_{\nu=1}^{L}Y_\nu\succeq 0,\ Y_\nu\succeq\pm\left[\widehat{L}_\nu^T\widehat{R}_\nu(y)+\widehat{R}_\nu^T(y)\widehat{L}_\nu\right],1\leq\nu\leq L\right\}. \tag{3.3.14}$$

This problem simplifies dramatically due to the following simple fact (see Exercise 3.6):

> (!) *Let $a,b$ be two vectors of the same dimension with $a\neq 0$. Then $Y\succeq ab^T+ba^T$ if and only if there exists $\lambda\geq 0$ such that $Y\succeq\lambda aa^T+\frac{1}{\lambda}bb^T$.*
> Here, by definition, $\frac{1}{0}bb^T$ is undefined when $b\neq 0$ and is the zero matrix when $b=0$.

By (!), a pair $(\tau,y)$ can be extended, by properly chosen $Y_\nu$, $\nu=1,...,L$, to a feasible solution of (3.3.14) if and only if there exist $\lambda_\nu\geq 0$, $1\leq\nu\leq L$, such that $\mathrm{Arrow}(WDy-b,\tau)-\rho\sum_\nu\left[\lambda_\nu\widehat{L}_\nu^T\widehat{L}_\nu+\lambda_\nu^{-1}\widehat{R}_\nu^T(y)\widehat{R}_\nu(y)\right]\succeq 0$, which, by the Schur Complement Lemma is equivalent to

$$\left[\begin{array}{c|c}\mathrm{Arrow}(WDy-b,\tau)-\sum\limits_\nu\rho\lambda_\nu\widehat{L}_\nu^T\widehat{L}_\nu & \rho[\widehat{R}_1^T(y),...,\widehat{R}_L^T(y)]\\ \hline \rho[\widehat{R}_1(y);...;\widehat{R}_L(y)] & \rho\mathrm{Diag}\{\lambda_1,...,\lambda_L\}\end{array}\right]\succeq 0.$$

Thus, problem (3.3.14) is equivalent to

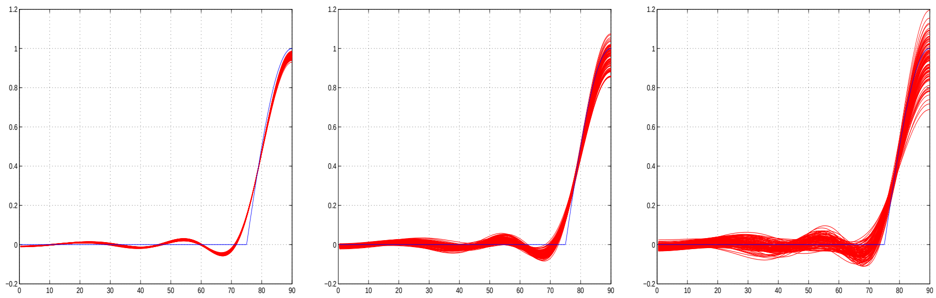
$$\min_{\tau,y,\gamma}\left\{\tau : \left[\begin{array}{c|c}\mathrm{Arrow}(WDy-b,\tau)-\sum\limits_\nu\gamma_\nu\widehat{L}_\nu^T\widehat{L}_\nu & \rho[\widehat{R}_1^T(y),...,\widehat{R}_L^T(y)]\\ \hline \rho[\widehat{R}_1(y);...;\widehat{R}_L(y)] & \mathrm{Diag}\{\gamma_1,...,\gamma_L\}\end{array}\right]\succeq 0\right\} \tag{3.3.15}$$

(we have set $\gamma_\nu=\rho\lambda_\nu$). Note that we managed to replace every matrix variable $Y_\nu$ in (3.3.14) with a *single scalar* variable $\lambda_\nu$ in (3.3.15). Note that this dramatic simplification is possible whenever all perturbation blocks are scalar.

With our particular $\widehat{L}_\nu$ and $\widehat{R}_\nu(y)$ the resulting problem (3.3.15) reads

$$\min_{\tau,y,\gamma}\left\{\tau : \left[\begin{array}{c|c|c}\tau-\sum_{\nu=1}^L\gamma_\nu & [WDy-b]^T & \\ \hline WDy-b & \tau I_m & \rho[y_1[WD]_1,...,y_L[WD]_L]\\ \hline & \rho[y_1[WD]_1,...,y_L[WD]_L]^T & \mathrm{Diag}\{\gamma_1,...,\gamma_L\}\end{array}\right]\succeq 0\right\}. \tag{3.3.16}$$

We have solved (3.3.16) at the uncertainty level $\rho=0.01$, thus getting a robust design. The optimal value in (3.3.16) is 0.02132 – while being approximately 2 times worse than the nominal optimal value, it still is pretty small. We then tested the robust design against actuation errors of magnitude $\rho=0.01$ and larger. The results, summarized in figure 3.2 and table 3.2, allow for the same conclusions as in the case of LP-based design, see p. 22. Recall that (3.3.16) is not the "true" RC of our uncertain problem, is just a safe approximation, tight within the factor $\pi/2$, of this RC. All we can conclude from this is that the value $\tau_*=0.02132$ of $\tau$ yielded by the approximation (that is, the guaranteed value of the objective at our robust design, the uncertainty level being 0.01) is in-between the true robust optimal values $\mathrm{Opt}_*(0.01)$ and $\mathrm{Opt}_*(0.01\pi/2)$ at the uncertainty levels 0.01 and $0.01\pi/2$, respectively. This information does *not* allow for meaningful conclusions on how far away is $\tau_*$ from the true robust optimal value $\mathrm{Opt}_*(0.01)$; at this point, all we can say in this respect is that $\mathrm{Opt}_*(0,01)$ is at least the nominal optimal value $\mathrm{Opt}_*(0)=0.0112$, and thus the loss in optimality caused by our approximation is at most by factor $\tau_*/\mathrm{Opt}_*(0)=1.90$. In particular, we cannot exclude that with our approximation, we lose as much as 90% in the value of the objective. The reality, however, is by far not so bad. Note that our perturbation set – the 10-dimensional box – is a



$\rho = 0.01$ $\rho = 0.05$ $\rho = 0.1$

Figure 3.2: "Dream and reality," robust optimal design: samples of 100 of actual diagrams (red) for different uncertainty levels. Blue: the target diagram.

| | Reality | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\rho = 0.01$ | | | $\rho = 0.05$ | | | $\rho = 0.1$ | | |
| | min | mean | max | min | mean | max | min | mean | max |
| $\|\cdot\|_{2,w}$-distance to target | 0.021 | 0.021 | 0.021 | 0.021 | 0.023 | 0.030 | 0.021 | 0.030 | 0.048 |
| energy concentration | 96.5% | 96.7% | 96.9% | 93.0% | 95.8% | 96.8% | 80.6% | 92.9% | 96.7% |

Table 3.2: Quality of robust antenna design. Data over 100 samples of actuation errors per each uncertainty level $\rho$.

For comparison: for nominal design, with the uncertainty level as small as $\rho = 0.001$, the average $\|\cdot\|_{2,w}$-distance of the actual diagram to target is as large as 4.69, and the expected energy concentration is as low as 19.5%.

convex hull of 1024 vertices, so we can think about our uncertainty as of the scenario one (section 3.2.1) generated by 1024 scenarios. This number is still within the grasp of the straightforward scheme proposed in section 3.2.1, and thus we can find in a reasonable time the true robust optimal value $\mathrm{Opt}_*(0.01)$, which turns out to be 0.02128. We see that the actual loss in the value of the objective caused by approximation its really small – it is less than 0.2%.

**Least Squares Inequality with Structured Norm-Bounded Uncertainty, Complex Case**

The uncertain Least Squares inequality (3.3.4) with structured norm-bounded perturbations makes sense in the case of complex left hand side data as well as in the case of real data. Surprisingly, in the complex case the RC admits a better in tightness factor safe tractable approximation than in the real case (specifically, the tightness factor $\frac{\pi}{2} = 1.57...$ stated in Theorem 3.4 in the complex case improves to $\frac{4}{\pi} = 1.27...$). Consider an uncertain Least Squares inequality (3.3.4) where $A(\eta) \in \mathbb{C}^{m\times n}$, $b(\eta) \in \mathbb{C}^m$ and the perturbations are structured norm-bounded and *complex*, meaning that (cf. (3.3.1), (3.3.2))

$$\begin{array}{ll}(a) & \mathcal{Z}_\rho^{\rm left} = \rho\mathcal{Z}_1^{\rm left} = \left\{\eta = (\eta^1,...,\eta^N): \begin{array}{l}\eta^\nu \in \mathbb{C}^{p_\nu\times q_\nu},\ \nu=1,...,N\\ \|\eta^\nu\|_{2,2}\le\rho,\ \nu=1,...,N\\ \eta^\nu=\theta_\nu I_{p_\nu},\theta_\nu\in\mathbb{C},\ \nu\in\mathcal{I}_{\rm S}\end{array}\right\},\\ (b) & A(\zeta)y+b(\zeta)=[A^{\rm n}y+b^{\rm n}]+\sum\limits_{\nu=1}^N L_\nu^H(y)\eta^\nu R_\nu(y),\end{array} \tag{3.3.17}$$

where $L_\nu(y)$, $R_\nu(y)$ are affine in $[\Re(y);\Im(y)]$ matrices with complex entries such that for every $\nu$ at least one of these matrices is independent on $y$ and is nonzero, and $B^H$ denotes the Hermitian conjugate of a complex-valued matrix $B$: $(B^H)_{ij} = \overline{B_{ji}}$, where $\overline{z}$ is the complex conjugate of a complex number $z$.

Observe that by exactly the same reasons as in the real case, we can assume w.l.o.g. that all scalar perturbation blocks are $1\times 1$, or, equivalently, that there are no scalar perturbation blocks at all, so that from now on we assume that $\mathcal{I}_{\rm S} = \emptyset$.

The derivation of the approximation is similar to the one in the real case. Specifically, we start with the evident observation that for a complex $k$-dimensional vector $u$ and a real $t$ the relation

$$\|u\|_2 \le t$$

is equivalent to the fact that the Hermitian matrix

$$\mathrm{Arrow}(u,t) = \left[\begin{array}{c|c} t & u^H \\ \hline u & tI_k\end{array}\right]$$

is $\succeq 0$; this fact is readily given by the complex version of the Schur Complement Lemma: *a Hermitian block matrix* $\left[\begin{array}{c|c} P & Q^H \\ \hline Q & R\end{array}\right]$ *with* $R \succ 0$ *is positive semidefinite if and only if the Hermitian matrix* $P - Q^H R^{-1} Q$ *is positive semidefinite* (cf. the proof of Lemma 3.1). It follows that $(y,\tau)$ is robust feasible for the uncertain Least Squares inequality in question if and only if

$$\begin{array}{c}\underbrace{\mathrm{Arrow}(A^{\rm n}y+b^{\rm n},\tau)}_{B_0(y,\tau)}+\sum\limits_{\nu=1}^N B_\nu(y,\eta^\nu)\succeq 0\,\forall\,(\eta:\|\eta^\nu\|_{2,2}\le\rho\,\forall\nu\le N)\\ \left[B_\nu(y,\eta^\nu)=\widehat{L}_\nu^H\eta^\nu\widehat{R}_\nu(y)+\widehat{R}_\nu^H(y)[\eta^\nu]^H\widehat{L}_\nu,\ \nu=1,...,N\right]\end{array} \tag{3.3.18}$$

where $\widehat{L}_\nu$ are constant matrices, and $\widehat{R}(y)$ are affine in $[\Re(y);\Im(y)]$ matrices readily given by $L_\nu(y)$, $R_\nu(y)$ (cf. (3.3.7) and take into account that we are in the situation $\mathcal{I}_{\rm S}=\emptyset$). It follows that whenever, for a given $(y,\tau)$, one can find Hermitian matrices $Y_\nu$ such that

$$Y_\nu \succeq B_\nu(y,\eta^\nu)\ \forall(\eta^\nu\in\mathbb{C}^{p_\nu\times q_\nu}:\|\eta^\nu\|_{2,2}\leq 1),\ \nu=1,...,N, \tag{3.3.19}$$

and $B_0(y,\tau)\succeq\rho\sum\limits_{\nu=1}^{N}Y_\nu$, the pair $(y,\tau)$ is robust feasible.
Same as in the real case, applying the $\mathcal{S}$-Lemma, (which works in the complex case as well as in the real one), a matrix $Y_\nu$ satisfies (3.3.19) if and only if

$$\exists\lambda_\nu\geq 0:\left[\begin{array}{c|c} Y_\nu-\lambda_\nu\widehat{L}_\nu^H\widehat{L}_\nu & -\widehat{R}_\nu^H(y)\\ \hline -\widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu}\end{array}\right],$$

where $k_\nu$ is the number of rows in $\widehat{R}_\nu(y)$. We have arrived at the first part of the following statement:

**Theorem 3.5** *The explicit system of LMIs*

$$\begin{array}{c}\left[\begin{array}{c|c} Y_\nu-\lambda_\nu\widehat{L}_\nu^H\widehat{L}_\nu & \widehat{R}_\nu^H(y)\\ \hline \widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu}\end{array}\right]\succeq 0,\ \nu=1,...,N,\\ \mathrm{Arrow}(A^{\rm n}y+b^{\rm n},\tau)-\rho\sum\limits_{\nu=1}^{N}Y_\nu\succeq 0\end{array} \tag{3.3.20}$$

*(for notation, see (3.3.18)) in the variables $\{Y_i=Y_i^H\},\lambda_\nu,y,\tau$ is a safe tractable approximation of the RC of the uncertain Least Squares inequality* (3.3.4), (3.3.17)*. The tightness factor of this approximation never exceeds $4/\pi$, and is equal to 1 when $N=1$.*

**Proof** is completely similar to the one of Theorem 3.4, modulo replacing the real case of the Matrix Cube Theorem (Theorem A.7) with its complex case (Theorem A.6).

#### From Uncertain Least Squares to Uncertain CQI

Let us come back to the real case. We have already built a tight approximation for the RC of a Least Squares inequality with structured norm-bounded uncertainty in the left hand side data. Our next goal is to extend this approximation to the case of uncertain CQI with side-wise uncertainty.

**Theorem 3.6** *Consider the uncertain CQI* (3.2.3) *with side-wise uncertainty, where the left hand side uncertainty is the structured norm-bounded one given by* (3.3.1), (3.3.2)*, and the right hand side perturbation set is given by a conic representation (cf. Theorem 1.1)*

$$\mathcal{Z}_\rho^{\rm right}=\rho\mathcal{Z}_1^{\rm right},\ \mathcal{Z}_1^{\rm right}=\{\chi:\exists u:P\chi+Qu+p\in\mathbf{K}\}, \tag{3.3.21}$$

*where $0\in\mathcal{Z}_1^{\rm right}$, $\mathbf{K}$ is a closed convex pointed cone and the representation is strictly feasible unless $\mathbf{K}$ is a polyhedral cone given by an explicit finite list of linear inequalities, and $0\in\mathcal{Z}_1^{\rm right}$.*

*For $\rho > 0$, the explicit system of LMIs*

$$\begin{array}{ll}(a) & \tau + \rho p^T v \le \delta(y),\ P^T v = \sigma(y),\ Q^T v = 0,\ v \in \mathbf{K}_* \\ (b.1) & Y_\nu \succeq \pm(\widehat{L}_\nu^T \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y)\widehat{L}_\nu),\ \nu \in \mathcal{I}_{\rm S} \\ (b.2) & \left[\begin{array}{c|c} Y_\nu - \lambda_\nu \widehat{L}_\nu^T \widehat{L}_\nu & \widehat{R}_\nu^T(y) \\ \hline \widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu}\end{array}\right] \succeq 0,\ \nu \notin \mathcal{I}_{\rm S} \\ (b.3) & \mathrm{Arrow}(A^{\rm n} y + b^{\rm n}, \tau) - \rho \sum\limits_{\nu=1}^N Y_\nu \succeq 0 \end{array} \tag{3.3.22}$$

*(for notation, see (3.3.7)) in variables $Y_1, ..., Y_N, \lambda_\nu, y, \tau, v$ is a safe tractable approximation of the RC of* (3.2.4)*. This approximation is exact when $N = 1$, and is tight within the factor $\frac{\pi}{2}$ otherwise.*

**Proof.** Since the uncertainty is side-wise, $y$ is robust feasible for (3.2.4), (3.3.1), (3.3.2), (3.3.21), the uncertainty level being $\rho > 0$, if and only if there exists $\tau$ such that

$$\begin{array}{ll}(c) & \sigma^T(\chi)y + \delta(\chi) \ge \tau\ \forall \chi \in \rho\mathcal{Z}_1^{\rm right}, \\ (d) & \|A(\eta)y + b(\eta)\|_2 \le \tau\ \forall \eta \in \rho\mathcal{Z}_1^{\rm left}.\end{array}$$

When $\rho > 0$, we have

$$\rho\mathcal{Z}_1^{\rm right} = \{\chi : \exists u : P(\chi/\rho) + Qu + p \in \mathbf{K}\} = \{\chi : \exists u' : P\chi + Qu' + \rho p \in \mathbf{K}\};$$

from the resulting conic representation of $\rho\mathcal{Z}_1^{\rm right}$, same as in the proof of Theorem 1.1, we conclude that the relations (3.3.22.$a$) represent equivalently the requirement ($c$), that is, $(y, \tau)$ satisfies ($c$) if and only if $(y, \tau)$ can be extended, by properly chosen $v$, to a solution of (3.3.22.$a$). By Theorem 3.4, the possibility to extend $(y, \tau)$ to a feasible solution of (3.3.22.$b$) is a sufficient condition for the validity of ($d$). Thus, the $(y, \tau)$ component of a feasible solution to (3.3.22) satisfies ($c$), ($d$), meaning that $y$ is robust feasible at the level of uncertainty $\rho$. Thus, (3.3.22) is a safe approximation of the RC in question.

The fact that the approximation is precise when there is only one left hand side perturbation block is readily given by Theorem 3.2 and Remark 3.1 allowing us to treat this block as full. It remains to verify that the tightness factor of the approximation is at most $\frac{\pi}{2}$, that is, to check that if a given $y$ cannot be extended to a feasible solution of the approximation for the uncertainty level $\rho$, then $y$ is not robust feasible for the uncertainty level $\frac{\pi}{2}\rho$ (see comments after Definition 3.3). To this end, let us set

$$\tau_y(r) = \inf_\chi \left\{\sigma^T(\chi)y + \delta(\chi) : \chi \in r\mathcal{Z}_1^{\rm right}\right\}.$$

Since $0 \in \mathcal{Z}_1^{\rm right}$ by assumption, $\tau_y(r)$ is nonincreasing in $r$. Clearly, $y$ is robust feasible at the uncertainty level $r$ if and only if

$$\|A(\eta)y + b(\eta)\|_2 \le \tau_y(r)\ \forall \eta \in r\mathcal{Z}_1^{\rm left}. \tag{3.3.23}$$

Now assume that a given $y$ cannot be extended to a feasible solution of (3.3.22) for the uncertainty level $\rho$. Let us set $\tau = \tau_y(\rho)$; then $(y, \tau)$ can be extended, by a properly chosen $v$, to a feasible solution of (3.3.22.$a$). Indeed, the latter system expresses equivalently the fact that

$(y,\tau)$ satisfies $(c)$, which indeed is the case for our $(y,\tau)$. Now, since $y$ cannot be extended to a feasible solution to (3.3.22) at the uncertainty level $\rho$, and the pair $(y,\tau)$ can be extended to a feasible solution of (3.3.22.$a$), we conclude that $(y,\tau)$ cannot be extended to a feasible solution of (3.3.22.$b$). By Theorem 3.4, the latter implies that $y$ is *not* robust feasible for the semi-infinite Least Squares constraint

$$\|A(\eta)y + b(\eta)\|_2 \leq \tau = \tau_y(\rho)\ \forall \eta \in \frac{\pi}{2}\rho\mathcal{Z}_1^{\rm left}.$$

Since $\tau_y(r)$ is nonincreasing in $r$, we conclude that $y$ does *not* satisfy (3.3.23) when $r = \frac{\pi}{2}\rho$, meaning that $y$ is not robust feasible at the level of uncertainty $\frac{\pi}{2}\rho$. □

### Convex Quadratic Constraint with Structured Norm-Bounded Uncertainty

Consider an uncertain convex quadratic constraint

$$\begin{array}{ll}(a) & y^TA^T(\zeta)A(\zeta)y \leq 2y^Tb(\zeta) + c(\zeta)\\ & \Updownarrow \\ (b) & \|[2A(\zeta)y; 1 - 2y^Tb(\zeta) - c(\zeta)]\|_2 \leq 1 + 2y^Tb(\zeta) + c(\zeta),\end{array} \tag{3.2.13}$$

where $A(\zeta)$ is $k \times n$ and the uncertainty is structured norm-bounded (cf. (3.2.14)), meaning that

$$\begin{array}{ll}(a) & \mathcal{Z}_\rho = \rho\mathcal{Z}_1 = \left\{\zeta = (\zeta^1,...,\zeta^N): \begin{array}{c}\zeta^\nu \in \mathbb{R}^{p_\nu \times q_\nu}\\ \|\zeta^\nu\|_{2,2} \leq \rho, 1 \leq \nu \leq N\\ \zeta^\nu = \theta_\nu I_{p_\nu}, \theta_\nu \in \mathbb{R}, \nu \in \mathcal{I}_{\rm S}\end{array}\right\},\\ (b) & \left[\begin{array}{c}A(\zeta)y\\ y^Tb(\zeta)\\ c(\zeta)\end{array}\right] = \left[\begin{array}{c}A^{\rm n}y\\ y^Tb^{\rm n}\\ c^{\rm n}\end{array}\right] + \sum\limits_{\nu=1}^N L_\nu^T(y)\zeta^\nu R_\nu(y)\end{array} \tag{3.3.24}$$

where, for every $\nu$, $L_\nu(y)$, $R_\nu(y)$ are matrices of appropriate sizes depending affinely on $y$ and such that at least one of the matrices is constant. Same as above, we can assume w.l.o.g. that all scalar perturbation blocks are $1 \times 1$: $p_\nu = k_\nu = 1$ for all $\nu \in \mathcal{I}_{\rm S}$.

Note that the equivalence in (3.2.13) means that we still are interested in an uncertain CQI with structured norm-bounded left hand side uncertainty. The uncertainty, however, is *not* side-wise, that is, we are in the situation we could not handle before. We can handle it now due to the fact that the uncertain CQI possesses a favorable structure inherited from the original convex quadratic form of the constraint.

We are about to derive a tight tractable approximation of the RC of (3.2.13), (3.3.24). The construction is similar to the one we used in the unstructured case $N = 1$, see section 3.2.4. Specifically, let us set $L_\nu(y) = [L_{\nu,A}(y), L_{\nu,b}(y), L_{\nu,c}(y)]$, where $L_{\nu,b}(y)$, $L_{\nu,c}(y)$ are the last two columns in $L_\nu(y)$, and let

$$\begin{array}{c}\widetilde{L}_\nu^T(y) = \left[L_{\nu,b}^T(y) + \frac{1}{2}L_{\nu,c}^T(y); L_{\nu,A}^T(y)\right],\ \widetilde{R}_\nu(y) = [R_\nu(y), 0_{q_\nu \times k}],\\ \mathcal{A}(y) = \left[\begin{array}{c|c}2y^Tb^{\rm n} + c^{\rm n} & [A^{\rm n}y]^T\\ \hline A^{\rm n}y & I\end{array}\right],\end{array} \tag{3.3.25}$$

so that $\mathcal{A}(y)$, $\widetilde{L}_\nu(y)$ and $\widetilde{R}_\nu(y)$ are affine in $y$ and at least one of the latter two matrices is constant.

We have

$$
\begin{aligned}
& y^T A^T(\zeta)A(\zeta)y \le 2y^T b(\zeta) + c(\zeta)\ \forall \zeta \in \mathcal{Z}_\rho \\
\Leftrightarrow\ & \left[\begin{array}{c|c} 2y^T b(\zeta) + c(\zeta) & [A(\zeta)y]^T \\ \hline A(\zeta)y & I \end{array}\right] \succeq 0\ \forall \zeta \in \mathcal{Z}_\rho \text{ [Schur Complement Lemma]} \\
\Leftrightarrow\ & \overbrace{\left[\begin{array}{c|c} 2y^T b^{\rm n} + c^{\rm n} & [A^{\rm n} y]^T \\ \hline A^{\rm n} y & I \end{array}\right]}^{\mathcal{A}(y)} \\
& + \sum_{\nu=1}^N \underbrace{\left[\begin{array}{c|c} [2L_{\nu,b}(y) + L_{\nu,c}(y)]^T \zeta^\nu R_\nu(y) & [L_{\nu,A}^T(y)\zeta^\nu R_\nu(y)]^T \\ \hline L_{\nu,A}^T(y)\zeta^\nu R_\nu(y) & \end{array}\right]}_{=\widetilde{L}_\nu^T(y)\zeta^\nu \widetilde{R}_\nu(y) + \widetilde{R}_\nu^T(y)[\zeta^\nu]^T \widetilde{L}_\nu(y)} \succeq 0\ \forall \zeta \in \mathcal{Z}_\rho \\
& \text{[by (3.3.24)]} \\
\Leftrightarrow\ & \mathcal{A}(y) + \sum_{\nu=1}^N \left[\widetilde{L}_\nu^T(y)\zeta^\nu \widetilde{R}_\nu(y) + \widetilde{R}_\nu^T(y)[\zeta^\nu]^T \widetilde{L}_\nu(y)\right] \succeq 0\ \forall \zeta \in \mathcal{Z}_\rho.
\end{aligned}
$$

Taking into account that for every $\nu$ at least one of the matrices $\widetilde{L}_\nu(y)$, $\widetilde{R}_\nu(y)$ is independent of $y$ and swapping, if necessary, $\zeta^\nu$ and $[\zeta^\nu]^T$, we can rewrite the last condition in the chain as

$$
\mathcal{A}(y) + \sum_{\nu=1}^N \left[\widehat{L}_\nu^T \zeta^\nu \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y)[\zeta^\nu]^T \widehat{L}_\nu\right] \succeq 0\ \forall (\zeta : \|\zeta^\nu\|_{2,2} \le \rho) \tag{3.3.26}
$$

where $\widehat{L}_\nu$, $\widehat{R}_\nu(y)$ are readily given matrices and $\widehat{R}_\nu(y)$ is affine in $y$. (Recall that we are in the situation where all scalar perturbation blocks are $1 \times 1$ ones, and we can therefore skip the explicit indication that $\zeta^\nu = \theta_\nu I_{p_\nu}$ for $\nu \in \mathcal{I}_{\rm S}$). Observe also that similarly to the case of a Least Squares inequality, all matrices $\left[\widehat{L}_\nu^T \zeta^\nu \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y)[\zeta^\nu]^T \widehat{L}_\nu\right]$ are of rank at most 2. Finally, we lose nothing by assuming that $\widehat{L}_\nu$ are nonzero for all $\nu$.

Proceeding exactly in the same fashion as in the case of the uncertain Least Squares inequality with structured norm-bounded perturbations, we arrive at the following result (cf. Theorem 3.4):

**Theorem 3.7** *The explicit system of LMIs*

$$
\begin{array}{c}
Y_\nu \succeq \pm(\widehat{L}_\nu^T \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y)\widehat{L}_\nu),\ \nu \in \mathcal{I}_{\rm S} \\
\left[\begin{array}{c|c} Y_\nu - \lambda_\nu \widehat{L}_\nu^T \widehat{L}_\nu & \widehat{R}_\nu^T(y) \\ \hline \widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu} \end{array}\right] \succeq 0,\ \nu \notin \mathcal{I}_{\rm S} \\
\mathcal{A}(y) - \rho \sum_{\nu=1}^L Y_\nu \succeq 0
\end{array} \tag{3.3.27}
$$

*($k_\nu$ is the number of rows in $\widehat{R}_\nu$) in variables $Y_1, ..., Y_N, \lambda_\nu, y$ is a safe tractable approximation of the RC of the uncertain convex quadratic constraint* (3.2.13), (3.3.24). *The tightness factor of this approximation never exceeds $\pi/2$, and equals 1 when $N = 1$.*

**Complex case.** The situation considered in section 3.3.1 admits a complex data version as well. Consider a convex quadratic constraint with complex-valued variables and a complex-valued structured norm-bounded uncertainty:

$$\begin{array}{l} y^H A^H(\zeta)A(\zeta)y \leq \Re\{2y^H b(\zeta)+c(\zeta)\} \\ \zeta\in\mathcal{Z}_\rho=\rho\mathcal{Z}_1=\left\{\zeta=(\zeta^1,...,\zeta^N): \begin{array}{l} \zeta^\nu\in\mathbb{C}^{p_\nu\times q_\nu},\ 1\leq\nu\leq N \\ \|\zeta^\nu\|_{2,2}\leq\rho,\ 1\leq\nu\leq N \\ \nu\in\mathcal{I}_{\rm S}\Rightarrow\zeta^\nu=\theta_\nu I_{p_\nu},\theta_\nu\in\mathbb{C}\end{array}\right\} \\ \left[\begin{array}{c} A(\zeta)y \\ y^H b(\zeta) \\ c(\zeta)\end{array}\right]=\left[\begin{array}{c} A^{\rm n}y \\ y^H b^{\rm n} \\ c^{\rm n}\end{array}\right]+\sum\limits_{\nu=1}^N L_\nu^H(y)\zeta^\nu R_\nu(y), \end{array} \tag{3.3.28}$$

where $A^{\rm n}\in\mathbb{C}^{k\times m}$ and the matrices $L_\nu(y)$, $R_\nu(y)$ are affine in $[\Re(y);\Im(y)]$ and such that for every $\nu$, either $L_\nu(y)$, or $R_\nu(y)$ are independent of $y$. Same as in the real case we have just considered, we lose nothing when assuming that all scalar perturbation blocks are $1\times 1$, which allows us to treat these blocks as full. Thus, the general case can be reduced to the case where $\mathcal{I}_{\rm S}=\emptyset$, which we assume from now on (cf. section 3.3.1).

In order to derive a safe approximation of the RC of (3.3.28), we can act exactly in the same fashion as in the real case to arrive at the equivalence

$$\begin{array}{l} y^H A^H(\zeta)A(\zeta)y\leq\Re\{2y^H b(\zeta)+c(\zeta)\}\ \forall\zeta\in\mathcal{Z}_\rho \\ \Leftrightarrow\ \overbrace{\left[\begin{array}{c|c}\Re\{2y^H b^{\rm n}+c^{\rm n}\} & [A^{\rm n}y]^H \\ \hline A^{\rm n}y & I\end{array}\right]}^{\mathcal{A}(y)} \\ \quad+\sum\limits_{\nu=1}^N\left[\begin{array}{c|c}\Re\{2y^H L_{\nu,b}(y)\zeta^\nu R_\nu(y)+L_{\nu,c}(y)\zeta^\nu R_\nu(y)\} & R_\nu^H[\zeta^\nu]^H L_{\nu,A}(y) \\ \hline L_{\nu,A}^H(y)\zeta^\nu R_\nu(y) & \end{array}\right]\succeq 0 \\ \qquad\qquad\forall(\zeta:\|\zeta^\nu\|_{2,2}\leq\rho,1\leq\nu\leq N)\end{array}$$

where $L_\nu(y)=[L_{\nu,A}(y),L_{\nu,b}(y),L_{\nu,c}(y)]$ and $L_{\nu,b}(y)$, $L_{\nu,c}(y)$ are the last two columns in $L_\nu(y)$.

Setting

$$\widetilde{L}_\nu^H(y)=\left[L_{\nu,b}^H(y)+\frac{1}{2}L_{\nu,c}^H(y);L_{\nu,A}^H(y)\right],\ \widetilde{R}_\nu(y)=[R_\nu(y),0_{q_\nu\times k}]$$

(cf. (3.3.25)), we conclude that the RC of (3.3.28) is equivalent to the semi-infinite LMI

$$\begin{array}{l}\mathcal{A}(y)+\sum\limits_{\nu=1}^N\left[\widetilde{L}_\nu^H(y)\zeta^\nu\widetilde{R}_\nu(y)+\widetilde{R}_\nu^H(y)[\zeta^\nu]^H\widetilde{L}_\nu(y)\right]\succeq 0 \\ \qquad\forall(\zeta:\|\zeta^\nu\|_{2,2}\leq\rho,1\leq\nu\leq N).\end{array} \tag{3.3.29}$$

As always, swapping, if necessary, $\zeta^\nu$ and $[\zeta^\nu]^H$ we may rewrite the latter semi-infinite LMI equivalently as

$$\begin{array}{l}\mathcal{A}(y)+\sum\limits_{\nu=1}^N\left[\widehat{L}_\nu^H\zeta^\nu\widehat{R}_\nu(y)+\widehat{R}_\nu^H(y)[\zeta^\nu]^H\widehat{L}_\nu\right]\succeq 0 \\ \qquad\forall(\zeta:\|\zeta^\nu\|_{2,2}\leq\rho,1\leq\nu\leq N),\end{array}$$

where $\widehat{R}_\nu(y)$ are affine in $[\Re(y);\Im(y)]$ and $\widehat{L}_\nu$ are nonzero. Applying the Complex case Matrix Cube Theorem (see the proof of Theorem 3.5), we finally arrive at the following result:

**Theorem 3.8** *The explicit system of LMIs*

$$\begin{array}{l}\left[\begin{array}{c|c} Y_\nu-\lambda_\nu\widehat{L}_\nu^H\widehat{L}_\nu & \widehat{R}_\nu^H(y) \\ \hline \widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu}\end{array}\right]\succeq 0,\ \nu=1,...,N, \\ \left[\begin{array}{c|c}\Re\{2y^H b^{\rm n}+c^{\rm n}\} & [A^{\rm n}y]^H \\ \hline A^{\rm n}y & I\end{array}\right]-\rho\sum\limits_{\nu=1}^N Y_\nu\succeq 0\end{array} \tag{3.3.30}$$

*($k_\nu$ is the number of rows in $\widehat{R}_\nu(y)$) in variables $Y_1 = Y_1^H, ..., Y_N = Y_N^H, \lambda_\nu \in \mathbb{R}, y \in \mathbb{C}^m$ is a safe tractable approximation of the RC of the uncertain convex quadratic inequality* (3.3.28)*. The tightness of this approximation is $\leq \frac{4}{\pi}$, and is equal to 1 when $N = 1$.*

### 3.3.2 The Case of ∩-Ellipsoidal Uncertainty

Consider the case where the uncertainty in CQI (3.2.3) is side-wise with the right hand side uncertainty exactly as in section 3.2.2, and with ∩-*ellipsoidal left hand side perturbation set*, that is,

$$\mathcal{Z}_\rho^{\text{left}} = \left\{\eta : \eta^T Q_j \eta \leq \rho^2,\, j = 1, ..., J\right\}, \tag{3.3.31}$$

where $Q_j \succeq 0$ and $\sum\limits_{j=1}^{J} Q_j \succ 0$. When $Q_j \succ 0$ for all $j$, $\mathcal{Z}_\rho^{\text{left}}$ is the intersection of $J$ ellipsoids centered at the origin. When $Q_j = a_j a_j^T$ are rank 1 matrices, $\mathcal{Z}^{\text{left}}$ is a polyhedral set symmetric w.r.t. origin and given by $J$ inequalities of the form $|a_j^T \eta| \leq \rho$, $j = 1, ..., J$. The requirement $\sum\limits_{j=1}^{J} Q_j \succ 0$ implies that $\mathcal{Z}_\rho^{\text{left}}$ is bounded (indeed, every $\eta \in \mathcal{Z}_\rho^{\text{left}}$ belongs to the ellipsoid $\eta^T(\sum_j Q_j)\eta \leq J\rho^2$).

We have seen in section 3.2.3 that the case $J = 1$, (i.e., of an ellipsoid $\mathcal{Z}_\rho^{\text{left}}$ centered at the origin), is a particular case of unstructured norm-bounded perturbation, so that in this case the RC is computationally tractable. The case of general ∩-ellipsoidal uncertainty includes the situation when $\mathcal{Z}_\rho^{\text{left}}$ is a box, where the RC is computationally intractable. However, we intend to demonstrate that with ∩-ellipsoidal left hand side perturbation set, the RC of (3.2.4) admits a safe tractable approximation tight within the “nearly constant” factor $\sqrt{(O(\ln J))}$.

#### Approximating the RC of Uncertain Least Squares Inequality

Same as in section 3.3.1, the side-wise nature of uncertainty reduces the task of approximating the RC of uncertain CQI (3.2.4) to a similar task for the RC of the uncertain Least Squares inequality (3.3.4). Representing

$$A(\zeta)y + b(\zeta) = \underbrace{[A^{\rm n} y + b^{\rm n}]}_{\beta(y)} + \underbrace{\sum_{\ell=1}^{L} \eta_\ell [A^\ell y + b^\ell]}_{\alpha(y)\eta} \tag{3.3.32}$$

where $L = \dim \eta$, observe that the RC of (3.3.4), (3.3.31) is equivalent to the system of constraints

$$\tau \geq 0 \;\&\; \|\beta(y) + \alpha(y)\eta\|_2^2 \leq \tau^2 \;\forall(\eta : \eta^T Q_j \eta \leq \rho^2,\, j = 1, ..., J)$$

or, which is clearly the same, to the system

$$\begin{array}{ll} (a) & \mathcal{A}_\rho \equiv \max\limits_{\eta,t} \left\{\eta^T \alpha^T(y)\alpha(y)\eta + 2t\beta^T(y)\alpha(y)\eta : \eta^T Q_j \eta \leq \rho^2 \,\forall j,\, t^2 \leq 1\right\} \\ & \leq \tau^2 - \beta^T(y)\beta(y) \\ (b) & \tau \geq 0. \end{array} \tag{3.3.33}$$

Next we use Lagrangian relaxation to derive the following result:

(!) Assume that for certain nonnegative reals $\gamma$, $\gamma_j$, $j = 1, ..., J$, the homogeneous quadratic form in variables $\eta, t$

$$\gamma t^2 + \sum_{j=1}^{J} \gamma_j \eta^T Q_j \eta - \left[\eta^T \alpha^T(y)\alpha(y)\eta + 2t\beta^T(y)\alpha(y)\eta\right] \tag{3.3.34}$$

is nonnegative everywhere. Then

$$\begin{array}{l}\mathcal{A}_\rho \equiv \max\limits_{\eta,t} \left\{\eta^T \alpha^T(y)\alpha(y)\eta + 2t\beta^T(y)\alpha(y)\eta : \eta^T Q_j \eta \le \rho^2,\ t^2 \le 1\right\} \\ \le \gamma + \rho^2 \sum\limits_{j=1}^{J} \gamma_j.\end{array} \tag{3.3.35}$$

Indeed, let $F = \{(\eta, t) : \eta^T Q_j \eta \le \rho^2, j = 1, ..., J, t^2 \le 1\}$. We have

$$\begin{array}{rcl}\mathcal{A}_\rho &=& \max\limits_{(\eta,t)\in F} \left\{\eta^T \alpha^T(y)\alpha(y)\eta + 2t\beta^T(y)\alpha(y)\eta\right\} \\ &\le& \max\limits_{(\eta,t)\in F} \left\{\gamma t^2 + \sum\limits_{j=1}^{J} \gamma_j \eta^T Q_j \eta\right\} \\ && \text{[since the quadratic form (3.3.34) is nonnegative everywhere]} \\ &\le& \gamma + \rho^2 \sum\limits_{j=1}^{J} \gamma_j \\ && \text{[due to the origin of } F \text{ and to } \gamma \ge 0,\ \gamma_j \ge 0].\end{array}$$

From (!) it follows that if $\gamma \ge 0$, $\gamma_j \ge 0$, $j = 1, ..., J$ are such that the quadratic form (3.3.34) is nonnegative everywhere, or, which is the same, such that

$$\left[\begin{array}{c|c} \gamma & -\beta^T(y)\alpha(y) \\ \hline -\alpha^T(y)\beta(y) & \sum\limits_{j=1}^{J} \gamma_j Q_j - \alpha^T(y)\alpha(y) \end{array}\right] \succeq 0$$

and

$$\gamma + \rho^2 \sum_{j=1}^{J} \gamma_j \le \tau^2 - \beta^T(y)\beta(y),$$

then $(y, \tau)$ satisfies (3.3.33.$a$). Setting $\nu = \gamma + \beta^T(y)\beta(y)$, we can rewrite this conclusion as follows: if there exist $\nu$ and $\gamma_j \ge 0$ such that

$$\left[\begin{array}{c|c} \nu - \beta^T(y)\beta(y) & -\beta^T(y)\alpha(y) \\ \hline -\alpha^T(y)\beta(y) & \sum\limits_{j=1}^{J} \gamma_j Q_j - \alpha^T(y)\alpha(y) \end{array}\right] \succeq 0$$

and

$$\nu + \rho^2 \sum_{j=1}^{J} \gamma_j \le \tau^2,$$

then $(y, \tau)$ satisfies (3.3.33.$a$).

Assume for a moment that $\tau > 0$. Setting $\lambda_j = \gamma_j/\tau$, $\mu = \nu/\tau$, the above conclusion can be rewritten as follows: if there exist $\mu$ and $\lambda_j \geq 0$ such that

$$\left[\begin{array}{c|c} \mu - \tau^{-1}\beta^T(y)\beta(y) & -\tau^{-1}\beta^T(y)\alpha(y) \\ \hline -\tau^{-1}\alpha^T(y)\beta(y) & \sum_{j=1}^{J}\lambda_j Q_j - \tau^{-1}\alpha^T(y)\alpha(y) \end{array}\right] \succeq 0$$

and

$$\mu + \rho^2 \sum_{j=1}^{J} \lambda_j \leq \tau,$$

then $(y, \tau)$ satisfies (3.3.33.$a$).

By the Schur Complement Lemma, the latter conclusion can further be reformulated as follows: if $\tau > 0$ and there exist $\mu, \lambda_j$ satisfying the relations

$$\begin{array}{ll} (a) & \left[\begin{array}{c|c|c} \mu & & \beta^T(y) \\ \hline & \sum_{j=1}^{J}\lambda_j Q_j & \alpha^T(y) \\ \hline \beta(y) & \alpha(y) & \tau I \end{array}\right] \succeq 0 \\ (b) & \mu + \rho^2 \sum_{j=1}^{J} \lambda_j \leq \tau \quad (c)\ \lambda_j \geq 0,\ j = 1, ..., J \end{array} \tag{3.3.36}$$

then $(y, \tau)$ satisfies (3.3.33.$a$). Note that in fact our conclusion is valid for $\tau \leq 0$ as well. Indeed, assume that $\tau \leq 0$ and $\mu, \lambda_j$ solve (3.3.36). Then clearly $\tau = 0$ and therefore $\alpha(y) = 0$, $\beta(y) = 0$, and thus (3.3.33.$a$) is valid. We have proved the first part of the following statement:

**Theorem 3.9** *The explicit system of constraints* (3.3.36) *in variables* $y$, $\tau$, $\mu$, $\lambda_1, ..., \lambda_J$ *is a safe tractable approximation of the RC of the uncertain Least Squares constraint* (3.3.4) *with* $\cap$*-ellipsoidal perturbation set* (3.3.31). *The approximation is exact when* $J = 1$, *and in the case of* $J > 1$ *the tightness factor of this approximation does not exceed*

$$\Omega(J) \leq 9.19\sqrt{\ln(J)}. \tag{3.3.37}$$

**Proof.** The fact that (3.3.36) is a safe approximation of the RC of (3.3.4), (3.3.31) is readily given by the reasoning preceding Theorem 3.9. To prove that the approximation is tight within the announced factor, note that the Approximate $\mathcal{S}$-Lemma (Theorem A.8) as applied to the quadratic forms in variables $x = [\eta; t]$

$$\begin{aligned} & x^T A x \equiv \left\{\eta^T \alpha^T(y)\alpha(y)\eta + 2t\beta^T(y)\alpha(y)\eta\right\},\ x^T B x \equiv t^2, \\ & x^T B_j x \equiv \eta^T Q_j \eta,\ 1 \leq j \leq J, \end{aligned}$$

states that if $J = 1$, then $(y, \tau)$ can be extended to a solution of (3.3.36) if and only if $(y, \tau)$ satisfies (3.3.33), that is, if and only if $(y, \tau)$ is robust feasible; thus, our approximation of the RC of (3.3.4), (3.3.31) is exact when $J = 1$. Now let $J > 1$, and suppose that $(y, \tau)$ cannot be extended to a feasible solution of (3.3.36). Due to the origin of this system, it follows that

$$\begin{aligned} & \mathrm{SDP}(\rho) \equiv \min_{\lambda, \{\lambda_j\}} \left\{\lambda + \rho^2 \sum_{j=1}^{J} \lambda_j : \lambda B + \sum_j \lambda_j B_j \succeq A, \lambda \geq 0, \lambda_j \geq 0\right\} \\ & > \tau^2 - \beta^T(y)\beta(y). \end{aligned} \tag{3.3.38}$$

By the Approximate $\mathcal{S}$-Lemma, with appropriately chosen $\Omega(J) \leq 9.19\sqrt{\ln(J)}$ we have $\mathcal{A}_{\Omega(J)\rho} \geq \mathrm{SDP}(\rho)$, which combines with (3.3.38) to imply that $\mathcal{A}_{\Omega(J)\rho} > \tau^2 - \beta^T(y)\beta(y)$, meaning that $(y,\tau)$ is not robust feasible at the uncertainty level $\Omega(J)\rho$ (cf. (3.3.33)). Thus, the tightness factor of our approximation does not exceed $\Omega(J)$. □

### From Uncertain Least Squares to Uncertain CQI

The next statement can obtained from Theorem 3.9 in the same fashion as Theorem 3.6 has been derived from Theorem 3.4.

**Theorem 3.10** *Consider uncertain CQI* (3.2.3) *with side-wise uncertainty, where the left hand side perturbation set is the* $\cap$*-ellipsoidal set* (3.3.31)*, and the right hand side perturbation set is as in Theorem 3.6. For* $\rho > 0$*, the explicit system of LMIs*

$$\begin{array}{ll}(a) & \tau + \rho p^T v \leq \delta(y),\ P^T v = \sigma(y),\ Q^T v = 0,\ v \in \mathbf{K}_* \\[2mm] (b.1) & \left[\begin{array}{c|c|c} \mu & & \beta^T(y) \\ \hline & \sum\limits_{j=1}^J \lambda_j Q_j & \alpha^T(y) \\ \hline \beta(y) & \alpha(y) & I \end{array}\right] \succeq 0 \\[2mm] (b.2) & \mu + \rho^2 \sum\limits_{j=1}^J \lambda_j \leq \tau,\ \lambda_j \succeq 0\, \forall j \end{array} \qquad (3.3.39)$$

*in variables* $y, v, \mu, \lambda_j, \tau$ *is a safe tractable approximation of the RC of the uncertain CQI. This approximation is exact when* $J = 1$ *and is tight within the factor* $\Omega(J) \leq 9.19\sqrt{\ln(J)}$ *when* $J > 1$.

### Convex Quadratic Constraint with ∩-Ellipsoidal Uncertainty

Now consider approximating the RC of an uncertain convex quadratic inequality

$$\begin{array}{c} y^T A^T(\zeta) A(\zeta) y \leq 2y^T b(\zeta) + c(\zeta) \\ \left[(A(\zeta), b(\zeta), c(\zeta)) = (A^{\mathrm{n}}, b^{\mathrm{n}}, c^{\mathrm{n}}) + \sum\limits_{\ell=1}^L \zeta_\ell (A^\ell, b^\ell, c^\ell)\right] \end{array} \qquad (3.3.40)$$

with ∩-ellipsoidal uncertainty:

$$\mathcal{Z}_\rho = \rho \mathcal{Z}_1 = \{\zeta \in \mathbb{R}^L : \zeta^T Q_j \zeta \leq \rho^2\} \quad [Q_j \succeq 0, \sum_j Q_j \succ 0] \qquad (3.3.41)$$

Observe that

$$\begin{array}{rcl} A(\zeta)y & = & \alpha(y)\zeta + \beta(y), \\ & & \alpha(y)\zeta = [A^1 y, ..., A^L y],\ \beta(y) = A^{\mathrm{n}} y \\ 2y^T b(\zeta) + c(\zeta) & = & 2\sigma^T(y)\zeta + \delta(y), \\ & & \sigma(y) = [y^T b^1 + c^1; ...; y^T b^L + c^L],\ \delta(y) = y^T b^{\mathrm{n}} + c^{\mathrm{n}} \end{array} \qquad (3.3.42)$$

so that the RC of (3.3.40), (3.3.41) is the semi-infinite inequality

$$\zeta^T \alpha^T(y)\alpha(y)\zeta + 2\zeta^T \left[\alpha^T(y)\beta(y) - \sigma(y)\right] \leq \delta(y) - \beta^T(y)\beta(y)\ \forall \zeta \in \mathcal{Z}_\rho,$$

or, which is the same, the semi-infinite inequality

$$\begin{array}{l}\mathcal{A}_\rho(y) \equiv \max\limits_{\zeta\in\mathcal{Z}_\rho, t^2\le 1} \zeta^T\alpha^T(y)\alpha(y)\zeta + 2t\zeta^T\left[\alpha^T(y)\beta(y)-\sigma(y)\right]\\ \le \delta(y) - \beta^T(y)\beta(y).\end{array} \tag{3.3.43}$$

Same as in section 3.3.2, we have

$$\begin{array}{rcl}\mathcal{A}_\rho(y) & \le & \inf\limits_{\lambda,\{\lambda_j\}}\left\{\lambda+\rho^2\sum\limits_{j=1}^J\lambda_j : \begin{array}{r}\lambda\ge 0, \lambda_j \ge 0, j=1,...,J\\ \forall(t,\zeta):\\ \lambda t^2 + \zeta^T(\sum\limits_{j=1}^J \lambda_j Q_j)\zeta \ge \zeta^T\alpha^T(y)\alpha(y)\zeta\\ +2t\zeta^T\left[\alpha^T(y)\beta(y)-\sigma(y)\right]\end{array}\right\}\\ & = & \inf\limits_{\lambda,\{\lambda_j\}}\left\{\lambda+\rho^2\sum\limits_{j=1}^J\lambda_j : \lambda\ge 0, \lambda_j\ge 0, j=1,...,J,\right.\\ & & \left.\left[\begin{array}{c|c}\lambda & -[\beta^T(y)\alpha(y)-\sigma^T(y)]\\ \hline -[\alpha^T(y)\beta(y)-\sigma(y)] & \sum\limits_j \lambda_j Q_j - \alpha^T(y)\alpha(y)\end{array}\right]\succeq 0\right\}.\end{array} \tag{3.3.44}$$

We conclude that the condition

$$\begin{array}{l}\exists(\lambda\ge 0,\{\lambda_j\ge 0\}):\\ \left\{\begin{array}{l}\lambda+\rho^2\sum\limits_{j=1}^J\lambda_j \le \delta(y)-\beta^T(y)\beta(y)\\ \left[\begin{array}{c|c}\lambda & -[\beta^T(y)\alpha(y)-\sigma^T(y)]\\ \hline -[\alpha^T(y)\beta(y)-\sigma(y)] & \sum\limits_j \lambda_j Q_j - \alpha^T(y)\alpha(y)\end{array}\right]\succeq 0\end{array}\right.\end{array}$$

is sufficient for $y$ to be robust feasible. Setting $\mu = \lambda + \beta^T(y)\beta(y)$, this sufficient condition can be rewritten equivalently as

$$\exists(\{\lambda_j\ge 0\},\mu):\left\{\begin{array}{l}\mu+\rho^2\sum\limits_{j=1}^J\lambda_j\le\delta(y)\\ \left[\begin{array}{c|c}\mu-\beta^T(y)\beta(y) & -[\beta^T(y)\alpha(y)-\sigma^T(y)]\\ \hline -[\alpha^T(y)\beta(y)-\sigma(y)] & \sum\limits_j \lambda_j Q_j - \alpha^T(y)\alpha(y)\end{array}\right]\succeq 0\end{array}\right. \tag{3.3.45}$$

We have

$$\begin{array}{l}\left[\begin{array}{c|c}\mu-\beta^T(y)\beta(y) & -[\beta^T(y)\alpha(y)-\sigma^T(y)]\\ \hline -[\alpha^T(y)\beta(y)-\sigma(y)] & \sum\limits_j \lambda_j Q_j - \alpha^T(y)\alpha(y)\end{array}\right]\\ = \left[\begin{array}{c|c}\mu & \sigma^T(y)\\ \hline \sigma(y) & \sum\limits_{j=1}^J\lambda_j Q_j\end{array}\right] - \left[\begin{array}{c}\beta^T(y)\\ \alpha^T(y)\end{array}\right]\left[\begin{array}{c}\beta^T(y)\\ \alpha^T(y)\end{array}\right]^T,\end{array}$$

so that the Schur Complement Lemma says that

$$\begin{array}{l}\left[\begin{array}{c|c}\mu-\beta^T(y)\beta(y) & -[\beta^T(y)\alpha(y)-\sigma^T(y)]\\ \hline -[\alpha^T(y)\beta(y)-\sigma(y)] & \sum\limits_j \lambda_j Q_j - \alpha^T(y)\alpha(y)\end{array}\right]\succeq 0\\ \Leftrightarrow \left[\begin{array}{c|c|c}\mu & \sigma^T(y) & \beta^T(y)\\ \hline \sigma(y) & \sum\limits_j\lambda_j Q_j & \alpha^T(y)\\ \hline \beta(y) & \alpha(y) & I\end{array}\right]\succeq 0.\end{array}$$

The latter observation combines with the fact that (3.3.45) is a sufficient condition for the robust feasibility of $y$ to yield the first part of the following statement:

**Theorem 3.11** *The explicit system of LMIs in variables $y, \mu, \lambda_j$:*

$$\begin{array}{ll}(a) & \left[\begin{array}{c|c|c} \mu & \sigma^T(y)] & \beta^T(y) \\ \hline \sigma(y) & \sum\limits_j \lambda_j Q_j & \alpha^T(y) \\ \hline \beta(y) & \alpha(y) & I \end{array}\right] \succeq 0 \\ (b) & \mu + \rho^2 \sum\limits_{j=1}^{J} \lambda_j \leq \delta(y) \quad (c)\ \lambda_j \geq 0,\ j = 1, ..., J \end{array} \tag{3.3.46}$$

*(for notation, see* (3.3.42)*) is a safe tractable approximation of the RC of* (3.3.40), (3.3.41)*. The tightness factor of this approximation equals 1 when $J = 1$ and does not exceed $\Omega(J) \leq 9.19\sqrt{\ln(J)}$ when $J > 1$.*

The proof of this theorem is completely similar to the proof of Theorem 3.9.

## 3.4 Uncertain Semidefinite Problems with Tractable RCs

In this section, we focus on uncertain Semidefinite Optimization (SDO) problems for which tractable Robust Counterparts can be derived.

### 3.4.1 Uncertain Semidefinite Problems

Recall that a *semidefinite program* (SDP) is a conic optimization program

$$\begin{array}{c} \min\limits_x \left\{ c^T x + d : \mathcal{A}_i(x) \equiv \sum\limits_{j=1}^{n} x_j A^{ij} - B_i \in \mathbf{S}_+^{k_i},\ i = 1, ..., m \right\} \\ \Updownarrow \\ \min\limits_x \left\{ c^T x + d : \mathcal{A}_i(x) \equiv \sum\limits_{j=1}^{n} x_j A^{ij} - B_i \succeq 0,\ i = 1, ..., m \right\} \end{array} \tag{3.4.1}$$

where $A^{ij}$, $B_i$ are symmetric matrices of sizes $k_i \times k_i$, $\mathbf{S}_+^k$ is the cone of real symmetric positive semidefinite $k \times k$ matrices, and $A \succeq B$ means that $A, B$ are symmetric matrices of the same sizes such that the matrix $A - B$ is positive semidefinite. A constraint of the form $\mathcal{A}x - B \equiv \sum\limits_j x_j A^j - B \succeq 0$ with symmetric $A^j$, $B$ is called a *Linear Matrix Inequality* (LMI); thus, an SDP is the problem of minimizing a linear objective under finitely many LMI constraints. Another, sometimes more convenient, setting of a semidefinite program is in the form of (3.1.2), that is,

$$\min_x \left\{ c^T x + d : A_i x - b_i \in \mathbf{Q}_i,\ i = 1, ..., m \right\}, \tag{3.4.2}$$

where nonempty sets $\mathbf{Q}_i$ are given by explicit finite lists of LMIs:

$$\mathbf{Q}_i = \{u \in \mathbb{R}^{p_i} : \mathcal{Q}_{i\ell}(u) \equiv \sum_{s=1}^{p_i} u_s Q^{si\ell} - Q^{i\ell} \succeq 0,\ \ell = 1, ..., L_i\}.$$

Note that (3.4.1) is a particular case of (3.4.2) where $\mathbf{Q}_i = \mathbf{S}_+^{k_i}$, $i = 1, ..., m$.

The notions of the *data* of a semidefinite program, of an *uncertain* semidefinite problem and of its (exact or approximate) *Robust Counterparts* are readily given by specializing the general descriptions from sections 3.1, 3.1.4, to the case when the underlying cones are the cones of positive semidefinite matrices. In particular,

• The *natural data* of a semidefinite program (3.4.2) is the collection

$$(c, d, \{A_i, b_i\}_{i=1}^m),$$

while the right hand side sets $\mathbf{Q}_i$ are treated as the problem's structure;

• An *uncertain* semidefinite problem is a collection of problems (3.4.2) with common structure and natural data running through an *uncertainty set*; we always assume that the data are affinely parameterized by *perturbation vector* $\zeta \in \mathbb{R}^L$ running through a given closed and convex *perturbation set* $\mathcal{Z}$ such that $0 \in \mathcal{Z}$:

$$\begin{array}{rcl} [c; d] & = & [c^{\rm n}; d^{\rm n}] + \sum\limits_{\ell=1}^L \zeta_\ell [c^\ell; d^\ell]; \\ {[A_i, b_i]} & = & [A_i^{\rm n}, b_i^{\rm n}] + \sum\limits_{\ell=1}^L \zeta_\ell [A_i^\ell, b_i^\ell],\ i = 1, ..., m \end{array} \tag{3.4.3}$$

• The Robust Counterpart of uncertain SDP (3.4.2), (3.4.3) at a perturbation level $\rho > 0$ is the semi-infinite optimization program

$$\min_{y=(x,t)} \left\{ t : \left. \begin{array}{l} [[c^{\rm n}]^T x + d^{\rm n}] + \sum\limits_{\ell=1}^L \zeta_\ell [[c^\ell]^T x + d^\ell] \leq t \\ {[A_i^{\rm n} x + b_i^{\rm n}]} + \sum\limits_{\ell=1}^L \zeta_\ell [A_i^\ell x + b_i^\ell] \in \mathbf{Q}_i,\ i = 1, ..., m \end{array} \right\} \forall \zeta \in \rho\mathcal{Z} \right\} \tag{3.4.4}$$

• A *safe tractable approximation* of the RC of uncertain SDP (3.4.2), (3.4.3) is a finite system $\mathcal{S}_\rho$ of explicitly computable convex constraints in variables $y = (x, t)$ (and possibly additional variables $u$) depending on $\rho > 0$ as a parameter, such that the projection $\widehat{Y}_\rho$ of the solution set of the system onto the space of $y$ variables is contained in the feasible set $Y_\rho$ of (3.4.4). Such an approximation is called *tight* within factor $\vartheta \geq 1$, if $Y_\rho \supset \widehat{Y}_\rho \supset Y_{\vartheta\rho}$. In other words, $\mathcal{S}_\rho$ is a $\vartheta$-tight safe approximation of (3.4.4), if:

1. Whenever $\rho > 0$ and $y$ are such that $y$ can be extended, by a properly chosen $u$, to a solution of $\mathcal{S}_\rho$, $y$ is robust feasible at the uncertainty level $\rho$, (i.e., $y$ is feasible for (3.4.4)).

2. Whenever $\rho > 0$ and $y$ are such that $y$ can<u>not</u> be extended to a feasible solution to $\mathcal{S}_\rho$, $y$ is <u>not</u> robust feasible at the uncertainty level $\vartheta\rho$, (i.e., $y$ violates some of the constraints in (3.4.4) when $\rho$ is replaced with $\vartheta\rho$).

### 3.4.2 Tractability of RCs of Uncertain Semidefinite Problems

Building the RC of an uncertain semidefinite problem reduces to building the RCs of the uncertain constraints constituting the problem, so that the tractability issues in Robust Semidefinite Optimization reduce to those for the Robust Counterpart

$$\mathcal{A}_\zeta(y) \equiv \mathcal{A}^{\rm n}(y) + \sum_{\ell=1}^L \zeta_\ell \mathcal{A}_\ell(y) \succeq 0 \ \forall \zeta \in \rho\mathcal{Z} \tag{3.4.5}$$

of a single uncertain LMI

$$\mathcal{A}_\zeta(y) \equiv \mathcal{A}^{\mathrm{n}}(y) + \sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}_\ell(y) \succeq 0; \tag{3.4.6}$$

here $\mathcal{A}^n(x)$, $\mathcal{A}_\ell(x)$ are symmetric matrices affinely depending on the design vector $y$.

More often than not the RC of an uncertain LMI is computationally intractable. Indeed, we saw in section 3 that intractability is typical already for the RCs of uncertain conic quadratic inequalities, and the latter are very special cases of uncertain LMIs (due to the fact that Lorentz cones are cross-sections of semidefinite cones, see Lemma 3.1). In the relatively simple case of uncertain CQIs, we met just 3 generic cases where the RCs were computationally tractable, specifically, the cases of

1. Scenario perturbation set (section 3.2.1);
2. Unstructured norm-bounded uncertainty (section 3.2.3);
3. Simple ellipsoidal uncertainty (section 3.2.5).

The RC associated with a scenario perturbation set is tractable for an arbitrary uncertain conic problem on a tractable cone; in particular, the RC of an uncertain LMI with scenario perturbation set is computationally tractable. Specifically, if $\mathcal{Z}$ in (3.4.5) is given as $\mathrm{Conv}\{\zeta^1, ..., \zeta^N\}$, then the RC (3.4.5) is nothing but the explicit system of LMIs

$$\mathcal{A}^{\mathrm{n}}(y) + \sum_{\ell=1}^{L} \zeta_\ell^i \mathcal{A}_\ell(y) \succeq 0, \; i = 1, ..., N. \tag{3.4.7}$$

The fact that the simple ellipsoidal uncertainty ($\mathcal{Z}$ is an ellipsoid) results in a tractable RC is specific for Conic Quadratic Optimization. In the LMI case, (3.4.5) can be NP-hard even with an ellipsoid in the role of $\mathcal{Z}$. In contrast to this, the case of unstructured norm-bounded perturbations remains tractable in the LMI situation. This is the only nontrivial tractable case we know. We are about to consider this case in full details.

### Unstructured Norm-Bounded Perturbations

**Definition 3.5** *We say that uncertain LMI* (3.4.6) *is with unstructured norm-bounded perturbations, if*

1. *The perturbation set $\mathcal{Z}$ (see* (3.4.3)*) is the set of all $p \times q$ matrices $\zeta$ with the usual matrix norm $\|\cdot\|_{2,2}$ not exceeding 1;*
2. *"The body" $\mathcal{A}_\zeta(y)$ of* (3.4.6) *can be represented as*

$$\mathcal{A}_\zeta(y) \equiv \mathcal{A}^{\mathrm{n}}(y) + \left[L^T(y)\zeta R(y) + R^T(y)\zeta^T L(y)\right], \tag{3.4.8}$$

*where both $L(\cdot)$, $R(\cdot)$ are affine and at least one of these matrix-valued functions is in fact independent of $y$.*

**Example 3.3** Consider the situation where $\mathcal{Z}$ is the unit Euclidean ball in $\mathbb{R}^L$ (or, which is the same, the set of $L \times 1$ matrices of $\|\cdot\|_{2,2}$-norm not exceeding 1), and

$$\mathcal{A}_\zeta(y) = \left[\begin{array}{c|c} a(y) & \zeta^T B^T(y) + b^T(y) \\ \hline B(y)\zeta + b(y) & A(y) \end{array}\right], \tag{3.4.9}$$

where $a(\cdot)$ is an affine scalar function, and $b(\cdot)$, $B(\cdot)$, $A(\cdot)$ are affine vector- and matrix-valued functions with $A(\cdot) \in \mathbf{S}^M$. Setting $R(y) \equiv R = [1, 0_{1\times M}]$, $L(y) = [0_{L\times 1}, B^T(y)]$, we have

$$\mathcal{A}_\zeta(y) = \underbrace{\left[\begin{array}{c|c} a(y) & b^T(y) \\ \hline b(y) & A(y) \end{array}\right]}_{\mathcal{A}^{\rm n}(y)} + L^T(y)\zeta R(y) + R^T(y)\zeta^T L(y),$$

thus, we are in the case of an unstructured norm-bounded uncertainty.

A closely related example is given by the LMI reformulation of an uncertain Least Squares inequality with unstructured norm-bounded uncertainty, see section 3.2.3.

Let us derive a tractable reformulation of an uncertain LMI with unstructured norm-bounded uncertainty. W.l.o.g. we may assume that $R(y) \equiv R$ is independent of $y$ (otherwise we can swap $\zeta$ and $\zeta^T$, swapping simultaneously $L$ and $R$) and that $R \neq 0$. We have

$$\begin{array}{cc} & y \text{ is robust feasible for (3.4.6), (3.4.8) at uncertainty level } \rho \\ \Leftrightarrow & \xi^T[\mathcal{A}^{\rm n}(y) + L^T(y)\zeta R + R^T\zeta^T L(y)]\xi \geq 0 \quad \forall \xi \quad \forall(\zeta : \|\zeta\|_{2,2} \leq \rho) \\ \Leftrightarrow & \xi^T\mathcal{A}^{\rm n}(y)\xi + 2\xi^T L^T(y)\zeta R\xi \geq 0 \quad \forall \xi \quad \forall(\zeta : \|\zeta\|_{2,2} \leq \rho) \\ \Leftrightarrow & \xi^T\mathcal{A}^{\rm n}(y)\xi + 2\underbrace{\min_{\|\zeta\|_{2,2}\leq\rho}\xi^T L^T(y)\zeta R\xi}_{=-\rho\|L(y)\xi\|_2\|R\xi\|_2} \geq 0 \quad \forall \xi \\ \Leftrightarrow & \xi^T\mathcal{A}^{\rm n}(y)\xi - 2\rho\|L(y)\xi\|_2\|R\xi\|_2 \geq 0 \; \forall \xi \\ \Leftrightarrow & \xi^T\mathcal{A}^{\rm n}(y)\xi + 2\rho\eta^T L(y)\xi \geq 0 \quad \forall(\xi, \eta : \eta^T\eta \leq \xi^T R^T R\xi) \\ \Leftrightarrow & \exists \lambda \geq 0 : \left[\begin{array}{c|c} & \rho L(y) \\ \hline \rho L^T(y) & \mathcal{A}^{\rm n}(y) \end{array}\right] \succeq \lambda \left[\begin{array}{c|c} -I_p & \\ \hline & R^T R \end{array}\right] \; [\mathcal{S}\text{-Lemma}] \\ \Leftrightarrow & \exists \lambda : \left[\begin{array}{c|c} \lambda I_p & \rho L(y) \\ \hline \rho L^T(y) & \mathcal{A}^{\rm n}(y) - \lambda R^T R \end{array}\right] \succeq 0. \end{array}$$

We have proved the following statement:

**Theorem 3.12** *The RC*

$$\mathcal{A}^{\rm n}(y) + L^T(y)\zeta R + R^T\zeta^T L(y) \succeq 0 \quad \forall(\zeta \in \mathbb{R}^{p\times q} : \|\zeta\|_{2,2} \leq \rho) \tag{3.4.10}$$

*of uncertain LMI* (3.4.6) *with unstructured norm-bounded uncertainty* (3.4.8) *(where, w.l.o.g., we assume that $R \neq 0$) can be represented equivalently by the LMI*

$$\left[\begin{array}{c|c} \lambda I_p & \rho L(y) \\ \hline \rho L^T(y) & \mathcal{A}^{\rm n}(y) - \lambda R^T R \end{array}\right] \succeq 0 \tag{3.4.11}$$

*in variables $y, \lambda$.*

### Application: Robust Structural Design

**Structural Design problem.** Consider a "linearly elastic" mechanical system $S$ that, mathematically, can be characterized by:

1. A linear space $\mathbb{R}^M$ of *virtual displacements* of the system.

2. A symmetric positive semidefinite $M \times M$ matrix $A$, called the *stiffness matrix* of the system.

The potential energy capacitated by the system when its displacement from the equilibrium is $v$ is

$$E = \frac{1}{2} v^T A v.$$

An external load applied to the system is given by a vector $f \in \mathbb{R}^M$. The associated *equilibrium displacement* $v$ of the system solves the linear equation

$$Av = f.$$

If this equation has no solutions, the load destroys the system — no equilibrium exists; if the solution is not unique, so is the equilibrium displacement. Both these "bad phenomena" can occur only when $A$ is not positive definite.

The *compliance* of the system under a load $f$ is the potential energy capacitated by the system in the equilibrium displacement $v$ associated with $f$, that is,

$$\mathrm{Compl}_f(A) = \frac{1}{2} v^T A v = \frac{1}{2} v^T f.$$

An equivalent way to define compliance is as follows. Given external load $f$, consider the concave quadratic form

$$f^T v - \frac{1}{2} v^T A v$$

on the space $\mathbb{R}^M$ of virtual displacements. It is easily seen that this form either is unbounded above, (which is the case when no equilibrium displacements exist), or attains its maximum. In the latter case, the compliance is nothing but the maximal value of the form:

$$\mathrm{Compl}_f(A) = \sup_{v \in \mathbb{R}^M} \left[ f^T v - \frac{1}{2} v^T A v \right],$$

and the equilibrium displacements are exactly the maximizers of the form.

There are good reasons to treat the compliance as the measure of rigidity of the construction with respect to the corresponding load — the less the compliance, the higher the rigidity. A typical *Structural Design* problem is as follows:

**Structural Design:** *Given*

- *the space $\mathbb{R}^M$ of virtual displacements of the construction,*
- *the stiffness matrix $A = A(t)$ affinely depending on a vector $t$ of design parameters restricted to reside in a given convex compact set $\mathcal{T} \subset \mathbb{R}^N$,*
- *a set $\mathcal{F} \subset \mathbb{R}^M$ of external loads,*

*find a construction $t_*$ that is as rigid as possible w.r.t. the "most dangerous" load from $\mathcal{F}$, that is,*

$$t_* \in \operatorname*{Argmin}_{T \in \mathcal{T}} \left\{ \mathrm{Compl}_{\mathcal{F}}(t) \equiv \sup_{f \in \mathcal{F}} \mathrm{Compl}_f(A(t)) \right\}.$$

Next we present three examples of Structural Design.

**Example 3.4 Truss Topology Design.** A *truss* is a mechanical construction, like railroad bridge, electric mast, or the Eiffel Tower, comprised of thin elastic *bars* linked to each other at *nodes*. Some of the nodes are partially or completely fixed, so that their virtual displacements form proper subspaces in $\mathbb{R}^2$ (for planar constructions) or $\mathbb{R}^3$ (for spatial ones). An external load is a collection of external forces acting at the nodes. Under such a load, the nodes move slightly, thus causing elongations and compressions in the bars, until the construction achieves an equilibrium, where the tensions caused in the bars as a result of their deformations compensate the external forces. The compliance is the potential energy capacitated in the truss at the equilibrium as a result of deformations of the bars.

A mathematical model of the outlined situation is as follows.

• *Nodes and the space of virtual displacements.* Let $\mathcal{M}$ be the nodal set, that is, a finite set in $\mathbb{R}^d$ ($d = 2$ for planar and $d = 3$ for spatial trusses), and let $V_i \subset \mathbb{R}^d$ be the linear space of virtual displacements of node $i$. (This set is the entire $\mathbb{R}^d$ for non-supported nodes, is $\{0\}$ for fixed nodes and is something in-between these two extremes for partially fixed nodes.) The space $V = \mathbb{R}^M$ of virtual displacements of the truss is the direct product $V = V_1 \times ... \times V_m$ of the spaces of virtual displacements of the nodes, so that a virtual displacement of the truss is a collection of "physical" virtual displacements of the nodes.

Now, an external load applied to the truss can be thought of as a collection of external physical forces $f_i \in \mathbb{R}^d$ acting at nodes $i$ from the nodal set. We lose nothing when assuming that $f_i \in V_i$ for all $i$, since the component of $f_i$ orthogonal to $V_i$ is fully compensated by the supports that make the directions from $V_i$ the only possible displacements of node $i$. Thus, we can always assume that $f_i \in V_i$ for all $i$, which makes it possible to identify a load with a vector $f \in V$. Similarly, the collection of nodal reaction forces caused by elongations and compressions of the bars can be thought of as a vector from $V$.

• *Bars and the stiffness matrix.* Every bar $j$, $j = 1, ..., N$, in the truss links two nodes from the nodal set $\mathcal{M}$. Denoting by $t_j$ the volume of the $j$-th bar, a simple analysis, (where one assumes that the nodal displacements are small and neglects all terms of order of squares of these displacements), demonstrates that the collection of the reaction forces caused by a nodal displacement $v \in V$ can be represented as $A(t)v$, where

$$A(t) = \sum_{j=1}^{N} t_j b_j b_j^T \tag{3.4.12}$$

is the stiffness matrix of the truss. Here $b_j \in V$ is readily given by the characteristics of the material of the $j$-th bar and the "nominal," (i.e., in the unloaded truss), positions of the nodes linked by this bar.

In a typical Truss Topology Design (TTD) problem, one is given a *ground structure* — a set $\mathcal{M}$ of tentative nodes along with the corresponding spaces $V_i$ of virtual displacements and the list $\mathcal{J}$ of $N$ tentative bars, (i.e., a list of pairs of nodes that could be linked by bars), and the characteristics of the bar's material; these data determine, in particular, the vectors $b_j$. The design variables are the volumes $t_j$ of the tentative bars. The design specifications always include the natural restrictions $t_j \geq 0$ and an upper bound $w$ on $\sum_j t_j$, (which, essentially, is an upper bound on the total weight of the truss). Thus, $\mathcal{T}$ is always a subset of the standard simplex $\{t \in \mathbb{R}^N : t \geq 0, \sum_j t_j \leq w\}$. There could be other design specifications, like upper and lower bounds on the volumes of some bars. The scenario set $\mathcal{F}$ usually is either a singleton (*single-load TTD*) or a small collection of external loads (*multi-load TTD*). With this setup, one seeks for a design $t \in \mathcal{T}$, that results in the smallest possible worst case, i.e., maximal over the loads from $\mathcal{F}$ compliance.

When formulating a TTD problem, one usually starts with a dense nodal set and allows for all pair connections of the tentative nodes by bars. At an optimal solution to the associated TTD problem, usually a pretty small number of bars get positive volumes, so that the solution recovers not only the optimal bar sizing, but also the optimal topology of the construction.

**Example 3.5 Free Material Optimization.** In Free Material Optimization (FMO) one seeks to design a mechanical construction comprised of material continuously distributed over a given 2-D or 3-D domain $\Omega$, and the mechanical properties of the material are allowed to vary from point to point. The ultimate goal of the design is to build a construction satisfying a number of constraints (most notably, an upper bound on the total weight) and most rigid w.r.t. loading scenarios from a given sample.

After finite element discretization, this (originally infinite-dimensional) optimization problem becomes a particular case of the aforementioned Structural Design problem where:

- the space $V = \mathbb{R}^M$ of virtual displacements is the space of "physical displacements" of the vertices of the finite element cells, so that a displacement $v \in V$ is a collection of displacements $v_i \in \mathbb{R}^d$ of the vertices ($d = 2$ for planar and $d = 3$ for spatial constructions). Same as in the TTD problem, displacements of some of the vertices can be restricted to reside in proper linear subspaces of $\mathbb{R}^d$;

- external loads are collections of physical forces applied at the vertices of the finite element cells; same as in the TTD case, these collections can be identified with vectors $f \in V$;

- the stiffness matrix is of the form

$$A(t) = \sum_{j=1}^{N} \sum_{s=1}^{S} b_{js} t_j b_{js}^T, \tag{3.4.13}$$

  where $N$ is the number of finite element cells and $t_j$ is the *stiffness tensor* of the material in the $j$-th cell. This tensor can be identified with a $p \times p$ symmetric positive semidefinite matrix, where $p = 3$ for planar constructions and $p = 6$ for spatial ones. The number $S$ and the $M \times p$ matrices $b_{is}$ are readily given by the geometry of the finite element cells and the type of finite element discretization.

In a typical FMO problem, one is given the number of the finite element cells along with the matrices $b_{ij}$ in (3.4.13), and a collection $\mathcal{F}$ of external loads of interest. The design vectors are collections $t = (t_1, ..., t_N)$ of positive semidefinite $p \times p$ matrices, and the design specifications always include the natural restrictions $t_j \succeq 0$ and an upper bound $\sum_j c_j \mathrm{Tr}(t_j) \leq w$, $c_j > 0$, on the total weighted trace of $t_j$; this bound reflects, essentially, an upper bound on the total weight of the construction. Along with these restrictions, the description of the feasible design set $\mathcal{T}$ can include other constraints, such as bounds on the spectra of $t_j$, (i.e., lower bounds on the minimal and upper bounds on the maximal eigenvalues of $t_j$). With this setup, one seeks for a design $t \in \mathcal{T}$ that results in the smallest worst case, (i.e., the maximal over the loads from $\mathcal{F}$) compliance.

The design yielded by FMO usually cannot be implemented "as it is" — in most cases, it would be either impossible, or too expensive to use a material with mechanical properties varying from point to point. The role of FMO is in providing an engineer with an "educated guess" of what the optimal construction could possibly be; given this guess, engineers produce something similar from composite materials, applying existing design tools that take into account finer design specifications, (which may include nonconvex ones), than those taken into consideration by the FMO design model.

Our third example, due to C. Roos, has nothing in common with mechanics — it is about design of electrical circuits. Mathematically, however, it is modeled as a Structural Design problem.

**Example 3.6** Consider an electrical circuit comprised of resistances and sources of current. Mathematically, such a circuit can be thought of as a graph with nodes $1,...,n$ and a set $E$ of oriented arcs. Every arc $\gamma$ is assigned with its *conductance* $\sigma_\gamma \geq 0$ (so that $1/\sigma_\gamma$ is the resistance of the arc). The nodes are equipped with external sources of current, so every node $i$ is assigned with a real number $f_i$ — the current supplied by the source. The steady state functioning of the circuit is characterized by currents $\jmath_\gamma$ in the arcs and potentials $v_i$ at the nodes, (these potentials are defined up to a common additive constant). The potentials and the currents can be found from the Kirchhoff laws, specifically, as follows. Let $G$ be the node-arc incidence matrix, so that the columns in $G$ are indexed by the nodes, the rows are indexed by the arcs, and $G_{\gamma i}$ is 1, $-1$ or 0, depending on whether the arc $\gamma$ starts at node $i$, ends at this node, or is not incident to the node, respectively. The first Kirchhoff law states that sum of all currents in the arcs leaving a given node minus the sum of all currents in the arcs entering the node is equal to the external current at the node. Mathematically, this law reads

$$G^T \jmath = f,$$

where $f = (f_1, ..., f_n)$ and $\jmath = \{\jmath_\gamma\}_{\gamma \in E}$ are the vector of external currents and the vector of currents in the arcs, respectively. The second law states that the current in an arc $\gamma$ is $\sigma_\gamma$ times the arc voltage — the difference of potentials at the nodes linked by the arc. Mathematically, this law reads

$$\jmath = \Sigma G v, \Sigma = \mathrm{Diag}\{\sigma_\gamma, \gamma \in E\}.$$

Thus, the potentials are given by the relation

$$G^T \Sigma G v = f.$$

Now, the heat $H$ dissipated in the circuit is the sum, over the arcs, of the products of arc currents and arc voltages, that is,

$$H = \sum_\gamma \sigma_\gamma ((Gv)_\gamma)^2 = v^T G^T \Sigma G v.$$

In other words, the heat dissipated in the circuit, the external currents forming a vector $f$, is the maximum of the convex quadratic form

$$2v^T f - v^T G^T \Sigma G v$$

over all $v \in \mathbb{R}^n$, and the steady state potentials are exactly the maximizers of this quadratic form. In other words, the situation is as if we were speaking about a mechanical system with stiffness matrix $A(\sigma) = G^T \Sigma G$ affinely depending on the vector $\sigma \geq 0$ of arc conductances subject to external load $f$, with the steady-state potentials in the role of equilibrium displacements, and the dissipated heat in this state in the role of (twice) the compliance.

It should be noted that the "stiffness matrix" in our present situation is degenerate — indeed, we clearly have $G\mathbf{1} = 0$, where $\mathbf{1}$ is the vector of ones, ("when the potentials of all nodes are equal, the currents in the arcs should be zero"), whence $A(\sigma)\mathbf{1} = 0$ as well. As a result, the necessary condition for the steady state to exist is $f^T\mathbf{1} = 0$, that is, the total sum of all external currents should be zero — a fact we could easily foresee. Whether this necessary condition is also sufficient depends on the topology of the circuit.

A straightforward "electrical" analogy of the Structural Design problem would be to build a circuit of a given topology, (i.e., to equip the arcs of a given graph with nonnegative conductances forming a design vector $\sigma$), satisfying specifications $\sigma \in \mathcal{S}$ in a way that minimizes the maximal steady-state dissipated heat, the maximum being taken over a given family $\mathcal{F}$ of vectors of external currents.

**Structural Design as an uncertain Semidefinite problem.** The aforementioned Structural Design problem can be easily posed as an SDP. The key element in the transformation of the problem is the following semidefinite representation of the compliance:

$$\text{Compl}_f(A) \le \tau \Leftrightarrow \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A \end{array}\right] \succeq 0. \tag{3.4.14}$$

Indeed,

$$\begin{array}{cl} & \text{Compl}_f(A) \le \tau \\ \Leftrightarrow & f^T v - \frac{1}{2} v^T A v \ge \tau \quad \forall v \in \mathbb{R}^M \\ \Leftrightarrow & 2\tau s^2 - 2s f^T v + v^T A v \ge 0 \quad \forall([v,s] \in \mathbb{R}^{M+1}) \\ \Leftrightarrow & \left[\begin{array}{c|c} 2\tau & -f^T \\ \hline -f & A \end{array}\right] \succeq 0 \\ \Leftrightarrow & \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A \end{array}\right] \succeq 0 \end{array}$$

where the last $\Leftrightarrow$ follows from the fact that

$$\left[\begin{array}{c|c} 2\tau & -f^T \\ \hline -f & A \end{array}\right] = \left[\begin{array}{c|c} 1 & \\ \hline & -I \end{array}\right] \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A \end{array}\right] \left[\begin{array}{c|c} 1 & \\ \hline & -I \end{array}\right]^T.$$

Thus, the Structural Design problem can be posed as

$$\min_{\tau,t} \left\{ \tau : \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0 \quad \forall f \in \mathcal{F}, t \in \mathcal{T} \right\}. \tag{3.4.15}$$

Assuming that the set $\mathcal{T}$ of feasible designs is LMI representable, problem (3.4.15) is nothing but the RC of the uncertain semidefinite problem

$$\min_{\tau,t} \left\{ \tau : \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0,\, t \in \mathcal{T} \right\}, \tag{3.4.16}$$

where the only uncertain data is the load $f$, and this data varies in a given set $\mathcal{F}$ (or, which is the same, in its closed convex hull $\text{cl Conv}(\mathcal{F})$). Thus, in fact we are speaking about the RC of a *single-load* Structural Design problem, with the load in the role of uncertain data varying in the uncertainty set $\mathcal{U} = \text{cl Conv}(\mathcal{F})$.

In actual design the set $\mathcal{F}$ of loads of interest is finite and usually quite small. For example, when designing a bridge for cars, an engineer is interested in a quite restricted family of scenarios, primarily in the load coming from many cars uniformly distributed along the bridge (this is, essentially, what happens in rush hours), and, perhaps, in a few other scenarios (like loads coming from a single heavy car in various positions). With finite $\mathcal{F} = \{f^1, ..., f^k\}$, we are in the situation of a scenario uncertainty, and the RC of (3.4.16) is the explicit semidefinite program

$$\min_{\tau,t} \left\{ \tau : \left[\begin{array}{c|c} 2\tau & [f^i]^T \\ \hline f^i & A(t) \end{array}\right] \succeq 0, i = 1, ..., k,\, t \in \mathcal{T} \right\}.$$

Note, however, that in reality the would-be construction will be affected by small "occasional" loads (like side wind in the case of a bridge), and the construction should be stable with respect to these loads. It turns out, however, that the latter requirement is not necessarily satisfied by the "nominal" construction that takes into consideration only the loads of primary interest. As an instructive example, consider the design of a console.

**Example 3.7** Figure 3.3.(c) represents optimal single-load design of a console with a $9 \times 9$ nodal grid on 2-D plane; nodes from the very left column are fixed, the remaining nodes are free, and the single scenario load is the unit force $f$ acting down and applied at the mid-node of the very right column (see figure 3.3.(a)). We allow nearly all tentative bars (numbering 2,039), except for (clearly redundant) bars linking fixed nodes or long bars that pass through more than two nodes and thus can be split into shorter ones (figure 3.3.(b)). The set $\mathcal{T}$ of admissible designs is given solely by the weight restriction:

$$\mathcal{T} = \{t \in \mathbb{R}^{2039} : t \geq 0, \sum_{i=1}^{2039} t_i \leq 1\}$$

(compliance is homogeneous of order 1 w.r.t. $t$: $\mathrm{Compl}_f(\lambda t) = \lambda \mathrm{Compl}_f(t)$, $\lambda > 0$, so we can normalize the weight bound to be 1).

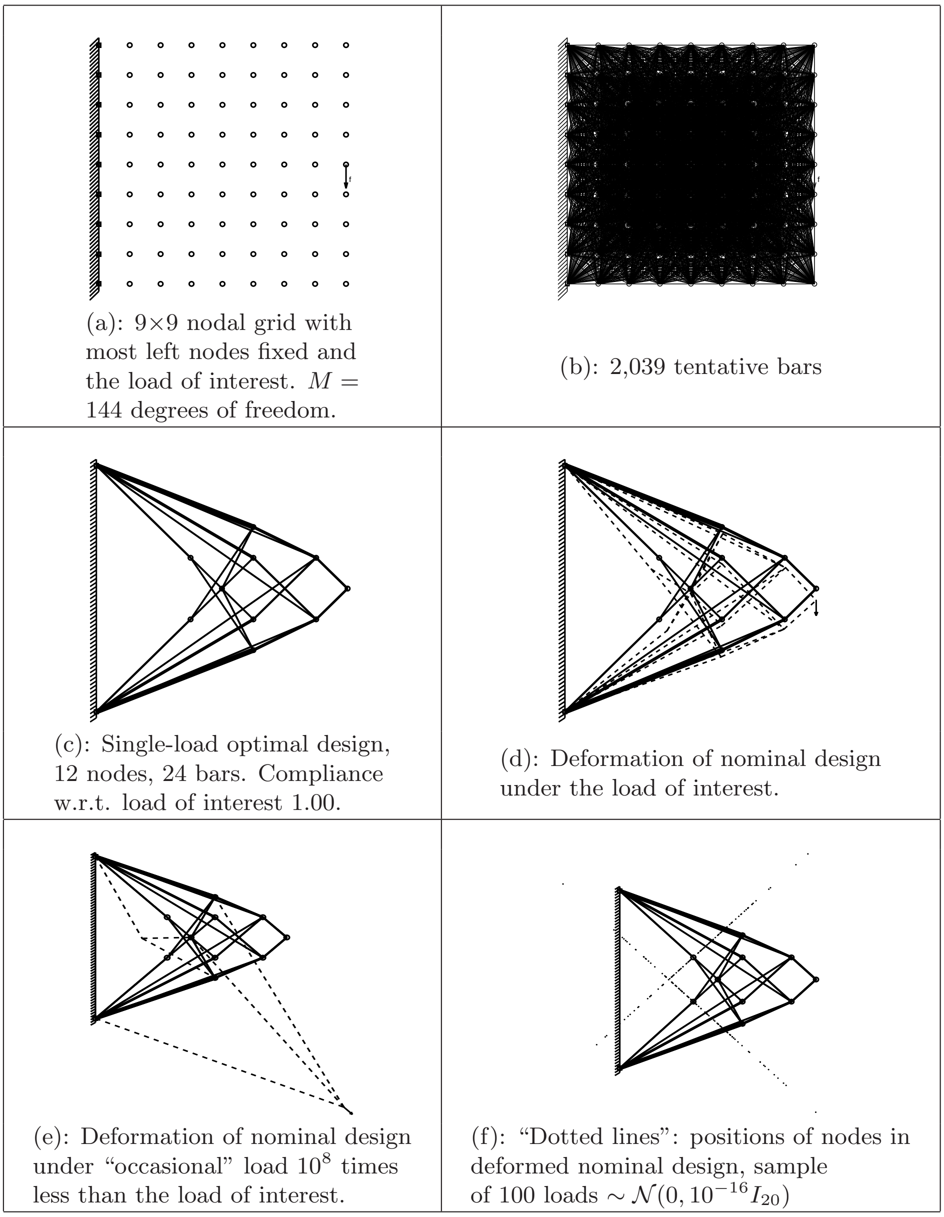
The compliance, in an appropriate scale, of the resulting nominally optimal truss (12 nodes, 24 bars) w.r.t. the scenario load $f$ is 1.00. At the same time, the construction turns out to be highly unstable w.r.t. small "occasional" loads distributed along the 10 free nodes used by the nominal design. For example, the mean compliance of the nominal design w.r.t. a random load $h \sim \mathcal{N}(0, 10^{-9} I_{20})$ is 5.406 (5.4 times larger than the nominal compliance), while the "typical" norm $\|h\|_2$ of this random load is $10^{-4.5}\sqrt{20}$ — more than three orders of magnitude less than the norm $\|f\|_2 = 1$ of the scenario load. The compliance of the nominally optimal truss w.r.t. a "bad" load $g$ that is $10^4$ times smaller than $f$ ($\|g\|_2 = 10^{-4}\|f\|_2$) is 27.6 — by factor 27 larger than the compliance w.r.t. $f$! Figure 3.3.(e) shows the deformation of the nominal design under the load $10^{-4}g$ (that is, the load that is $10^8$ (!) times smaller than the scenario load). One can compare this deformation with the one under the load $f$ (figure 3.3.(d)). Figure 3.3.(f) depicts shifts of the nodes under a sample of 100 random loads $h \sim \mathcal{N}(0, 10^{-16} I_{20})$ — loads of norm by 7 plus orders of magnitude less than $\|f\|_2 = 1$.

To prevent the optimal design from being crushed by a small load that is outside of the set $\mathcal{F}$ of loading scenarios, it makes sense to extend $\mathcal{F}$ to a more "massive" set, primarily by adding to $\mathcal{F}$ all loads of magnitude not exceeding a given "small" uncertainty level $\rho$. A challenge here is to decide where the small loads can be applied. In problems like TTD, it does not make sense to require the would-be construction to be capable of carrying small loads distributed along *all* nodes of the ground structure; indeed, not all of these nodes should be present in the final design, and of course there is no reason to bother about forces acting at non-existing nodes. The difficulty is that we do not know in advance which nodes will be present in the final design. One possibility to resolve this difficulty to some extent is to use a two-stage procedure as follows:

• at the first stage, we seek for the "nominal" design — the one that is optimal w.r.t. the "small" set $\mathcal{F}$ comprised of the scenario loads and, perhaps, all loads of magnitude $\leq \rho$ acting along the same nodes as the scenario loads — these nodes definitely will be present in the resulting design;

• at the second stage, we solve the problem again, with the nodes actually used by the nominal design in the role of our new nodal set $\mathcal{M}^+$, and extend $\mathcal{F}$ to the set $\mathcal{F}^+$ by taking the union of $\mathcal{F}$ and the Euclidean ball $B_\rho$ of all loads $g$, $\|g\|_2 \leq \rho$, acting along $\mathcal{M}^+$.

We have arrived at the necessity to solve (3.4.15) in the situation where $\mathcal{F}$ is the union of a finite set $\{f^1, ..., f^k\}$ and a Euclidean ball. This is a particular case of the situation when $\mathcal{F}$ is



(a): 9×9 nodal grid with most left nodes fixed and the load of interest. $M =$ 144 degrees of freedom.

(b): 2,039 tentative bars

(c): Single-load optimal design, 12 nodes, 24 bars. Compliance w.r.t. load of interest 1.00.

(d): Deformation of nominal design under the load of interest.

(e): Deformation of nominal design under "occasional" load $10^8$ times less than the load of interest.

(f): "Dotted lines": positions of nodes in deformed nominal design, sample of 100 loads $\sim \mathcal{N}(0, 10^{-16} I_{20})$

Figure 3.3: Nominal design.

the union of $S < \infty$ ellipsoids

$$E_s = \{f = f^s + B_s\zeta^s : \zeta^s \in \mathbb{R}^{k_s}, \|\zeta^s\|_2 \leq 1\}$$

or, which is the same, $\mathcal{Z}$ is the convex hull of the union of $S$ ellipsoids $E_1, ..., E_S$. The associated "uncertainty-immunized" Structural Design problem (3.4.15) — the RC of (3.4.16) with $\mathcal{Z}$ in the role of $\mathcal{F}$ — is clearly equivalent to the problem

$$\min_{t,\tau}\left\{\tau : \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t)\end{array}\right] \succeq 0 \quad \forall f \in E_s,\ s = 1, ..., S; t \in \mathcal{T}\right\}. \tag{3.4.17}$$

In order to build a tractable equivalent of this semi-infinite semidefinite problem, we need to build a tractable equivalent to a semi-infinite LMI of the form

$$\left[\begin{array}{c|c} 2\tau & \zeta^T B^T + f^T \\ \hline B\zeta + f & A(t)\end{array}\right] \succeq 0 \quad \forall(\zeta \in \mathbb{R}^k : \|\zeta\|_2 \leq \rho). \tag{3.4.18}$$

But such an equivalent is readily given by Theorem 3.12 (cf. Example 3.3). Applying the recipe described in this Theorem, we end up with a representation of (3.4.18) as the following LMI in variables $\tau$, $t$, $\lambda$:

$$\left[\begin{array}{c|c|c} \lambda I_k & & \rho B^T \\ \hline & 2\tau - \lambda & f^T \\ \hline \rho B & f & A(t)\end{array}\right] \succeq 0. \tag{3.4.19}$$

Observe that when $f = 0$, (3.4.19) simplifies to

$$\left[\begin{array}{c|c} 2\tau I_k & \rho B^T \\ \hline \rho B & A(t)\end{array}\right] \succeq 0. \tag{3.4.20}$$

**Example 3.7 continued.** Let us apply the outlined methodology to the Console example (Example 3.7). In order to immunize the design depicted on figure 3.3.(c) against small occasional loads, we start with reducing the initial $9 \times 9$ nodal set to the set of 12 nodes $\mathcal{M}^+$ (figure 3.4.(a)) used by the nominal design, and allow for $N = 54$ tentative bars on this reduced nodal set (figure 5.1.(b)) (we again allow for all pair connections of nodes, except for connections of two fixed nodes and for long bars passing through more than two nodes). According to the outlined methodology, we should then extend the original singleton $\mathcal{F} = \{f\}$ of scenario loads to the larger set $\mathcal{F}^+ = \{f\} \cup B_\rho$, where $B_\rho$ is the Euclidean ball of radius $\rho$, centered at the origin in the $(M = 20)$-dimensional space of virtual displacements of the reduced planar nodal set. With this approach, an immediate question would be how to specify $\rho$. In order to avoid an ad hoc choice of $\rho$, we modify our approach as follows. Recalling that the compliance of the nominally optimal design w.r.t. the scenario load is 1.00, let us impose on our would-be "immunized" design the restriction that its worst case compliance w.r.t. the extended scenario set $\mathcal{F}_\rho = \{f\} \cup B_\rho$ should be at most $\tau_* = 1.025$, (i.e., 2.5% more than the optimal nominal compliance), and maximize under this restriction the radius $\rho$. In other words, we seek for a truss of the same unit weight as the nominally optimal one with "nearly optimal" rigidity w.r.t. the scenario load $f$ and as large as possible worst-case rigidity w.r.t. occasional loads of a given magnitude. The resulting problem is the semi-infinite semidefinite program

$$\max_{t,\rho}\left\{\rho : \begin{array}{l} \left[\begin{array}{c|c} 2\tau_* & f^T \\ \hline f & A(t)\end{array}\right] \succeq 0 \\ \left[\begin{array}{c|c} 2\tau_* & \rho h^T \\ \hline \rho h & A(t)\end{array}\right] \succeq 0 \quad \forall(h : \|h\|_2 \leq 1) \\ t \succeq 0, \sum_{i=1}^N t_i \leq 1 \end{array}\right\}.$$

This semi-infinite program is equivalent to the usual semidefinite program

$$\max_{t,\rho}\left\{\rho:\ \begin{array}{c}\left[\begin{array}{c|c}2\tau_* & f^T\\ \hline f & A(t)\end{array}\right]\succeq 0\\ \left[\begin{array}{c|c}2\tau_* I_M & \rho I_M\\ \hline \rho I_M & A(t)\end{array}\right]\succeq 0\\ t\succeq 0,\sum_{i=1}^N t_i\leq 1\end{array}\right\} \tag{3.4.21}$$

(cf. (3.4.20)).

Computation shows that for Example 3.7, the optimal value in (3.4.21) is $\rho_* = 0.362$; the *robust design* yielded by the optimal solution to the problem is depicted in figure 3.4.(c). Along with the differences in sizing of bars, note the difference in the structures of the robust and the nominal design (figure 3.5). Observe that passing from the nominal to the robust design, we lose just 2.5% in the rigidity w.r.t. the scenario load and gain a dramatic improvement in the capability to carry occasional loads. Indeed, the compliance of the robust truss w.r.t. *every* load $g$ of the magnitude $\|g\|_2 = 0.36$ (36% of the magnitude of the load of interest) is at most 1.025; the similar quantity for the nominal design is as large as $1.65\times10^9$ ! An additional evidence of the dramatic advantages of the robust design as compared to the nominal one can be obtained by comparing the pictures (d) through (f) in figure 3.3 with their counterparts in figure 3.4.

### Applications in Robust Control

A major source of uncertain Semidefinite problems is Robust Control. An instructive example is given by Lyapunov Stability Analysis/Synthesis.

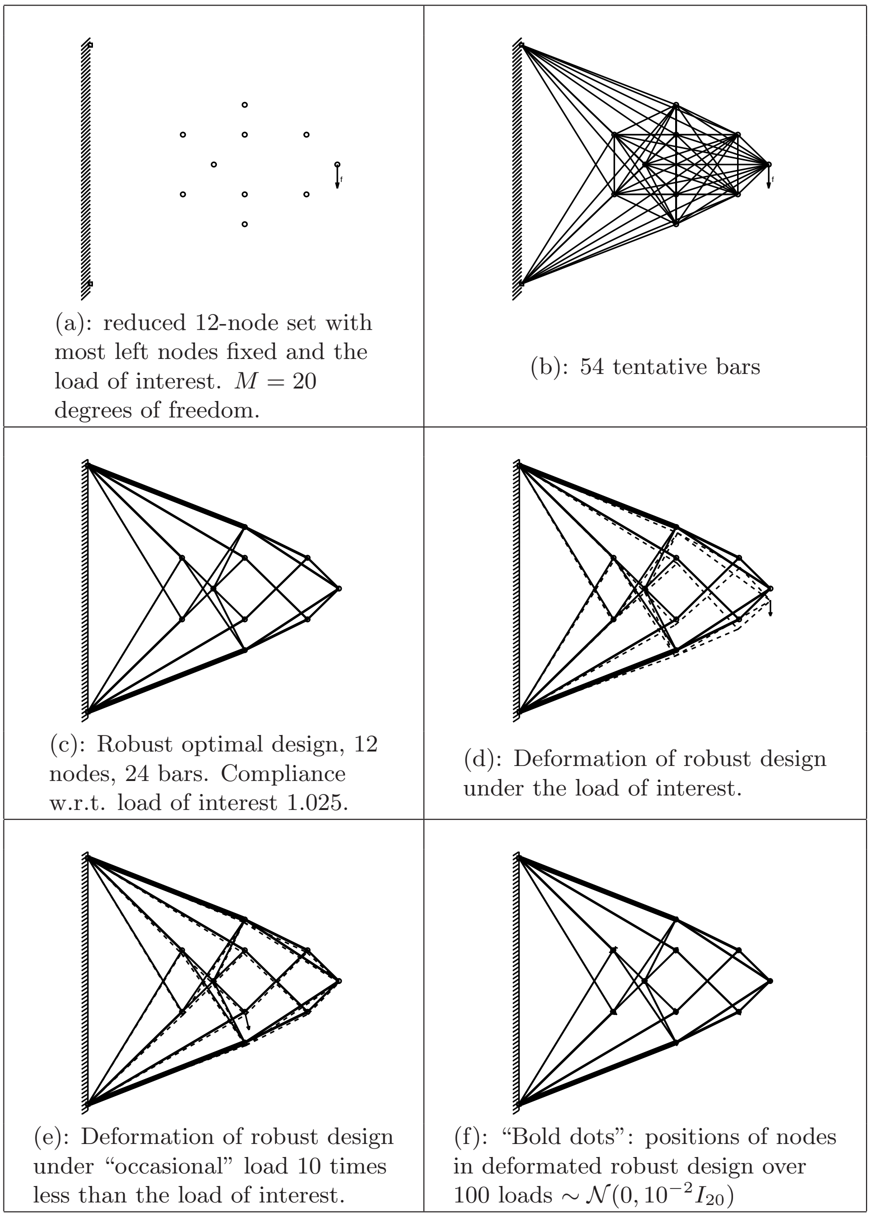
**Lyapunov Stability Analysis.** Consider a time-varying linear dynamical system "closed" by a linear output-based feedback:

$$\begin{array}{ll}(a) & \dot{x}(t) = A_t x(t) + B_t u(t) + R_t d_t \text{ [open loop system, or } plant\text{]}\\ (b) & y(t) = C_t x(t) + D_t d_t \text{ [output]}\\ (c) & u(t) = K_t y(t) \text{ [output-based feedback]}\\ & \Downarrow\\ (d) & \dot{x}(t) = [A_t + B_t K_t C_t]x(t) + [R_t + B_t K_t D_t]d_t \text{ [closed loop system]}\end{array} \tag{3.4.22}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $d_t \in \mathbb{R}^p$, $y(t) \in \mathbb{R}^q$ are respectively, the state, the control, the external disturbance, and the output at time $t$, $A_t$, $B_t$, $R_t$, $C_t$, $D_t$ are matrices of appropriate sizes specifying the dynamics of the system; and $K_t$ is the feedback matrix. We assume that the dynamical system in question is *uncertain*, meaning that we do not know the dependencies of the matrices $A_t$,...,$K_t$ on $t$; all we know is that the collection $M_t = (A_t, B_t, C_t, D_t, R_t, K_t)$ of all these matrices stays all the time within a given compact uncertainty set $\mathcal{M}$. For our further purposes, it makes sense to think that there exists an underlying time-invariant "nominal" system corresponding to known nominal values $A^{\rm n}$,...,$K^{\rm n}$ of the matrices $A_t, ..., K_t$, while the actual dynamics corresponds to the case when the matrices drift (perhaps, in a time-dependent fashion) around their nominal values.

An important desired property of a linear dynamical system is its *stability* — the fact that every state trajectory $x(t)$ of (every realization of) the closed loop system converges to 0 as $t \to \infty$, provided that the external disturbances $d_t$ are identically zero. For a time-invariant linear system

$$\dot{x} = Q^{\rm n} x,$$



(a): reduced 12-node set with most left nodes fixed and the load of interest. $M = 20$ degrees of freedom.

(b): 54 tentative bars

(c): Robust optimal design, 12 nodes, 24 bars. Compliance w.r.t. load of interest 1.025.

(d): Deformation of robust design under the load of interest.

(e): Deformation of robust design under "occasional" load 10 times less than the load of interest.

(f): "Bold dots": positions of nodes in deformated robust design over 100 loads $\sim \mathcal{N}(0, 10^{-2} I_{20})$

Figure 3.4: Robust design.

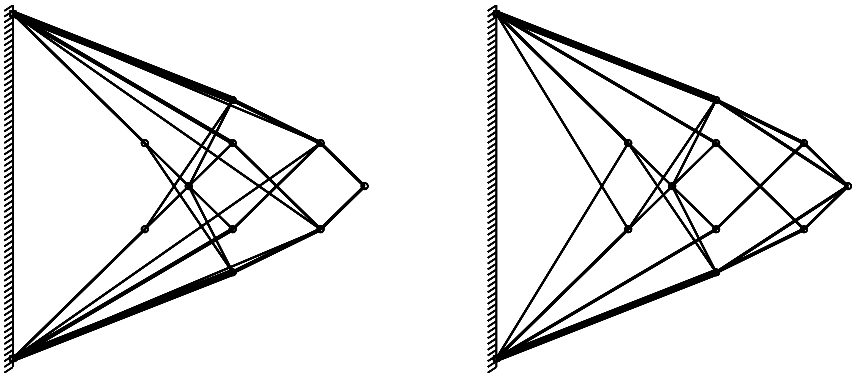


Figure 3.5: Nominal (left) and robust (right) designs.

the necessary and sufficient stability condition is that all eigenvalues of $A$ have negative real parts or, equivalently, that there exists a *Lyapunov Stability Certificate* (LSC) — a positive definite symmetric matrix $X$ such that

$$[Q^{\rm n}]^T X + XQ^{\rm n} \prec 0.$$

For uncertain system (3.4.22), a *sufficient* stability condition is that all matrices

$$Q \in \mathcal{Q} = \{Q = A^M + B^M K^M C^M : M \in \mathcal{M}\}$$

have a common LSC $X$, that is, there exists $X \succ 0$ such that

$$\begin{array}{ll} (a) & Q^T X + XQ^T \prec 0 \quad \forall Q \in \mathcal{Q} \\ & \Updownarrow \\ (b) & [A^M + B^M K^M C^M]^T X + X[A^M + B^M K^M C^M] \prec 0 \quad \forall M \in \mathcal{M}; \end{array} \tag{3.4.23}$$

here $A^M$,...,$K^M$ are the components of a collection $M \in \mathcal{M}$.

The fact that the existence of a common LSC for all matrices $Q \in \mathcal{Q}$ is sufficient for the stability of the closed loop system is nearly evident. Indeed, since $\mathcal{M}$ is compact, for every feasible solution $X \succ 0$ of the semi-infinite LMI (3.4.23) one has

$$\forall M \in \mathcal{M} : [A^M + B^M K^M C^M]^T X + X[A^M + B^M K^M C^M] \prec -\alpha X \tag{$*$}$$

with appropriate $\alpha > 0$. Now let us look what happens with the quadratic form $x^T X x$ along a state trajectory $x(t)$ of (3.4.22). Setting $f(t) = x^T(t)Xx(t)$ and invoking (3.4.22.$d$), we have

$$\begin{array}{rl} f'(t) & = \dot{x}^T(t)Xx(t) + x(t)X\dot{x}(t) \\ & = x^T(t)\left[[A_t + B_t K_t C_t]^T X + X[A_t + B_t K_t C_t]\right] x(t) \\ & \leq -\alpha f(t), \end{array}$$

where the concluding inequality is due to $(*)$. From the resulting differential inequality

$$f'(t) \leq -\alpha f(t)$$

it follows that

$$f(t) \leq \exp\{-\alpha t\} f(0) \to 0, \ t \to \infty.$$

Recalling that $f(t) = x^T(t)Xx(t)$ and $X$ is positive definite, we conclude that $x(t) \to 0$ as $t \to \infty$.

Observe that the set $\mathcal{Q}$ is compact along with $\mathcal{M}$. It follows that $X$ is an LSC if and only if $X \succ 0$ and

$$\exists \beta > 0 : Q^T X + XQ \preceq -\beta I \quad \forall Q \in \mathcal{Q}$$

$$\Leftrightarrow \quad \exists \beta > 0 : Q^T X + XQ \preceq -\beta I \quad \forall Q \in \mathrm{Conv}(\mathcal{Q}).$$

Multiplying such an $X$ by an appropriate positive real, we can ensure that

$$X \succeq I \ \&\ Q^T X + XQ \preceq -I \quad \forall Q \in \mathrm{Conv}(\mathcal{Q}). \tag{3.4.24}$$

Thus, we lose nothing when requiring from an LSC to satisfy the latter system of (semi-infinite) LMIs, and from now on LSCs in question will be exactly the solutions of this system.

Observe that (3.4.24) is nothing but the RC of the uncertain system of LMIs

$$X \succeq I \ \&\ Q^T X + XQ \preceq -I, \tag{3.4.25}$$

the uncertain data being $Q$ and the uncertainty set being $\mathrm{Conv}(\mathcal{Q})$. Thus, RCs arise naturally in the context of Robust Control.

Now let us apply the results on tractability of the RCs of uncertain LMI in order to understand when the question of existence of an LSC for a given uncertain system (3.4.22) can be posed in a computationally tractable form. There are, essentially, two such cases — *polytopic* and *unstructured norm-bounded* uncertainty.

**Polytopic uncertainty.** By definition, polytopic uncertainty means that the set $\mathrm{Conv}(\mathcal{Q})$ is given as a convex hull of an explicit list of "scenarios" $Q^i$, $i = 1, ..., N$:

$$\mathrm{Conv}(\mathcal{Q}) = \mathrm{Conv}\{Q^1, ..., Q^N\}.$$

In our context this situation occurs when the components $A^M, B^M, C^M, K^M$ of $M \in \mathcal{M}$ run, independently of each other, through convex hulls of respective scenarios

$$S_A = \mathrm{Conv}\{A^1, ..., A^{N_A}\}, S_B = \mathrm{Conv}\{B^1, ..., B^{N_B}\},$$
$$S_C = \mathrm{Conv}\{C^1, ..., C^{N_C}\}, S_K = \mathrm{Conv}\{K^1, ..., K^{N_K}\};$$

in this case, the set $\mathrm{Conv}(\mathcal{Q})$ is nothing but the convex hull of $N = N_A N_B N_C N_K$ "scenarios" $Q^{ijk\ell} = A^i + B^j K^\ell C^k$, $1 \le i \le N_A$,...,$1 \le \ell \le N_K$.

Indeed, $\mathcal{Q}$ clearly contains all matrices $Q^{ijk\ell}$ and therefore $\mathrm{Conv}(\mathcal{Q}) \supset \mathrm{Conv}(\{Q^{ijk\ell}\})$. On the other hand, the mapping $(A, B, C, K) \mapsto A + BKC$ is polylinear, so that the image $\mathcal{Q}$ of the set $S_A \times S_B \times S_C \times S_K$ under this mapping is contained in the convex set $\mathrm{Conv}(\{Q^{ijk\ell}\})$, whence $\mathrm{Conv}(\{Q^{ijk\ell}\}) \supset \mathrm{Conv}(\mathcal{Q})$.

In the case in question we are in the situation of scenario perturbations, so that (3.4.25) is equivalent to the explicit system of LMIs

$$X \succeq I, [Q^i]^T X + XQ^i \preceq -I,\ i = 1, ..., N.$$

**Unstructured norm-bounded uncertainty.** Here

$$\mathrm{Conv}(\mathcal{Q}) = \{Q = Q^{\mathrm{n}} + U\zeta V : \zeta \in \mathbb{R}^{p\times q}, \|\zeta\|_{2,2} \le \rho\}.$$

In our context this situation occurs, e.g., when 3 of the 4 matrices $A^M$, $B^M$, $C^M$, $K^M$, $M \in \mathcal{M}$, are in fact certain, and the remaining matrix, say, $A^M$, runs through a set of the form $\{A^{\mathrm{n}} + G\zeta H : \zeta \in \mathbb{R}^{p\times q}, \|\zeta\|_{2,2} \le \rho\}$.

In the case of unstructured norm-bounded uncertainty, the semi-infinite LMI in (3.4.25) is of the form

$$\begin{array}{c} Q^T X + XQ \preceq -I \quad \forall Q \in \mathrm{Conv}(\mathcal{Q}) \\ \Updownarrow \\ \underbrace{-I - [Q^{\rm n}]^T X - XQ^{\rm n}}_{\mathcal{A}^{\rm n}(X)} + [\underbrace{-XU}_{L^T(X)} \zeta \underbrace{V}_{R} + R^T \zeta^T L(X)] \succeq 0 \\ \forall (\zeta \in \mathbb{R}^{p \times q}, \|\zeta\|_{2,2} \le \rho). \end{array}$$

Invoking Theorem 3.12, (3.4.25) is equivalent to the explicit system of LMIs

$$X \succeq I, \left[ \begin{array}{c|c} \lambda I_p & \rho U^T X \\ \hline \rho XU & -I - [Q^{\rm n}]^T X - XQ^{\rm n} - \lambda V^T V \end{array} \right] \succeq 0. \tag{3.4.26}$$

in variables $X, \lambda$.

**Lyapunov Stability Synthesis.** We have considered the *Stability Analysis* problem, where one, given an uncertain closed-loop dynamical system along with the associated uncertainty set $\mathcal{M}$, seeks to verify a sufficient stability condition. A more challenging problem is *Stability Synthesis*: given an uncertain open loop system (3.4.22.*a*–*b*) along with the associated compact uncertainty set $\widehat{\mathcal{M}}$ in the space of collections $\widehat{M} = (A, B, C, D, R)$, find a linear output-based feedback

$$u(t) = Ky(t)$$

*and* an LSC for the resulting closed loop system.

The Synthesis problem has a nice solution, due to [22], in the case of *state-based* feedback (that is, $C_t \equiv I$) and under the assumption that the feedback is implemented exactly, so that the state dynamics of the closed loop system is given by

$$\dot{x}(t) = [A_t + B_t K]x(t) + [R_t + B_t K D_t]d_t. \tag{3.4.27}$$

The pairs $(K, X)$ of "feedback – LSC" that we are looking for are exactly the feasible solutions to the system of semi-infinite matrix inequalities in variables $X, K$:

$$X \succ 0 \ \& \ [A + BK]^T X + X[A + BK] \prec 0 \ \ \forall [A, B] \in \mathcal{AB}; \tag{3.4.28}$$

here $\mathcal{AB}$ is the projection of $\widehat{M}$ on the space of $[A, B]$ data. The difficulty is that the system is *nonlinear* in the variables. As a remedy, let us carry out the nonlinear substitution of variables $X = Y^{-1}$, $K = ZY^{-1}$. With this substitution, (3.4.28) becomes a system in the new variables $Y, Z$:

$$Y \succ 0 \ \& \ [A + BZY^{-1}]^T Y^{-1} + Y^{-1}[A + BZY^{-1}] \prec 0 \ \ \forall [A, B] \in \mathcal{AB};$$

multiplying both sides of the second matrix inequality from the left and from the right by $Y$, we convert the system to the equivalent form

$$Y \succ 0, \ \& \ AY + YA^T + BZ + Z^T B^T \prec 0 \quad \forall [A, B] \in \mathcal{AB}.$$

Since $\mathcal{AB}$ is compact along with $\widehat{\mathcal{M}}$, the solutions to the latter system are exactly the pairs $(Y, Z)$ that can be obtained by scaling $(Y, Z) \mapsto (\lambda Y, \lambda Z)$, $\lambda > 0$, from the solutions to the system of semi-infinite LMIs

$$Y \succeq I \ \& \ AY + YA^T + BZ + Z^T B^T \preceq -I \quad \forall [A, B] \in \mathcal{AB} \tag{3.4.29}$$

in variables $Y, Z$. When the uncertainty $\mathcal{AB}$ can be represented either as a polytopic, or as unstructured norm-bounded, the system (3.4.29) of semi-infinite LMIs admits an equivalent tractable reformulation.

## 3.5 Approximating RCs of Uncertain Semidefinite Problems

### 3.5.1 Tight Tractable Approximations of RCs of Uncertain SDPs with Structured Norm-Bounded Uncertainty

We have seen that the possibility to reformulate the RC of an uncertain semidefinite program in a computationally tractable form is a "rare commodity," so that there are all reasons to be interested in the second best thing — in situations where the RC admits a tight tractable approximation. To the best of our knowledge, just one such case is known — the case of *structured norm-bounded uncertainty* we are about to consider in this section.

#### Uncertain LMI with Structured Norm-Bounded Perturbations

Consider an uncertain LMI

$$\mathcal{A}_\zeta(y) \succeq 0 \tag{3.4.6}$$

where the "body" $\mathcal{A}_\zeta(y)$ is bi-linear in the design vector $y$ and the perturbation vector $\zeta$. The definition of a structured norm-bounded perturbation follows the path we got acquainted with in section 3:

**Definition 3.6** *We say that the uncertain constraint* (3.4.6) *is affected by structured norm-bounded uncertainty with uncertainty level $\rho$, if*

1. *The perturbation set $\mathcal{Z}_\rho$ is of the form*

$$\mathcal{Z}_\rho = \left\{ \zeta = (\zeta^1, ..., \zeta^L) : \begin{array}{l} \zeta^\ell \in \mathbb{R}, |\zeta^\ell| \leq \rho, \, \ell \in \mathcal{I}_{\rm s} \\ \zeta^\ell \in \mathbb{R}^{p_\ell \times q_\ell} : \|\zeta^\ell\|_{2,2} \leq \rho, \, \ell \notin \mathcal{I}_{\rm s} \end{array} \right\} \tag{3.5.1}$$

2. *The body $\mathcal{A}_\zeta(y)$ of the constraint can be represented as*

$$\begin{array}{rcl} \mathcal{A}_\zeta(y) & = & \mathcal{A}^{\rm n}(y) + \sum\limits_{\ell \in \mathcal{I}_{\rm S}} \zeta^\ell \mathcal{A}_\ell(y) \\ & & + \sum\limits_{\ell \notin \mathcal{I}_{\rm s}} \left[ L_\ell^T(y) \zeta^\ell R_\ell + R_\ell^T [\zeta^\ell]^T L_\ell(y) \right], \end{array} \tag{3.5.2}$$

*where $\mathcal{A}_\ell(y)$, $\ell \in \mathcal{I}_{\rm s}$, and $L_\ell(y)$, $\ell \notin \mathcal{I}_{\rm s}$, are affine in $y$, and $R_\ell$, $\ell \notin \mathcal{I}_{\rm s}$, are nonzero.*

**Theorem 3.13** *Given uncertain LMI* (3.4.6) *with structured norm-bounded uncertainty* (3.5.1), (3.5.2)*, let us associate with it the following system of LMIs in variables $Y_\ell$, $\ell = 1, ..., L$, $\lambda_\ell$, $\ell \notin \mathcal{I}_{\rm s}$, $y$:*

$$\begin{array}{ll} (a) & Y_\ell \succeq \pm \mathcal{A}_\ell(y), \, \ell \in \mathcal{I}_{\rm s} \\ (b) & \left[ \begin{array}{c|c} \lambda_\ell I_{p_\ell} & L_\ell(y) \\ \hline L_\ell^T(y) & Y_\ell - \lambda_\ell R_\ell^T R_\ell \end{array} \right] \succeq 0, \, \ell \notin \mathcal{I}_{\rm s} \\ (c) & \mathcal{A}^{\rm n}(y) - \rho \sum\limits_{\ell=1}^{L} Y_\ell \succeq 0 \end{array} \tag{3.5.3}$$

*Then system* (3.5.3) *is a safe tractable approximation of the RC*

$$\mathcal{A}_\zeta(y) \succeq 0 \; \forall \zeta \in \mathcal{Z}_\rho \tag{3.5.4}$$

*of* (3.4.6), (3.5.1), (3.5.2)*, and the tightness factor of this approximation does not exceed* $\vartheta(\mu)$*, where* $\mu$ *is the smallest integer* $\geq 2$ *such that* $\mu \geq \max\limits_y \mathrm{Rank}(\mathcal{A}_\ell(y))$ *for all* $\ell \in \mathcal{I}_{\rm s}$*, and* $\vartheta(\cdot)$ *is a universal function of* $\mu$ *such that*

$$\vartheta(2) = \frac{\pi}{2},\ \vartheta(4) = 2,\ \vartheta(\mu) \leq \pi\sqrt{\mu/2},\ \mu > 2.$$

*The approximation is exact, if either* $L = 1$*, or all perturbations are scalar, (i.e.,* $\mathcal{I}_{\rm s} = \{1, ..., L\}$*) and all* $\mathcal{A}_\ell(y)$ *are of ranks not exceeding 1.*

**Proof.** Let us fix $y$ and observe that a collection $y$, $Y_1, ..., Y_L$ can be extended to a feasible solution of (3.5.3) if and only if

$$\forall \zeta \in \mathcal{Z}_\rho : \left\{ \begin{array}{l} -\rho Y_\ell \preceq \zeta^\ell \mathcal{A}_\ell(y),\ \ell \in \mathcal{I}_{\rm s}, \\ -\rho Y_\ell \preceq L_\ell^T(y)\zeta^\ell R_\ell + R_\ell^T[\zeta^\ell]^T L_\ell(y),\ \ell \notin \mathcal{I}_{\rm s} \end{array} \right.$$

(see Theorem 3.12). It follows that if, in addition, $Y_\ell$ satisfy (3.5.3.$c$), then $y$ is feasible for (3.5.4), so that (3.5.3) is a safe tractable approximation of (3.5.4). The fact that this approximation is tight within the factor $\vartheta(\mu)$ is readily given by the real case Matrix Cube Theorem (Theorem A.7). The fact that the approximation is exact when $L = 1$ is evident when $\mathcal{I}_{\rm s} = \{1\}$ and is readily given by Theorem 3.12 when $\mathcal{I}_{\rm s} = \emptyset$. The fact that the approximation is exact when all perturbations are scalar and all matrices $\mathcal{A}_\ell(y)$ are of ranks not exceeding 1 is evident. □

### Application: Lyapunov Stability Analysis/Synthesis Revisited

We start with the Analysis problem. Consider the uncertain time-varying dynamical system (3.4.22) and assume that the uncertainty set $\mathrm{Conv}(\mathcal{Q}) = \mathrm{Conv}(\{A^M + B^M K^M C^M\} : M \in \mathcal{M}\})$ in (3.4.24) is an *interval* uncertainty, meaning that

$$\begin{array}{c} \mathrm{Conv}(\mathcal{Q}) = Q^{\rm n} + \rho\mathcal{Z},\ \mathcal{Z} = \{\sum\limits_{\ell=1}^{L} \zeta_\ell U_\ell : \|\zeta\|_\infty \leq 1\}, \\ \mathrm{Rank}(U_\ell) \leq \mu,\ 1 \leq \ell \leq L. \end{array} \tag{3.5.5}$$

Such a situation (with $\mu = 1$) arises, e.g., when two of the 3 matrices $B_t, C_t, K_t$ are certain, and the remaining one of these 3 matrices, say, $K_t$, and the matrix $A_t$ are affected by entry-wise uncertainty:

$$\{(A^M, K^M) : M \in \mathcal{M}\} = \left\{ (A, K) : \begin{array}{l} |A_{ij} - A_{ij}^{\rm n}| \leq \rho\alpha_{ij} \forall (i,j) \\ |K_{pq} - K_{pq}^{\rm n}| \leq \rho\kappa_{pq} \forall (p,q) \end{array} \right\},$$

In this case, denoting by $B^{\rm n}$, $C^{\rm n}$ the (certain!) matrices $B_t$, $C_t$, we clearly have

$$\begin{array}{rl} \mathrm{Conv}(\mathcal{Q}) = & \underbrace{A^{\rm n} + B^{\rm n} K^{\rm n} C^{\rm n}}_{Q^{\rm n}} + \rho \Big\{ [\sum\limits_{i,j} \xi_{ij} [\alpha_{ij} e_i e_j^T] \\ & + \sum\limits_{p,q} \eta_{pq} [\kappa_{pq} B^{\rm n} f_p g_q^T C^{\rm n}]] : |\xi_{ij}| \leq 1, |\eta_{pq}| \leq 1 \Big\}, \end{array}$$

where $e_i$, $f_p$, $g_q$ are the standard basic orths in the spaces $\mathbb{R}^{\dim x}$, $\mathbb{R}^{\dim u}$ and $\mathbb{R}^{\dim y}$, respectively. Note that the matrix coefficients at the "elementary perturbations" $\xi_{ij}$, $\eta_{pq}$ are of rank 1, and these perturbations, independently of each other, run through $[-1, 1]$ — exactly as required in (3.5.5) for $\mu = 1$.

In the situation of (3.5.5), the semi-infinite Lyapunov LMI

$$Q^T X + XQ \preceq -I \;\forall Q \in \mathrm{Conv}(\mathcal{Q})$$

in (3.4.24) reads

$$\begin{array}{l}\underbrace{-I - [Q^{\mathrm{n}}]^T X - XQ^{\mathrm{n}}}_{\mathcal{A}^{\mathrm{n}}(X)} + \rho \sum\limits_{\ell=1}^{L} \zeta_\ell \underbrace{[-U_\ell^T X - XU_\ell]}_{\mathcal{A}_\ell(X)} \\ \succeq 0 \;\forall (\zeta : |\zeta_\ell| \leq 1, \, \ell = 1, ..., L).\end{array} \tag{3.5.6}$$

We are in the case of structured norm-bounded perturbations with $\mathcal{I}_{\mathrm{s}} = \{1, ..., L\}$. Noting that the ranks of all matrices $\mathcal{A}_\ell(X)$ never exceed $2\mu$ (since all $U_\ell$ are of ranks $\leq \mu$), the safe tractable approximation of (3.5.6) given by Theorem 3.13 is tight within the factor $\vartheta(2\mu)$. It follows, in particular, that *in the case of* (3.5.5) *with* $\mu = 1$, *we can find efficiently a lower bound, tight within the factor* $\pi/2$, *on the Lyapunov Stability Radius of the uncertain system* (3.4.22) (that is, on the supremum of those $\rho$ for which the stability of our uncertain dynamical system can be certified by an LSC). The lower bound in question is the supremum of those $\rho$ for which the approximation is feasible, and this supremum can be easily approximated to whatever accuracy by bisection.

We can process in the same fashion the Lyapunov Stability Synthesis problem in the presence of interval uncertainty. Specifically, assume that $C_t \equiv I$ and the uncertainty set $\mathcal{AB} = \{[A^M, B^M] : M \in \mathcal{M}\}$ underlying the Synthesis problem is an interval uncertainty:

$$\mathcal{AB} = [A^{\mathrm{n}}, B^{\mathrm{n}}] + \rho \{\sum_{\ell=1}^{L} \zeta_\ell U_\ell : \|\zeta\|_\infty \leq 1\}, \quad \mathrm{Rank}(U_\ell) \leq \mu \,\forall \ell. \tag{3.5.7}$$

We arrive at the situation of (3.5.7) with $\mu = 1$, e.g., when $\mathcal{AB}$ corresponds to entry-wise uncertainty:

$$\mathcal{AB} = [A^{\mathrm{n}}, B^{\mathrm{n}}] + \rho \{H \equiv [\delta A, \delta B] : |H_{ij}| \leq h_{ij} \,\forall i, j\}.$$

In the case of (3.5.7) the semi-infinite LMI in (3.4.29) reads

$$\begin{array}{l}\underbrace{-I - [A^{\mathrm{n}}, B^{\mathrm{n}}][Y; Z] - [Y; Z]^T [A^{\mathrm{n}}, B^{\mathrm{n}}]^T}_{\mathcal{A}^{\mathrm{n}}(Y,Z)} \\ + \rho \sum\limits_{\ell=1}^{L} \zeta_\ell \underbrace{[-U_\ell [Y; Z] - [Y; Z]^T U_\ell^T]}_{\mathcal{A}_\ell(Y,Z)} \succeq 0 \,\forall (\zeta : |\zeta_\ell| \leq 1, \, \ell = 1, ..., L).\end{array} \tag{3.5.8}$$

We again reach a situation of structured norm-bounded uncertainty with $\mathcal{I}_{\mathrm{s}} = \{1, ..., L\}$ and all matrices $\mathcal{A}_\ell(\cdot)$, $\ell = 1, ..., L$, being of ranks at most $2\mu$. Thus, Theorem 3.13 provides us with a tight, within factor $\vartheta(2\mu)$, safe tractable approximation of the Lyapunov Stability Synthesis problem.

**Illustration: Controlling a multiple pendulum.** Consider a multiple pendulum ("a train") depicted in figure 3.6. Denoting by $m_i$, $i = 1, ..., 4$, the masses of the "engine" ($i = 1$) and the "cars" ($i = 2, 3, 4$, counting from right to left), Newton's laws for the dynamical system in question read

$$\begin{array}{rcrrrr} m_1 \frac{d^2}{dt^2} x_1(t) & = & -\kappa_1 x_1(t) & +\kappa_1 x_2(t) & & +u(t) \\ m_2 \frac{d^2}{dt^2} x_2(t) & = & \kappa_1 x_1(t) & -(\kappa_1 + \kappa_2) x_2(t) & +\kappa_2 x_3(t) & \\ m_3 \frac{d^2}{dt^2} x_3(t) & = & & \kappa_2 x_2(t) & -(\kappa_2 + \kappa_3) x_3(t) & +\kappa_3 x_4(t) \\ m_4 \frac{d^2}{dt^2} x_4(t) & = & & & \kappa_3 x_3(t) & -\kappa_3 x_4(t), \end{array} \tag{3.5.9}$$

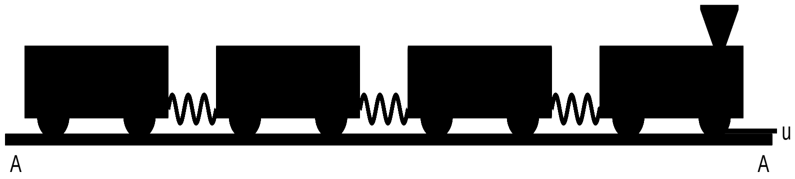



Figure 3.6: “Train”: 4 masses (3 “cars” and “engine”) linked by elastic springs and sliding without friction (aside of controlled force $u$) along “rail” AA.

where $x_i(t)$ are shifts of the engine and the cars from their respective positions in the state of rest (where nothing moves and the springs are neither shrunk nor expanded), and $\kappa_i$ are the elasticity constants of the springs (counted from right to left). Passing from masses $m_i$ to their reciprocals $\mu_i = 1/m_i$ and adding to the coordinates of the cars their velocities $v_i(t) = \dot{x}_i(t)$, we can rewrite (3.5.9) as the system of 8 linear differential equations:

$$
\begin{array}{rcl}
\dot{x}(t) & = & \underbrace{\left[\begin{array}{cccc|cccc}
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & & 1 \\ \hline
-\kappa_1\mu_1 & \kappa_1\mu_1 & & & & & & \\
\kappa_1\mu_2 & -[\kappa_1+\kappa_2]\mu_2 & \kappa_2\mu_2 & & & & & \\
 & \kappa_2\mu_3 & -[\kappa_2+\kappa_3]\mu_3 & \kappa_3\mu_3 & & & & \\
 & & \kappa_3\mu_4 & -\kappa_3\mu_4 & & & &
\end{array}\right]}_{A_\mu} x(t) \\
 & & + \underbrace{\left[\begin{array}{c} \\ \\ \\ \\ \hline \mu_1 \\ \\ \\ \\ \end{array}\right]}_{B_\mu} u(t)
\end{array}
\tag{3.5.10}
$$

where $x(t) = [x_1(t); x_2(t); x_3(t); x_4(t); v_1(t); v_2(t); v_3(t); v_4(t)]$. System (3.5.10) “as it is” (i.e., with trivial control $u(\cdot) \equiv 0$) is unstable; not only it has a solution that does not converge to 0 as $t \to \infty$, it has even an unbounded solution (specifically, one where $x_i(t) = vt$, $v_i(t) \equiv v$, which corresponds to uniform motion of the cars and the engine with no tensions in the springs). Let us look for a stabilizing state-based linear feedback controller

$$u(t) = Kx(t), \tag{3.5.11}$$

that is robust w.r.t. the masses of the cars and the engine when they vary in given segments $\Delta_i$, $i = 1, ..., 4$. To this end we can apply the Lyapunov Stability Synthesis machinery. Observe that to say that the masses $m_i$ run, independently of each other, through given segments is exactly the same as to say that their reciprocals $\mu_i$ run, independently of each other, through other given segments $\Delta'_i$; thus, our goal is as follows:

> **Stabilization:** *Given elasticity constants* $\kappa_i$ *and segments* $\Delta'_i \subset \{\mu > 0\}$, $i = 1, ..., 4$, *find a linear feedback* (3.5.11) *and a Lyapunov Stability Certificate* $X$ *for the*

*corresponding closed loop system* (3.5.10), (3.5.11), *with the uncertainty set for the system being*

$$\mathcal{AB} = \{[A_\mu, B_\mu] : \mu_i \in \Delta_i', \ i = 1, ..., 4\}.$$

Note that in our context the Lyapunov Stability Synthesis approach is, so to speak, "doubly conservative." First, the existence of a common LSC for all matrices $Q$ from a given compact set $\mathcal{Q}$ is only a *sufficient* condition for the stability of the uncertain dynamical system

$$\dot{x}(t) = Q_t x(t), \ \ Q_t \in \mathcal{Q} \forall t,$$

and as such this condition is conservative. Second, in our train example there are reasons to think of $m_i$ as of uncertain data (in reality the loads of the cars and the mass of the engine could vary from trip to trip, and we would not like to re-adjust the controller as long as these changes are within a reasonable range), but there is absolutely no reason to think of these masses as varying in time. Indeed, we could perhaps imagine a mechanism that makes the masses $m_i$ time-dependent, but with this mechanism our original model (3.5.9) becomes invalid — Newton's laws in the form of (3.5.9) are not applicable to systems with varying masses and at the very best they offer a reasonable approximation of the true model, provided that the changes in masses are slow. Thus, in our train example a common LSC for all matrices $Q = A + BK$, $[A, B] \in \mathcal{AB}$, would guarantee much more than required, namely, that all trajectories of the closed loop system "train plus feedback controller" converge to 0 as $t \to \infty$ even in the case when the parameters $\mu_i \in \Delta_i'$ vary in time at a high speed. This is much more than what we actually need — convergence to 0 of all trajectories in the case when $\mu_i \in \Delta_i'$ do not vary in time.

The system of semi-infinite LMIs we are about to process in the connection of the Lyapunov Stability Synthesis is

$$\begin{array}{ll} (a) & [A,B][Y;Z] + [Y;Z]^T[A,B]^T \preceq -\alpha Y, \ \forall [A,B] \in \mathcal{AB} \\ (b) & Y \succeq I \\ (c) & Y \leq \chi I, \end{array} \tag{3.5.12}$$

where $\alpha > 0$ and $\chi > 1$ are given. This system differs slightly from the "canonical" system (3.4.29), and the difference is twofold:

- [major] in (3.4.29), the semi-infinite Lyapunov LMI is written as

$$[A,B][Y;Z] + [Y;Z]^T[A,B]^T \preceq -I,$$

  which is just a convenient way to express the relation

$$[A,B][Y;Z] + [Y;Z]^T[A,B]^T \prec 0, \forall [A,B] \in \mathcal{AB}.$$

  Every feasible solution $[Y;Z]$ to this LMI with $Y \succ 0$ produces a stabilizing feedback $K = ZY^{-1}$ and the common LSC $X = Y^{-1}$ for all instances of the matrix $Q = A + BK$, $[A, B] \in \mathcal{AB}$, of the closed loop system, i.e.,

$$[A + BK]^T X + X[A + BK] \prec 0 \ \ \forall [A,B] \in \mathcal{AB}.$$

  The latter condition, however, says nothing about the corresponding decay rate. In contrast, when $[Y;Z]$ is feasible for (3.5.12.$a, b$), the associated stabilizing feedback $K = ZY^{-1}$ and LSC $X = Y^{-1}$ satisfy the relation

$$[A + BK]^T X + X[A + BK] \prec -\alpha X \ \ \forall [A,B] \in \mathcal{AB},$$

and this relation, as we have seen when introducing the Lyapunov Stability Certificate, implies that

$$x^T(t)Xx(t) \leq \exp\{-\alpha t\}x^T(0)Xx(0),\ t \geq 0,$$

which guarantees that the decay rate in the closed loop system is at least $\alpha$. In our illustration (same as in real life), we prefer to deal with this "stronger" form of the Lyapunov Stability Synthesis requirement, in order to have a control over the decay rate associated with the would-be controller.

- [minor] In (3.5.12) we impose an upper bound on the condition number (ratio of the maximal and minimal eigenvalues) of the would-be LSC; with normalization of $Y$ given by (3.5.12.$b$), this bound is ensured by (3.5.12.$c$) and is precisely $\chi$. The only purpose of this bound is to avoid working with extremely ill-conditioned positive definite matrices, which can cause numerical problems.

Now let us use Theorem 3.13 to get a tight safe tractable approximation of the semi-infinite system of LMIs (3.5.12). Denoting by $\mu_i^{\mathrm{n}}$ the midpoints of the segments $\Delta_i'$ and by $\delta_i$ the half-width of these segments, we have

$$\begin{array}{rcl} \mathcal{AB} & \equiv & \{[A_\mu, B_\mu] : \mu_i \in \Delta_i', i = 1, ..., 4\} \\ & = & \{[A_{\mu^{\mathrm{n}}}, B_{\mu^{\mathrm{n}}}] + \sum\limits_{\ell=1}^{4} \zeta_\ell U_\ell : |\zeta_\ell| \leq 1,\ \ell = 1, ..., 4\}, \\ U_\ell & = & \delta_\ell p_\ell q_\ell^T, \end{array}$$

where $p_\ell \in \mathbb{R}^8$ has the only nonzero entry, equal to 1, in the position $4 + \ell$, and

$$\left[\begin{array}{c} q_1^T \\ q_2^T \\ q_3^T \\ q_4^T \end{array}\right] = \left[\begin{array}{c|c|c|c|c|c|c|c|c} -\kappa_1 & \kappa_1 & & & & & & & 1 \\ \hline \kappa_1 & -[\kappa_1+\kappa_2] & \kappa_2 & & & & & & \\ \hline & \kappa_2 & -[\kappa_2+\kappa_3] & \kappa_3 & & & & & \\ \hline & & \kappa_3 & -\kappa_3 & & & & & \end{array}\right]$$

Consequently, the analogy of (3.5.12) with uncertainty level $\rho$ ((3.5.12) itself corresponds to $\rho = 1$) is the semi-infinite system of LMIs

$$\begin{array}{l} \underbrace{-\alpha Y - [A_{\mu^{\mathrm{n}}}, B_{\mu^{\mathrm{n}}}][Y; Z] - [Y; Z]^T [A_{\mu^{\mathrm{n}}}, B_{\mu^{\mathrm{n}}}]^T}_{\mathcal{A}^{\mathrm{n}}(Y,Z)} \\ +\rho \sum\limits_{\ell=1}^{4} \zeta_\ell (\underbrace{-\delta_\ell [p_\ell q_\ell^T [Y; Z] + [Y; Z]^T q_\ell p_\ell^T]}_{\mathcal{A}_\ell(Y,Z)}) \succeq 0\, \forall (\zeta : |\zeta_\ell| \leq 1,\ \ell = 1, ..., 4) \\ Y \succeq I_8,\ Y \preceq \chi I_8 \end{array} \qquad (3.5.13)$$

in variables $Y$, $Z$ (cf. (3.5.8)). The safe tractable approximation of this semi-infinite system of LMIs as given by Theorem 3.13 is the system of LMIs

$$\begin{array}{l} Y_\ell \succeq \pm \mathcal{A}_\ell(Y, Z),\ \ell = 1, ..., 4 \\ \mathcal{A}^{\mathrm{n}}(Y, Z) - \rho \sum\limits_{\ell=1}^{4} Y_\ell \succeq 0 \\ Y \succeq I_8,\ Y \preceq \chi I_8 \end{array} \qquad (3.5.14)$$

in variables $Y, Z, Y_1, ..., Y_4$. Since all $U_\ell$ are of rank 1 and therefore all $\mathcal{A}_\ell(Y, Z)$ are of rank $\leq 2$, Theorem 3.13 states that this safe approximation is tight within the factor $\pi/2$.

Of course, in our toy example no approximation is needed — the set $\mathcal{AB}$ is a polytopic uncertainty with just $2^4 = 16$ vertices, and we can straightforwardly convert (3.5.13) into an exactly equivalent system of 18 LMIs

$$\begin{array}{l}\mathcal{A}^{\rm n}(Y,Z) \succeq \rho \sum\limits_{\ell=1}^{4} \epsilon_\ell \mathcal{A}_\ell(Y,Z),\ \epsilon_\ell = \pm 1,\ \ell = 1,...,4 \\ Y \succeq I_8,\ Y \preceq \chi I_8\end{array}$$

in variables $Y, Z$. The situation would change dramatically if there were, say, 30 cars in our train rather than just 3. Indeed, in the latter case the precise "polytopic" approach would require solving a system of $2^{31} + 2 = 2,147,483,650$ LMIs of the size $62 \times 62$ in variables $Y \in \mathbf{S}^{62}, Z \in \mathbb{R}^{1\times 63}$, which is a bit too much... In contrast, the approximation (3.5.14) is a system of just $31 + 2 = 33$ LMIs of the size $62 \times 62$ in variables $\{Y_\ell \in \mathbf{S}^{62}\}_{\ell=1}^{31}$, $Y \in \mathbf{S}^{62}$, $Z \in \mathbb{R}^{1\times 63}$ (totally $(31+1)\frac{62\cdot 63}{2} + 63 = 60606$ scalar decision variables). One can argue that the latter problem still is too large from a practical perspective. But in fact it can be shown that in this problem, one can easily eliminate the matrices $Y_\ell$ (every one of them can be replaced with a *single* scalar decision variable, cf. Antenna example on p. 106), which reduces the design dimension of the approximation to $31 + \frac{62\cdot 63}{2} + 63 = 2047$. A convex problem of this size can be solved pretty routinely.

We are about to present numerical results related to stabilization of our toy 3-car train. The setup in our computations is as follows:

$$\begin{array}{c}\kappa_1 = \kappa_2 = \kappa_3 = 100.0; \alpha = 0.01; \chi = 10^8; \\ \Delta_1' = [0.5, 1.5], \Delta_2' = \Delta_3' = \Delta_4' = [1.5, 4.5],\end{array}$$

which corresponds to the mass of the engine varying in $[2/3, 2]$ and the masses of the cars varying in $[2/9, 2/3]$.

We computed, by a kind of bisection, the largest $\rho$ for which the approximation (3.5.14) is feasible; the optimal feedback we have found is

$$\begin{array}{rl}u = & 10^7 \big[ -0.2892x_1 - 2.5115x_2 + 6.3622x_3 - 3.5621x_4 \\ & \quad -0.0019v_1 - 0.0912v_2 - 0.0428v_3 + 0.1305v_4 \big],\end{array}$$

and the (lower bound on the) Lyapunov Stability radius of the closed loop system as yielded by our approximation is $\widehat{\rho} = 1.05473$. This bound is $> 1$, meaning that our feedback stabilizes the train in the above ranges of the masses of the engine and the cars (and in fact, even in slightly larger ranges $0.65 \leq m_1 \leq 2.11$, $0.22 \leq m_2, m_3, m_4 \leq 0.71$). An interesting question is by how much the *lower bound* $\widehat{\rho}$ is less than the Lyapunov Stability radius $\rho_*$ of the closed loop system. Theory guarantees that the ratio $\rho_*/\widehat{\rho}$ should be $\leq \pi/2 = 1.570...$. In our small problem we can compute $\rho_*$ by applying the polytopic uncertainty approach, that results in $\rho_* = 1.05624$. Thus, in reality $\rho_*/\widehat{\rho} \approx 1.0014$, much better than the theoretical bound 1.570.... In figure 3.7, we present sample trajectories of the closed loop system yielded by our design, the level of perturbations being 1.054 — pretty close to $\widehat{\rho} = 1.05473$.

## 3.6 Approximating Chance Constrained CQIs and LMIs

In the first reading this section can be skipped.

Below we develop safe tractable approximations of *chance constrained* randomly perturbed Conic Quadratic and Linear Matrix Inequalities. For omitted proofs, see [3].

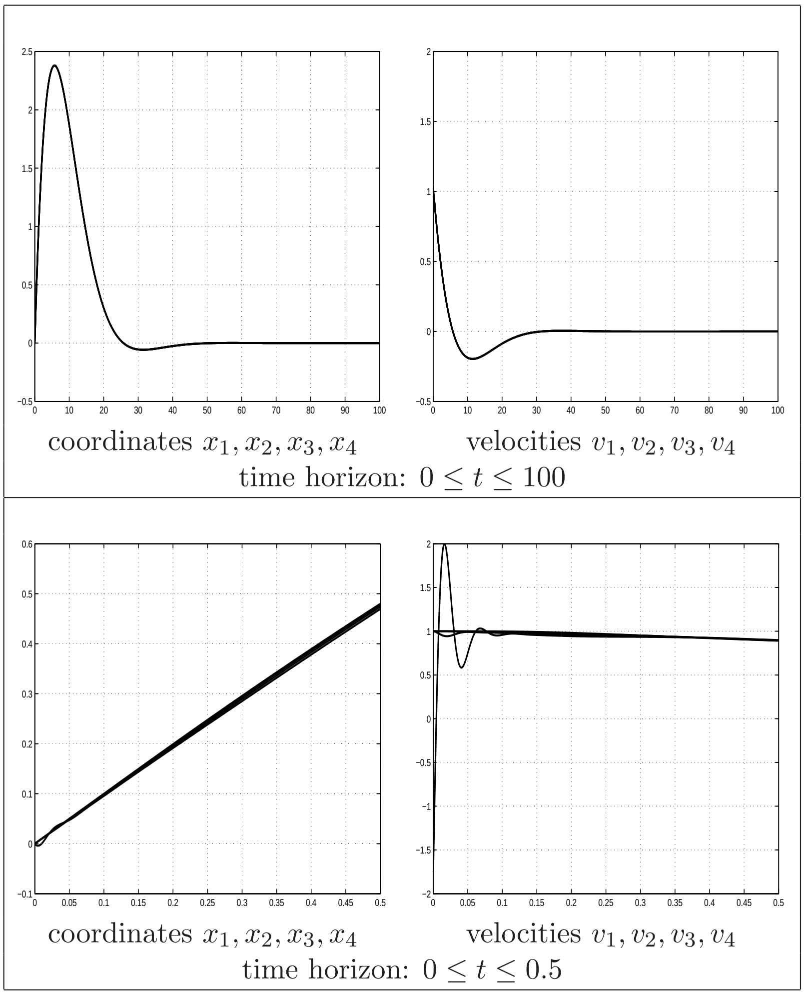



Figure 3.7: Sample trajectories of the 3-car train.

### 3.6.1 Chance Constrained LMIs

We have considered the Robust of uncertain conic quadratic and semidefinite programs. Now we intend to consider *randomly perturbed* CQPs and SDPs and to derive safe approximations of their chance constrained versions (cf. section 2.1). From this perspective, it is convenient to treat chance constrained CQPs as particular cases of chance constrained SDPs (such an option is given by Lemma 3.1), so that in the sequel we focus on chance constrained SDPs. Thus, we are interested in a randomly perturbed semidefinite program

$$\min_y \left\{ c^T y : \mathcal{A}^{\mathrm{n}}(y) + \rho \sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}^\ell(y) \succeq 0, y \in \mathcal{Y} \right\}, \tag{3.6.1}$$

where $\mathcal{A}^{\mathrm{n}}(y)$ and all $\mathcal{A}^\ell(y)$ are affine in $y$, $\rho \geq 0$ is the "perturbation level," $\zeta = [\zeta_1; ...; \zeta_L]$ is a random perturbation, and $\mathcal{Y}$ is a semidefinite representable set. We associate with this problem its *chance constrained* version

$$\min_y \left\{ c^T y : \mathrm{Prob}\left\{ \mathcal{A}^{\mathrm{n}}(y) + \rho \sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}^\ell(y) \succeq 0 \right\} \geq 1 - \epsilon, y \in \mathcal{Y} \right\} \tag{3.6.2}$$

where $\epsilon \ll 1$ is a given positive tolerance. Our goal is to build a computationally tractable safe approximation of (3.6.2). We start with assumptions on the random variables $\zeta_\ell$, which will be in force everywhere in the following:

> *Random variables $\zeta_\ell$, $\ell = 1, ..., L$, are independent with zero mean satisfying either*
> **A.I** ["bounded case"] $|\zeta_\ell| \leq 1$, $\ell = 1, ..., L$,
> *or*
> **A.II** ["Gaussian case"] $\zeta_\ell \sim \mathcal{N}(0, 1), \ell = 1, ..., L$.

Note that most of the results to follow can be extended to the case when $\zeta_\ell$ are independent with zero means and "light tail" distributions. We prefer to require more in order to avoid too many technicalities.

### Approximating Chance Constrained LMIs: Preliminaries

The problem we are facing is basically as follows:

> (?) *Given symmetric matrices $A, A_1$,...,$A_L$, find a verifiable sufficient condition for the relation*
> $$\text{Prob}\{\sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq A\} \geq 1 - \epsilon. \tag{3.6.3}$$

Since $\zeta$ is with zero mean, it is natural to require $A \succeq 0$ (this condition clearly is necessary when $\zeta$ is symmetrically distributed w.r.t. 0 and $\epsilon < 0.5$). Requiring a bit more, namely, $A \succ 0$, we can reduce the situation to the case when $A = I$, due to

$$\text{Prob}\{\sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq A\} = \text{Prob}\{\sum_{\ell=1}^{L} \zeta_\ell \underbrace{A^{-1/2} A_\ell A^{-1/2}}_{B_\ell} \preceq I\}. \tag{3.6.4}$$

Now let us try to guess a verifiable sufficient condition for the relation

$$\text{Prob}\{\sum_{\ell=1}^{L} \zeta_\ell B_\ell \preceq I\} \geq 1 - \epsilon. \tag{3.6.5}$$

First of all, we do not lose much when strengthening the latter relation to

$$\text{Prob}\{\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| \leq 1\} \geq 1 - \epsilon \tag{3.6.6}$$

(here and in what follows, $\|\cdot\|$ stands for the standard matrix norm $\|\cdot\|_{2,2}$). Indeed, the latter condition is nothing but

$$\text{Prob}\{-I \preceq \sum_{\ell=1}^{L} \zeta_\ell B_\ell \preceq I\} \geq 1 - \epsilon,$$

so that it implies (3.6.5). In the case of $\zeta$ symmetrically distributed w.r.t. the origin, we have a "nearly inverse" statement: the validity of (3.6.5) implies the validity of (3.6.6) with $\epsilon$ increased to $2\epsilon$.

The central observation is that whenever (3.6.6) holds true and the distribution of the random matrix

$$S = \sum_{\ell=1}^{L} \zeta_\ell B_\ell$$

is not pathological, we should have

$$\mathbf{E}\{\|S^2\|\} \leq O(1),$$

whence, by Jensen's Inequality,

$$\|\mathbf{E}\{S^2\}\| \leq O(1)$$

as well. Taking into account that $\mathbf{E}\{S^2\} = \sum_{\ell=1}^{L} \mathbf{E}\{\zeta_\ell^2\}B_\ell^2$, we conclude that when all quantities $\mathbf{E}\{\zeta_\ell^2\}$ are of order of 1, we should have $\|\sum_{\ell=1}^L B_\ell^2\| \leq O(1)$, or, which is the same,

$$\sum_{\ell=1}^{L} B_\ell^2 \preceq O(1)I. \tag{3.6.7}$$

By the above reasoning, (3.6.7) is a kind of a necessary condition for the validity of the chance constraint (3.6.6), at least for random variables $\zeta_\ell$ that are symmetrically distributed w.r.t. the origin and are "of order of 1." To some extent, this condition can be treated as nearly sufficient, as is shown by the following two theorems.

**Theorem 3.14** *Let $B_1, ..., B_L \in \mathbf{S}^m$ be deterministic matrices such that*

$$\sum_{\ell=1}^{L} B_\ell^2 \preceq I \tag{3.6.8}$$

*and $\Upsilon > 0$ be a deterministic real. Let, further, $\zeta_\ell$, $\ell = 1, ..., L$, be independent random variables taking values in $[-1, 1]$ such that*

$$\chi \equiv \mathrm{Prob}\left\{\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| \leq \Upsilon\right\} > 0. \tag{3.6.9}$$

*Then*

$$\forall \Omega > \Upsilon : \mathrm{Prob}\left\{\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| > \Omega\right\} \leq \frac{1}{\chi}\exp\{-(\Omega - \Upsilon)^2/16\}. \tag{3.6.10}$$

**Proof.** Let $Q = \{z \in \mathbb{R}^L : \|\sum_\ell z_\ell B_\ell\| \leq 1\}$. Observe that

$$\|[\sum_\ell z_\ell B_\ell]u\|_2 \leq \sum_\ell |z_\ell|\|B_\ell u\|_2 \leq \left(\sum_\ell z_\ell^2\right)^{1/2}\left(\sum_\ell u^T B_\ell^2 u\right)^{1/2} \leq \|z\|_2\|u\|_2,$$

where the concluding relation is given by (3.6.8). It follows that $\|\sum_\ell z_\ell B_\ell\| \leq \|z\|_2$, whence $Q$ contains the unit $\|\cdot\|_2$-ball $B$ centered at the origin in $\mathbb{R}^L$. Besides this, $Q$ is clearly closed, convex and symmetric w.r.t. the origin. Invoking the Talagrand Inequality (Theorem A.9), we have

$$\mathbf{E}\left\{\exp\{\mathrm{dist}^2_{\|\cdot\|_2}(\zeta, \Upsilon Q)/16\}\right\} \leq (\mathrm{Prob}\{\zeta \in \Upsilon Q\})^{-1} = \frac{1}{\chi}. \tag{3.6.11}$$

Now, when $\zeta$ is such that $\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| > \Omega$, we have $\zeta \notin \Omega Q$, whence, due to symmetry and convexity of $Q$, the set $(\Omega - \Upsilon)Q + \zeta$ does not intersect the set $\Upsilon Q$. Since $Q$ contains $B$, the set $(\Omega - \Upsilon)Q + \zeta$ contains $\|\cdot\|_2$-ball, centered at $\zeta$, of the radius $\Omega - \Upsilon$, and therefore this ball does not intersect $\Upsilon Q$ either, whence $\mathrm{dist}_{\|\cdot\|_2}(\zeta, \Upsilon Q) > \Omega - \Upsilon$. The resulting relation

$$\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| > \Omega \Leftrightarrow \zeta \notin \Omega Q \Rightarrow \mathrm{dist}_{\|\cdot\|_2}(\zeta, \Upsilon Q) > \Omega - \Upsilon$$

combines with (3.6.11) and the Tschebyshev Inequality to imply that

$$\mathrm{Prob}\{\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| > \Omega\} \leq \frac{1}{\chi}\exp\{-(\Omega - \Upsilon)^2/16\}. \qquad \square$$

**Theorem 3.15** *Let $B_1,...,B_L \in \mathbf{S}^m$ be deterministic matrices satisfying* (3.6.8) *and $\Upsilon > 0$ be a deterministic real. Let, further, $\zeta_\ell$, $\ell = 1,...,L$, be independent $\mathcal{N}(0,1)$ random variables such that* (3.6.9) *holds true with $\chi > 1/2$.*

*Then*

$$\begin{array}{l} \forall \Omega \geq \Upsilon : \mathrm{Prob}\{\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| > \Omega\} \\ \quad \leq \mathrm{Erf}\left(\mathrm{ErfInv}(1-\chi) + (\Omega - \Upsilon)\max[1, \Upsilon^{-1}\mathrm{ErfInv}(1-\chi)]\right) \\ \quad \leq \exp\{-\frac{\Omega^2\Upsilon^{-2}\mathrm{ErfInv}^2(1-\chi)}{2}\}, \end{array} \tag{3.6.12}$$

*where* $\mathrm{Erf}(\cdot)$ *and* $\mathrm{ErfInv}(\cdot)$ *are the error and the inverse error functions, see* (2.2.6), (2.2.7).

**Proof.** Let $Q = \{z \in \mathbb{R}^L : \|\sum_\ell z_\ell B_\ell\| \leq \Upsilon\}$. By the same argument as in the beginning of the proof of Theorem 3.14, $Q$ contains the centered at the origin $\|\cdot\|_2$-ball of the radius $\Upsilon$. Besides this, by definition of $Q$ we have $\mathrm{Prob}\{\zeta \in Q\} \geq \chi$. Invoking item (i) of Theorem A.10, $Q$ contains the centered at the origin $\|\cdot\|_2$-ball of the radius $r = \max[\mathrm{ErfInv}(1-\chi), \Upsilon]$, whence, by item (ii) of this Theorem, (3.6.12) holds true. □

The last two results are stated next in a form that is better suited for our purposes.

**Corollary 3.1** *Let $A, A_1, ..., A_L$ be deterministic matrices from $\mathbf{S}^m$ such that*

$$\exists \{Y_\ell\}_{\ell=1}^L : \left\{ \begin{array}{l} \left[\begin{array}{c|c} Y_\ell & A_\ell \\ \hline A_\ell & A \end{array}\right] \succeq 0,\ 1 \leq \ell \leq L \\ \sum_{\ell=1}^{L} Y_\ell \preceq A \end{array} \right., \tag{3.6.13}$$

*let $\Upsilon > 0$, $\chi > 0$ be deterministic reals and $\zeta_1, ..., \zeta_L$ be independent random variables satisfying either* **A.I***, or* **A.II***, and such that*

$$\mathrm{Prob}\{-\Upsilon A \preceq \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq \Upsilon A\} \geq \chi. \tag{3.6.14}$$

*Then*

(i) *When $\zeta_\ell$ satisfy* **A.I***, we have*

$$\forall \Omega > \Upsilon : \mathrm{Prob}\{-\Omega A \preceq \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq \Omega A\} \geq 1 - \frac{1}{\chi}\exp\{-(\Omega - \Upsilon)^2/16\}; \tag{3.6.15}$$

(ii) *When $\zeta_\ell$ satisfy* **A.II***, and, in addition, $\chi > 0.5$, we have*

$$\begin{array}{l} \forall \Omega > \Upsilon : \mathrm{Prob}\{-\Omega A \preceq \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq \Omega A\} \\ \quad \geq 1 - \mathrm{Erf}\left(\mathrm{ErfInv}(1-\chi) + (\Omega - \Upsilon)\max\left[1, \frac{\mathrm{ErfInv}(1-\chi)}{\Upsilon}\right]\right), \end{array} \tag{3.6.16}$$

*with* $\mathrm{Erf}(\cdot)$, $\mathrm{ErfInv}(\cdot)$ *given by* (2.2.6), (2.2.7).

**Proof.** Let us prove (i). Given positive $\delta$, let us set $A^\delta = A + \delta I$. Observe that the premise in (3.6.14) clearly implies that $A \succeq 0$, whence $A^\delta \succ 0$. Now let $Y_\ell$ be such that the conclusion in (3.6.13) holds true. Then $\left[\begin{array}{c|c} Y_\ell & A_\ell \\ \hline A_\ell & A^\delta \end{array}\right] \succeq 0$, whence, by the Schur Complement Lemma, $Y_\ell \succeq A_\ell[A^\delta]^{-1}A_\ell$, so that

$$\sum_\ell A_\ell[A^\delta]^{-1}A_\ell \preceq \sum_\ell Y_\ell \preceq A \preceq A^\delta.$$

We see that

$$\sum_\ell \big[\underbrace{[A^\delta]^{-1/2}A_\ell[A^\delta]^{-1/2}}_{B_\ell^\delta}\big]^2 \preceq I.$$

Further, relation (3.6.14) clearly implies that

$$\mathrm{Prob}\{-\Upsilon A^\delta \preceq \sum_\ell \zeta_\ell A_\ell \preceq \Upsilon A^\delta\} \geq \chi,$$

or, which is the same,

$$\mathrm{Prob}\{-\Upsilon I \preceq \sum_\ell \zeta_\ell B_\ell^\delta \preceq \Upsilon I\} \geq \chi.$$

Applying Theorem 3.14, we conclude that

$$\Omega > \Upsilon \Rightarrow \mathrm{Prob}\{-\Omega I \preceq \sum_\ell \zeta_\ell B_\ell^\delta \preceq \Omega I\} \geq 1 - \frac{1}{\chi}\exp\{-(\Omega-\Upsilon)^2/16\},$$

which in view of the structure of $B_\ell^\delta$ is the same as

$$\Omega > \Upsilon \Rightarrow \mathrm{Prob}\{-\Omega A^\delta \preceq \sum_\ell \zeta_\ell A_\ell \preceq \Omega A^\delta\} \geq 1 - \frac{1}{\chi}\exp\{-(\Omega-\Upsilon)^2/16\}. \tag{3.6.17}$$

For every $\Omega > \Upsilon$, the sets $\{\zeta : -\Omega A^{1/t} \preceq \sum\limits_\ell \zeta_\ell A_\ell \preceq \Omega A^{1/t}\}$, $t = 1, 2, ...$, shrink as $t$ grows, and their intersection over $t = 1, 2, ...$ is the set $\{\zeta : -\Omega A \preceq \sum\limits_\ell \zeta_\ell A_\ell \preceq \Omega A\}$, so that (3.6.17) implies (3.6.15), and (i) is proved. The proof of (ii) is completely similar, with Theorem 3.15 in the role of Theorem 3.14. □

**Comments.** When $A \succ 0$, invoking the Schur Complement Lemma, the condition (3.6.13) is satisfied iff it is satisfied with $Y_\ell = A_\ell A^{-1} A_\ell$, which in turn is the case iff $\sum\limits_\ell A_\ell A^{-1} A_\ell \preceq A$, or which is the same, iff $\sum\limits_\ell [A^{-1/2} A_\ell A^{-1/2}]^2 \preceq I$. Thus, *condition* (3.6.4), (3.6.7) *introduced in connection with Problem* (?), *treated as a condition on the variable symmetric matrices $A, A_1, ..., A_L$, is LMI-representable,* (3.6.13) *being the representation.* Further, (3.6.13) can be written as the following explicit LMI on the matrices $A, A_1, ..., A_L$:

$$\mathrm{Arrow}(A, A_1, ..., A_L) \equiv \left[\begin{array}{c|ccc} A & A_1 & \dots & A_L \\ \hline A_1 & A & & \\ \vdots & & \ddots & \\ A_L & & & A \end{array}\right] \succeq 0. \tag{3.6.18}$$

Indeed, when $A \succ 0$, the Schur Complement Lemma says that the matrix $\mathrm{Arrow}(A, A_1, ..., A_L)$ is $\succeq 0$ if and only if

$$\sum_\ell A_\ell A^{-1} A_\ell \preceq A,$$

and this is the case if and only if (3.6.13) holds. Thus, (3.6.13) and (3.6.18) are equivalent to each other when $A \succ 0$, which, by standard approximation argument, implies the equivalence of these two properties in the general case (that is, when $A \succeq 0$). It is worthy of noting that the set of matrices $(A, A_1, ..., A_L)$ satisfying (3.6.18) form a cone that can be considered as the matrix analogy of the Lorentz cone (look what happens when all the matrices are $1 \times 1$ ones).

### 3.6.2 The Approximation Scheme

To utilize the outlined observations and results in order to build a safe/"almost safe" tractable approximation of a chance constrained LMI in (3.6.2), we proceed as follows.

**1)** We introduce the following:

**Conjecture 3.1** *Under assumptions* **A.I** *or* **A.II***, condition* (3.6.13) *implies the validity of* (3.6.14) *with known in advance $\chi > 1/2$ and "a moderate" (also known in advance) $\Upsilon > 0$.*

With properly chosen $\chi$ and $\Upsilon$, this Conjecture indeed is true, see below. We, however, prefer not to stick to the corresponding worst-case-oriented values of $\chi$ and $\Upsilon$ and consider $\chi > 1/2$, $\Upsilon > 0$ as somehow chosen parameters of the construction to follow, and we proceed *as if* we know in advance that our conjecture, with the chosen $\Upsilon$, $\chi$, is true. Eventually we shall explain how to justify this tactics.

**2)** Trusting in Conjecture 3.1, we have at our disposal constants $\Upsilon > 0$, $\chi \in (0.5, 1]$ such that (3.6.13) implies (3.6.14). We claim that *modulo Conjecture 3.1, the following systems of LMIs in variables $y, U_1, ..., U_L$ are safe tractable approximations of the chance constrained LMI in* (3.6.2):

*In the case of* **A.I**:

$$\begin{array}{ll} (a) & \left[\begin{array}{c|c} U_\ell & \mathcal{A}^\ell(y) \\ \hline \mathcal{A}^\ell(y) & \mathcal{A}^{\rm n}(y) \end{array}\right] \succeq 0,\ 1 \le \ell \le L \\ (b) & \rho^2 \sum\limits_{\ell=1}^{L} U_\ell \preceq \Omega^{-2}\mathcal{A}^{\rm n}(y),\ \Omega = \Upsilon + 4\sqrt{\ln(\chi^{-1}\epsilon^{-1})}; \end{array} \tag{3.6.19}$$

*In the case of* **A.II**:

$$\begin{array}{ll} (a) & \left[\begin{array}{c|c} U_\ell & \mathcal{A}^\ell(y) \\ \hline \mathcal{A}^\ell(y) & \mathcal{A}^{\rm n}(y) \end{array}\right] \succeq 0,\ 1 \le \ell \le L \\ (b) & \rho^2 \sum\limits_{\ell=1}^{L} U_\ell \preceq \Omega^{-2}\mathcal{A}^{\rm n}(y),\ \Omega = \Upsilon + \frac{\max[\mathrm{ErfInv}(\epsilon) - \mathrm{ErfInv}(1-\chi), 0]}{\max[1, \Upsilon^{-1}\mathrm{ErfInv}(1-\chi)]} \\ & \le \Upsilon + \max\left[\mathrm{ErfInv}(\epsilon) - \mathrm{ErfInv}(1-\chi), 0\right]. \end{array} \tag{3.6.20}$$

Indeed, assume that $y$ can be extended to a feasible solution $(y, U_1, ..., U_L)$ of (3.6.19). Let us set $A = \Omega^{-1}\mathcal{A}^{\rm n}(y)$, $A_\ell = \rho\mathcal{A}^\ell(y)$, $Y_\ell = \Omega\rho^2 U_\ell$. Then $\left[\begin{array}{c|c} Y_\ell & A_\ell \\ \hline A_\ell & A \end{array}\right] \succeq 0$ and $\sum\limits_\ell Y_\ell \preceq A$ by (3.6.19). Applying Conjecture 3.1 to the matrices $A, A_1, ..., A_L$, we conclude that (3.6.14) holds true as well. Applying Corollary 3.1.(i), we get

$$\begin{array}{l} \mathrm{Prob}\left\{\rho\sum\limits_\ell \zeta_\ell \mathcal{A}^\ell(y) \not\preceq \mathcal{A}^{\rm n}(y)\right\} = \mathrm{Prob}\left\{\sum\limits_\ell \zeta_\ell A_\ell \not\preceq \Omega A\right\} \\ \le \chi^{-1}\exp\{-(\Omega-\Upsilon)^2/16\} = \epsilon, \end{array}$$

as claimed.

Relation (3.6.20) can be justified, modulo the validity of Conjecture 3.1, in the same fashion, with item (ii) of Corollary 3.1 in the role of item (i).

**3)** We replace the chance constrained LMI problem (3.6.2) with the outlined safe (modulo the validity of Conjecture 3.1) approximation, thus arriving at the approximating problem

$$\min_{y, \{U_\ell\}} \left\{ c^T y : \begin{array}{l} \left[\begin{array}{c|c} U_\ell & \mathcal{A}^\ell(y) \\ \hline \mathcal{A}^\ell(y) & \mathcal{A}^{\rm n}(y) \end{array}\right] \succeq 0,\ 1 \le \ell \le L \\ \\ \rho^2 \sum\limits_\ell U_\ell \preceq \Omega^{-2}\mathcal{A}^{\rm n}(y),\ y \in \mathcal{Y} \end{array} \right\}, \tag{3.6.21}$$

where $\Omega$ is given by the required tolerance *and our guesses for* $\Upsilon$ *and* $\chi$ according to (3.6.19) or (3.6.20), depending on whether we are in the case of a bounded random perturbation model (Assumption **A.I**) or a Gaussian one (Assumption **A.II**).

We solve the approximating SDO problem and obtain its optimal solution $y_*$. If (3.6.21) were indeed a safe approximation of (3.6.2), we would be done: $y_*$ would be a *feasible* suboptimal solution to the chance constrained problem of interest. However, since we are not sure of the validity of Conjecture 3.1, we need an additional phase — *post-optimality analysis* — aimed at justifying the feasibility of $y_*$ for the chance constrained problem. Note that *at this phase, we should not bother about the validity of Conjecture 3.1 in full generality — all we need is to justify the validity of the relation*

$$\mathrm{Prob}\{-\Upsilon A \preceq \sum_\ell \zeta_\ell A_\ell \preceq \Upsilon A\} \ge \chi \tag{3.6.22}$$

*for specific matrices*

$$A = \Omega^{-1}\mathcal{A}^{\rm n}(y_*),\ A_\ell = \rho\mathcal{A}^\ell(y_*),\ \ell = 1, ..., L, \tag{3.6.23}$$

*which we have in our disposal after $y_*$ is found, and which indeed satisfy* (3.6.13) (cf. "justification" of approximations (3.6.19), (3.6.20) in item 2)).

In principle, there are several ways to justify (3.6.22):

1. Under certain structural assumptions on the matrices $A$, $A_\ell$ and with properly chosen $\chi, \Upsilon$, our Conjecture 3.1 is provably true. Specifically, we shall see in section 3.6.4 that:
   (a) when $A$, $A_\ell$ are diagonal, (which corresponds to the semidefinite reformulation of a Linear Optimization problem), Conjecture 3.1 holds true with $\chi = 0.75$ and $\Upsilon = \sqrt{3\ln(8m)}$ (recall that $m$ is the size of the matrices $A, A_1, ..., A_L$);
   (b) when $A$, $A_\ell$ are arrow matrices, (which corresponds to the semidefinite reformulation of a conic quadratic problem), Conjecture 3.1 holds true with $\chi = 0.75$ and $\Upsilon = 4\sqrt{2}$.
2. Utilizing deep results from Functional Analysis, it can be proved (see [3, Proposition B.5.2]) that Conjecture 3.1 is true *for all matrices* $A, A_1, ..., A_L$ when $\chi = 0.75$ and $\Upsilon = 4\sqrt{\ln\max[m, 3]}$. It should be added that in order for our Conjecture 3.1 to be true for all $L$ and all $m \times m$ matrices $A, A_1, ..., A_L$ with $\chi$ not too small, $\Upsilon$ should be *at least* $O(1)\sqrt{\ln m}$ with appropriate positive absolute constant $O(1)$.

In view of the above facts, we could *in principle* avoid the necessity to rely on any conjecture. However, the "theoretically valid" values of $\Upsilon, \chi$ are *by definition* worst-case oriented and can be too conservative for the particular matrices we are interested in. The situation is even worse: these theoretically valid values reflect not the worst case "as it is," but rather our abilities to analyze this worst case and therefore are conservative estimates of the "true" (and already conservative) $\Upsilon, \chi$. This is why we prefer to use a technique that is based on *guessing* $\Upsilon, \chi$ and a subsequent "verification of the guess" by a *simulation-based* justification of (3.6.22).

**Comments.** Note that our proposed course of action is completely similar to what we did in section 2.2. The essence of the matter there was as follows: we were interested in building a safe approximation of the chance constraint

$$\sum_{\ell=1}^{L} \zeta_\ell a_\ell \le a \tag{3.6.24}$$

with deterministic $a, a_1, ..., a_L \in \mathbb{R}$ and random $\zeta_\ell$ satisfying Assumption **A.I**. To this end, we used the *provable fact* expressed by Proposition 2.1:

> *Whenever random variables $\zeta_1, ..., \zeta_L$ satisfy* **A.I** *and deterministic reals $b, a_1, ..., a_L$ are such that*
>
> $$\sqrt{\sum_{\ell=1}^{L} a_\ell^2} \le b,$$
>
> *or, which is the same,*
>
> $$\mathrm{Arrow}(b, a_1, ..., a_L) \equiv \left[\begin{array}{c|ccc} b & a_1 & \cdots & a_L \\ \hline a_1 & b & & \\ \vdots & & \ddots & \\ a_L & & & b \end{array}\right] \succeq 0,$$
>
> *one has*
>
> $$\forall \Omega > 0 : \mathrm{Prob}\left\{\sum_{\ell=1}^{L} \zeta_\ell a_\ell \le \Omega b\right\} \ge 1 - \psi(\Omega),$$
> $$\psi(\Omega) = \exp\{-\Omega^2/2\}.$$

As a result, the condition

$$\mathrm{Arrow}(\Omega^{-1}a, a_1, ..., a_L) \equiv \left[\begin{array}{c|ccc} \Omega^{-1}a & a_1 & ... & a_L \\ \hline a_1 & \Omega^{-1}a & & \\ \vdots & & \ddots & \\ a_L & & & \Omega^{-1}a \end{array}\right] \succeq 0$$

is sufficient for the validity of the chance constraint

$$\mathrm{Prob}\left\{\sum_\ell \zeta_\ell a_\ell \leq a\right\} \geq 1 - \psi(\Omega).$$

What we are doing under Assumption **A.I** now can be sketched as follows: we are interested in building a safe approximation of the chance constraint

$$\sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq A \tag{3.6.25}$$

with deterministic $A, A_1, ..., A_L \in \mathbf{S}^m$ and random $\zeta_\ell$ satisfying Assumption **A.I**. To this end, we use the following *provable fact* expressed by Theorem 3.14:

*Whenever random variables $\zeta_1, ..., \zeta_L$ satisfy* **A.I** *and deterministic symmetric matrices $B, A_1, ..., A_L$ are such that*

$$\mathrm{Arrow}(B, A_1, ..., A_L) \equiv \left[\begin{array}{c|ccc} B & A_1 & ... & A_L \\ \hline A_1 & B & & \\ \vdots & & \ddots & \\ A_L & & & B \end{array}\right] \succeq 0, \tag{!}$$

*and*

$$\mathrm{Prob}\{-\Upsilon B \preceq \sum_\ell \zeta_\ell A_\ell \preceq \Upsilon B\} \geq \chi \tag{$*$}$$

*with certain $\chi, \Upsilon > 0$, one has*

$$\begin{array}{c} \forall \Omega > \Upsilon : \mathrm{Prob}\left\{\sum\limits_{\ell=1}^{L} \zeta_\ell A_\ell \preceq \Omega B\right\} \geq 1 - \psi_{\Upsilon,\chi}(\Omega), \\ \psi_{\Upsilon,\chi}(\Omega) = \chi^{-1} \exp\{-(\Omega - \Upsilon)^2/16\}. \end{array}$$

As a result, the condition

$$\mathrm{Arrow}(\Omega^{-1}A, A_1, ..., A_L) \equiv \left[\begin{array}{c|ccc} \Omega^{-1}A & A_1 & ... & A_L \\ \hline A_1 & \Omega^{-1}A & & \\ \vdots & & \ddots & \\ A_L & & & \Omega^{-1}A \end{array}\right] \succeq 0$$

is a sufficient condition for the validity of the chance constraint

$$\mathrm{Prob}\left\{\sum_\ell \zeta_\ell A_\ell \preceq A\right\} \geq 1 - \psi_{\Upsilon,\chi}(\Omega),$$

*provided that $\Omega > \Upsilon$ and $\chi > 0$, $\Upsilon > 0$ are such that the matrices $B, A_1, ..., A_L$ satisfy* $(*)$.

The constructions are pretty similar; the only difference is that in the matrix case we need an additional "provided that," which is absent in the scalar case. In fact, it is automatically present in the scalar case: from the Tschebyshev Inequality it follows that when $B, A_1, ..., A_L$ are scalars, condition (!) implies the validity of ($*$) with, say, $\chi = 0.75$ and $\Upsilon = 2$. We now could apply the matrix-case result to recover the scalar-case, at the cost of replacing $\psi(\Omega)$ with $\psi_{2,0.75}(\Omega)$, which is not that big a loss.

Conjecture 3.1 *suggests* that in the matrix case we also should not bother much about "provided that" — it is automatically implied by (!), perhaps with a somehow worse value of $\Upsilon$, but still not too large. As it was already mentioned, we can prove certain versions of the Conjecture, and we can also verify its validity, for *guessed* $\chi$, $\Upsilon$ and matrices $B, A_1, ..., A_L$ that we are interested in, by simulation. The latter is the issue we consider next.

### Simulation-Based Justification of (3.6.22)

Let us start with the following simple situation: there exists a random variable $\xi$ taking value 1 with probability $p$ and value 0 with probability $1 - p$; we can simulate $\xi$, that is, for every sample size $N$, observe realizations $\xi^N = (\xi_1, ..., \xi_N)$ of $N$ independent copies of $\xi$. We do not know $p$, and our goal is to infer a reliable lower bound on this quantity from simulations. The simplest way to do this is as follows: given "reliability tolerance" $\delta \in (0, 1)$, a sample size $N$ and an integer $L$, $0 \le L \le N$, let

$$\widehat{p}_{N,\delta}(L) = \min\left\{q \in [0,1] : \sum_{k=L}^{N} \binom{N}{k} q^k (1-q)^{N-k} \ge \delta\right\}.$$

The interpretation of $\widehat{p}_{N,\delta}(L)$ is as follows: imagine we are flipping a coin, and let $q$ be the probability to get heads. We restrict $q$ to induce chances at least $\delta$ to get $L$ or more heads when flipping the coin $N$ times, and $\widehat{p}_{N,\delta}(L)$ is exactly the smallest of these probabilities $q$. Observe that

$$(L > 0, \widehat{p} = \widehat{p}_{N,\delta}(L)) \Rightarrow \sum_{k=L}^{N} \binom{N}{k} \widehat{p}^k (1-\widehat{p})^{N-k} = \delta \qquad (3.6.26)$$

and that $\widehat{p}_{N,\delta}(0) = 0$.

An immediate observation is as follows:

**Lemma 3.4** *For a fixed $N$, let $L(\xi^N)$ be the number of ones in a sample $\xi^N$, and let*

$$\widehat{p}(\xi^N) = \widehat{p}_{N,\delta}(L(\xi^N)).$$

*Then*

$$\text{Prob}\{\widehat{p}(\xi^N) > p\} \le \delta. \qquad (3.6.27)$$

**Proof.** Let

$$M(p) = \min\left\{\mu \in \{0, 1, ..., N\} : \sum_{k=\mu+1}^{N} \binom{N}{k} p^k (1-p)^{N-k} \le \delta\right\}$$

(as always, a sum over empty set of indices is 0) and let $\Theta$ be the event $\{\xi^N : L(\xi^N) > M(p)\}$, so that by construction

$$\text{Prob}\{\Theta\} \le \delta.$$

Now, the function

$$f(q) = \sum_{k=M(p)}^{N} \binom{N}{k} q^k (1-q)^{N-k}$$

is a nondecreasing function of $q \in [0, 1]$, and by construction $f(p) > \delta$; it follows that if $\xi^N$ is such that $\widehat{p} \equiv \widehat{p}(\xi^N) > p$, then $f(\widehat{p}) > \delta$ as well:

$$\sum_{k=M(p)}^{N} \binom{N}{k} \widehat{p}^k (1-\widehat{p})^{N-k} > \delta \qquad (3.6.28)$$

and, besides this, $L(\xi^N) > 0$ (since otherwise $\widehat{p} = \widehat{p}_{N,\delta}(0) = 0 \le p$). Since $L(\xi^N) > 0$, we conclude from (3.6.26) that

$$\sum_{k=L(\xi^N)}^{N} \binom{N}{k} \widehat{p}^k (1-\widehat{p})^{N-k} = \delta,$$

which combines with (3.6.28) to imply that $L(\xi^N) > M(p)$, that is, $\xi^N$ in question is such that the event $\Theta$ takes place. The bottom line is: the probability of the event $\widehat{p}(\xi^N) > p$ is at most the probability of $\Theta$, and the latter, as we remember, is $\le \delta$. $\square$

Lemma 3.4 says that the simulation-based (and thus random) quantity $\widehat{p}(\xi^N)$ is, with probability at least $1-\delta$, a *lower bound* for unknown probability $p \equiv \mathrm{Prob}\{\xi = 1\}$. When $p$ is not small, this bound is reasonably good already for moderate $N$, even when $\delta$ is extremely small, say, $\delta = 10^{-10}$. For example, here are simulation results for $p = 0.8$ and $\delta = 10^{-10}$:

| $N$ | 10 | 100 | 1,000 | 10,000 | 100,000 |
|---|---|---|---|---|---|
| $\widehat{p}$ | 0.06032 | 0.5211 | 0.6992 | 0.7814 | 0.7908 |

Coming back to our chance constrained problem (3.6.2), we can now use the outlined bounding scheme in order to carry out post-optimality analysis, namely, as follows:

**Acceptance Test**: *Given a reliability tolerance $\delta \in (0,1)$, guessed $\Upsilon$, $\chi$ and a solution $y_*$ to the associated problem* (3.6.21), *build the matrices* (3.6.23). *Choose an integer $N$, generate a sample of $N$ independent realizations $\zeta^1, ..., \zeta^N$ of the random vector $\zeta$, compute the quantity*

$$L = \mathrm{Card}\{i : -\Upsilon A \preceq \sum_{\ell=1}^{L} \zeta_\ell^i A_\ell \preceq \Upsilon A\}$$

*and set*

$$\widehat{\chi} = \widehat{p}_{N,\delta}(L).$$

*If $\widehat{\chi} \ge \chi$, accept $y_*$, that is, claim that $y_*$ is a feasible solution to the chance constrained problem of interest* (3.6.2).

By the above analysis, the random quantity $\widehat{\chi}$ is, with probability $\ge 1-\delta$, a lower bound on $p \equiv \mathrm{Prob}\{-\Upsilon A \preceq \sum_\ell \zeta_\ell A_\ell \preceq \Upsilon A\}$, so that the probability to accept $y_*$ in the case when $p < \chi$ is at most $\delta$. When this "rare event" does not occur, the relation (3.6.22) is satisfied, and therefore $y_*$ is indeed feasible for the chance constrained problem. In other words, the probability to accept $y_*$ when it is *not* a feasible solution to the problem of interest is at most $\delta$.

The outlined scheme does not say what to do if $y_*$ does *not* pass the Acceptance Test. A naive approach would be to check whether $y_*$ satisfies the chance constraint by direct simulation. This approach indeed is workable when $\epsilon$ is not too small (say, $\epsilon \ge 0.001$); for small $\epsilon$, however, it would require an unrealistically large simulation sample. A practical alternative is to resolve the approximating problem with $\Upsilon$ increased by a reasonable factor (say, 1.1 or 2), and to repeat this "trial and error" process until the Acceptance Test is passed.

### A Modification

The outlined approach can be somehow streamlined when applied to a slightly modified problem (3.6.2), specifically, to the problem

$$\max_{\rho,y}\left\{\rho : \mathrm{Prob}\left\{\mathcal{A}^{\mathrm{n}}(y) + \rho\sum_{\ell=1}^{L}\zeta_\ell \mathcal{A}^\ell(y) \succeq 0\right\} \ge 1-\epsilon, c^T y \le \tau_*, y \in \mathcal{Y}\right\} \tag{3.6.29}$$

where $\tau_*$ is a given upper bound on the original objective. Thus, now we want to maximize the level of random perturbations under the restrictions that $y \in \mathcal{Y}$ satisfies the chance constraint and is not too bad in terms of the original objective.

Approximating this problem by the method we have developed in the previous section, we end up with the problem

$$\min_{\beta,y,\{U_\ell\}} \left\{ \beta : \begin{array}{c} \left[\begin{array}{c|c} U_\ell & \mathcal{A}^\ell(y) \\ \hline \mathcal{A}^\ell(y) & \mathcal{A}^{\rm n}(y) \end{array}\right] \succeq 0,\ 1 \leq \ell \leq L \\ \\ \sum_\ell U_\ell \preceq \beta \mathcal{A}^{\rm n}(y),\ c^T y \leq \tau_*,\ y \in \mathcal{Y} \end{array} \right\} \tag{3.6.30}$$

(cf. (3.6.21); in terms of the latter problem, $\beta = (\Omega\rho)^{-2}$, so that maximizing $\rho$ is equivalent to minimizing $\beta$). Note that this problem remains the same whatever our guesses for $\Upsilon, \chi$. Further, (3.6.30) is a so called *GEVP* — Generalized Eigenvalue problem; while not being exactly a semidefinite program, it can be reduced to a "short sequence" of semidefinite programs via bisection in $\beta$ and thus is efficiently solvable. Solving this problem, we arrive at a solution $\beta_*, y_*, \{U_\ell^*\}$; all we need is to understand what is the "feasibility radius" $\rho_*(y_*)$ of $y_*$ — the largest $\rho$ for which $(y_*, \rho)$ satisfies the chance constraint in (3.6.29). As a matter of fact, we cannot compute this radius efficiently; what we will actually build is a reliable *lower bound* on the feasibility radius. This can be done by a suitable modification of the Acceptance Test. Let us set

$$A = \mathcal{A}^{\rm n}(y_*), A_\ell = \beta_*^{-1/2} \mathcal{A}^\ell(y_*),\ \ell = 1, ..., L; \tag{3.6.31}$$

note that these matrices satisfy (3.6.13). We apply to the matrices $A, A_1, ..., A_L$ the following procedure:

**Randomized $r$-procedure**:

Input: A collection of symmetric matrices $A, A_1, ..., A_L$ satisfying (3.6.13) and $\epsilon, \delta \in (0, 1)$.

Output: A random $r \geq 0$ such that with probability at least $1 - \delta$ one has

$$\mathrm{Prob}\{\zeta : -A \preceq r \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq A\} \geq 1 - \epsilon. \tag{3.6.32}$$

*Description*:

1. We choose a $K$-point grid $\Gamma = \{\omega_1 < \omega_2 < ... < \omega_K\}$ with $\omega_1 \geq 1$ and a reasonably large $\omega_K$, e.g., the grid
$$\omega_k = 1.1^k$$
and choose $K$ large enough to ensure that Conjecture 3.1 holds true with $\Upsilon = \omega_K$ and $\chi = 0.75$; note that $K = O(1)\ln(\ln m)$ will do;
2. We simulate $N$ independent realizations $\zeta^1, ..., \zeta^N$ of $\zeta$ and compute the integers
$$L_k = \mathrm{Card}\{i : -\omega_k A \preceq \sum_{\ell=1}^{L} \zeta_\ell^i A_\ell \preceq \omega_k A\}.$$
We then compute the quantities
$$\widehat{\chi}_k = \widehat{p}_{N,\delta/K}(L_k),\ k = 1, ..., K,$$
where $\delta \in (0, 1)$ is the chosen in advance "reliability tolerance."

Setting
$$\chi_k = \mathrm{Prob}\{-\omega_k A \preceq \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq \omega_k A\},$$
we infer from Lemma 3.4 that
$$\widehat{\chi}_k \leq \chi_k,\ k = 1, ..., K \tag{3.6.33}$$
with probability at least $1 - \delta$.

3. We define a function $\psi(s)$, $s \geq 0$, as follows.
   *In the bounded case (Assumption* **A.I***)*, we set
$$\psi_k(s) = \begin{cases} 1, \ s \leq \omega_k \\ \min\left[1, \widehat{\chi}_k^{-1} \exp\{-(s-\omega_k)^2/16\}\right], \ s > \omega_k; \end{cases}$$
   *In the Gaussian case (Assumption* **A.II***)*, we set
$$\psi_k(s) = \begin{cases} 1, \ \text{if } \widehat{\chi}_k \leq 1/2 \text{ or } s \leq \omega_k, \\ \mathrm{Erf}\big(\mathrm{ErfInv}(1-\widehat{\chi}_k) \\ +(s-\omega_k)\max[1, \omega_k^{-1}\mathrm{ErfInv}(1-\widehat{\chi}_k)]\big), \\ \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$
   In both cases, we set
$$\psi(s) = \min_{1 \leq k \leq K} \psi_k(s).$$
   We claim that
   (!) *When* (3.6.33) *takes place* (recall that this happens with probability at least $1-\delta$), $\psi(s)$ *is, for all* $s \geq 0$, *an upper bound on* $1 - \mathrm{Prob}\{-sA \preceq \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq sA\}$.
   Indeed, in the case of (3.6.33), the matrices $A, A_1, ..., A_L$ (they from the very beginning are assumed to satisfy (3.6.13)) satisfy (3.6.14) with $\Upsilon = \omega_k$ and $\chi = \widehat{\chi}_k$; it remains to apply Corollary 3.1.
4. We set
$$s_* = \inf\{s \geq 0 : \psi(s) \leq \epsilon\}, \quad r = \frac{1}{s_*}$$
   and claim that with this $r$, (3.6.32) holds true.

Let us justify the outlined construction. Assume that (3.6.33) takes place. Then, by (!), we have
$$\mathrm{Prob}\{-sA \preceq \sum_\ell \zeta_\ell \preceq sA\} \geq 1 - \psi(s).$$
Now, the function $\psi(s)$ is clearly continuous; it follows that when $s_*$ is finite, we have $\psi(s_*) \leq \epsilon$, and therefore (3.6.32) holds true with $r = 1/s_*$. If $s_* = +\infty$, then $r = 0$, and the validity of (3.6.32) follows from $A \succeq 0$ (the latter is due to the fact that $A, A_1, ..., A_L$ satisfy (3.6.13)).

When applying the Randomized $r$-procedure to matrices (3.6.31), we end up with $r = r_*$ satisfying, with probability at least $1-\delta$, the relation (3.6.32), and with our matrices $A, A_1, ..., A_L$ this relation reads
$$\mathrm{Prob}\{-\mathcal{A}^{\mathrm{n}}(y_*) \preceq r_*\beta_*^{-1/2}\sum_{\ell=1}^{L}\zeta_\ell \mathcal{A}^\ell(y_*) \preceq \mathcal{A}^{\mathrm{n}}(y_*)\} \geq 1 - \epsilon.$$
Thus, setting
$$\widehat{\rho} = \frac{r_*}{\sqrt{\beta_*}},$$
we get, with probability at least $1-\delta$, a valid lower bound on the feasibility radius $\rho_*(y_*)$ of $y_*$.

### Illustration: Example 3.7 Revisited

Let us come back to the robust version of the Console Design problem (section 3.4.2, Example 3.7), where we were looking for a console capable (i) to withstand in a nearly optimal fashion a given load of interest, and (ii) to withstand equally well (that is, with the same or smaller compliance) every "occasional load"

$g$ from the Euclidean ball $B_\rho = \{g : \|g\|_2 \leq \rho\}$ of loads distributed along the 10 free nodes of the construction. Formally, our problem was

$$\max_{t,r} \left\{ r : \begin{array}{l} \left[\begin{array}{c|c} 2\tau_* & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0 \\ \left[\begin{array}{c|c} 2\tau_* & rh^T \\ \hline rh & A(t) \end{array}\right] \succeq 0 \ \forall (h : \|h\|_2 \leq 1) \\ t \geq 0, \sum_{i=1}^N t_i \leq 1 \end{array} \right\}, \tag{3.6.34}$$

where $\tau_* > 0$ and the load of interest $f$ are given and $A(t) = \sum\limits_{i=1}^N t_i b_i b_i^T$ with $N = 54$ and known $(\mu = 20)$-dimensional vectors $b_i$. Note that what is now called $r$ was called $\rho$ in section 3.4.2.

Speaking about a console, it is reasonable to assume that in reality the "occasional load" vector is random $\sim \mathcal{N}(0, \rho^2 I_\mu)$ and to require that the construction should be capable of carrying such a load with the compliance $\leq \tau_*$ with probability at least $1 - \epsilon$, with a very small value of $\epsilon$, say, $\epsilon = 10^{-10}$. Let us now look for a console that satisfies these requirements with the largest possible value of $\rho$. The corresponding chance constrained problem is

$$\max_{t,\rho} \left\{ \rho : \begin{array}{l} \left[\begin{array}{c|c} 2\tau_* & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0 \\ \mathrm{Prob}_{h \sim \mathcal{N}(0, I_{20})} \left\{ \left[\begin{array}{c|c} 2\tau_* & \rho h^T \\ \hline \rho h & A(t) \end{array}\right] \succeq 0 \right\} \geq 1 - \epsilon \\ t \geq 0, \sum_{i=1}^N t_i \leq 1 \end{array} \right\}, \tag{3.6.35}$$

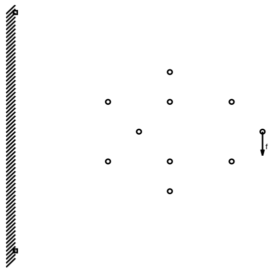
and its approximation (3.6.30) is

$$\min_{t,\beta,\{U_\ell\}_{\ell=1}^{20}} \left\{ \beta : \begin{array}{l} \left[\begin{array}{c|c} 2\tau_* & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0 \\ \left[\begin{array}{c|c} U_\ell & E_\ell \\ \hline E_\ell & Q(t) \end{array}\right] \succeq 0,\ 1 \leq \ell \leq \mu = 20 \\ \sum\limits_{\ell=1}^{\mu} U_\ell \preceq \beta Q(t),\ \ t \geq 0, \sum_{i=1}^N t_i \leq 1 \end{array} \right\}, \tag{3.6.36}$$

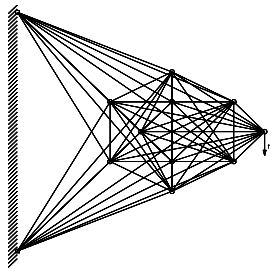
where $E_\ell = e_0 e_\ell^T + e_\ell e_0^T$, $e_0, ..., e_\mu$ are the standard basic orths in $\mathbb{R}^{\mu+1} = \mathbb{R}^{21}$, and $Q(t)$ is the matrix $\mathrm{Diag}\{2\tau_*, A(t)\} \in \mathbf{S}^{\mu+1} = \mathbf{S}^{21}$.

Note that the matrices participating in this problem are simple enough to allow us to get without much difficulty a "nearly optimal" description of theoretically valid values of $\Upsilon, \chi$ (see section 3.6.4). Indeed, here Conjecture 3.1 is valid with every $\chi \in (1/2, 1)$ provided that $\Upsilon \geq O(1)(1-\chi)^{-1/2}$. Thus, after the optimal solution $t_{\rm ch}$ to the approximating problem is found, we can avoid the simulation-based identification of a lower bound $\widehat{\rho}$ on $\rho_*(t_{\rm ch})$ (that is, on the largest $\rho$ such that $(t_{\rm ch}, \rho)$ satisfies the chance constraint in (3.6.35)) and can get a 100%-reliable lower bound on this quantity, while the simulation-based technique is capable of providing no more than a $(1-\delta)$-reliable lower bound on $\rho_*(t_{\rm ch})$ with perhaps small, but positive $\delta$. It turns out, however, that in our particular problem this 100%-reliable lower bound on $\rho_*(y_*)$ is significantly (by factor about 2) smaller than the $(1-\delta)$-reliable bound given by the outlined approach, even when $\delta$ is as small as $10^{-10}$. This is why in the experiment we are about to discuss, we used the simulation-based lower bound on $\rho_*(t_{\rm ch})$.

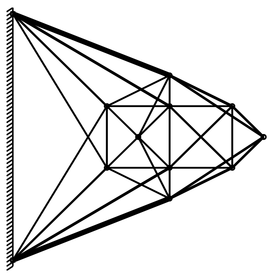
The results of our experiment are as follows. The console given by the optimal solution to (3.6.36), let it be called the *chance constrained* design, is presented in figure 3.8 (cf. figures 3.3, 3.4 representing the nominal and the robust designs, respectively). The lower bounds on the feasibility radius for the chance constrained design associated with $\epsilon = \delta = 10^{-10}$ are presented in table 3.3; the plural ("bounds") comes from the fact that we worked with three different sample sizes $N$ shown in table 3.3. Note that we can apply the outlined techniques to bound from below the feasibility radius of the *robust* design $t_{\rm rb}$ — the one given by the optimal solution to (3.6.34), see figure 3.4; the resulting bounds are presented in table 3.3.



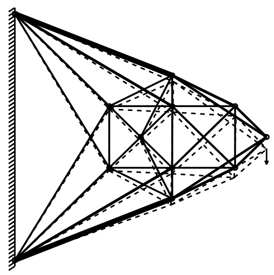
(a): reduced 12-node set with most left nodes fixed and the load of interest. $\mu = 20$ degrees of freedom.



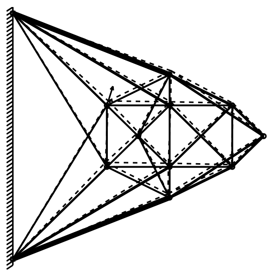
(b): 54 tentative bars



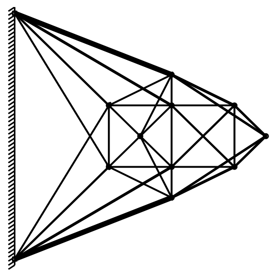
(c): Chance constrained design, 12 nodes, 33 bars. Compliance w.r.t. load of interest 1.025.



(d): Deformation of the design under the load of interest.



(e): Deformation of the design under "occasional" load 10 times less than the load of interest.



(f): "Bold dots": positions of nodes in deformed design, sample of 100 loads $\sim \mathcal{N}(0, 10^{-2} I_{20})$

Figure 3.8: Chance constrained design.

| Design | Lower bound on feasibility radius | | |
|---|---|---|---|
| | $N = 10,000$ | $N = 100,000$ | $N = 1,000,000$ |
| chance constrained $t_{\rm ch}$ | 0.0354 | 0.0414 | 0.0431 |
| robust $t_{\rm rb}$ | 0.0343 | 0.0380 | 0.0419 |

Table 3.3: $(1-10^{-10})$-confident lower bounds on feasibility radii for the chance constrained and the robust designs.

Finally, we note that we can exploit the specific structure of the particular problem in question to get alternative lower bounds on the feasibility radii of the chance constrained and the robust designs. Recall that the robust design ensures that the compliance of the corresponding console w.r.t. *any* load $g$ of Euclidean norm $\leq r_*$ is at most $\tau_*$; here $r_* \approx 0.362$ is the optimal value in (3.6.34). Now, if $\rho$ is such that $\mathrm{Prob}_{h\sim\mathcal{N}(0,I_{20})}\{\rho\|h\|_2 > r_*\} \leq \epsilon = 10^{-10}$, then clearly $\rho$ is a 100%-reliable lower bound on the feasibility radius of the robust design. We can easily compute the largest $\rho$ satisfying the latter condition; it turns out to be 0.0381, 9% less than the best simulation-based lower bound. Similar reasoning can be applied to the chance constrained design $t_{\rm ch}$: we first find the largest $r = r_+$ for which $(t_{\rm ch}, r)$ is feasible for (3.6.34) (it turns out that $r_+ = 0.321$), and then find the largest $\rho$ such that $\mathrm{Prob}_{h\sim\mathcal{N}(0,I_{20})}\{\rho\|h\|_2 > r_+\} \leq \epsilon = 10^{-10}$, ending up with the lower bound 0.0337 on the feasibility radius of the chance constrained design (25.5% worse than the best related bound in table 3.3).

### 3.6.3 Gaussian Majorization

Under favorable circumstances, we can apply the outlined approximation scheme to random perturbations that do not fit exactly neither Assumption **A.I**, nor Assumption **A.II.** As an instructive example, consider the case where the random perturbations $\zeta_\ell$, $\ell = 1, ..., L$, in (3.6.1) are independent and symmetrically and *unimodally* distributed w.r.t. 0. Assume also that we can point out scaling factors $\sigma_\ell > 0$ such that the distribution of each $\zeta_\ell$ is less diffuse than the Gaussian $\mathcal{N}(0, \sigma_\ell^2)$ distribution (see Definition 2.2). Note that in order to build a safe tractable approximation of the chance constrained LMI

$$\mathrm{Prob}\left\{\mathcal{A}^{\rm n}(y) + \sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}_\ell(y) \succeq 0\right\} \geq 1 - \epsilon, \tag{3.6.2}$$

or, which is the same, the constraint

$$\mathrm{Prob}\left\{\mathcal{A}^{\rm n}(y) + \sum_{\ell=1}^{L} \widetilde{\zeta}_\ell \widetilde{\mathcal{A}}^\ell(y) \succeq 0\right\} \geq 1 - \epsilon \qquad \left[\begin{array}{c} \widetilde{\zeta}_\ell = \sigma_\ell^{-1}\zeta_\ell \\ \widetilde{\mathcal{A}}^\ell(y) = \sigma_\ell \mathcal{A}^\ell(y) \end{array}\right]$$

it suffices to build such an approximation for the symmetrized version

$$\mathrm{Prob}\{-\mathcal{A}^{\rm n}(y) \preceq \sum_{\ell=1}^{L} \widetilde{\zeta}_\ell \widetilde{\mathcal{A}}^\ell(y) \preceq \mathcal{A}^{\rm n}(y)\} \geq 1 - \epsilon \tag{3.6.37}$$

of the constraint. Observe that the random variables $\widetilde{\zeta}_\ell$ are independent and possess symmetric and unimodal w.r.t. 0 distributions that are less diffuse than the $\mathcal{N}(0,1)$ distribution. Denoting by $\eta_\ell$, $\ell = 1, ..., L$, independent $\mathcal{N}(0,1)$ random variables and invoking the Majorization Theorem (Theorem 2.6), we see that the validity of the chance constraint

$$\mathrm{Prob}\{-\mathcal{A}^{\rm n}(y) \preceq \sum_{\ell=1}^{L} \eta_\ell \widetilde{\mathcal{A}}^\ell(y) \preceq \mathcal{A}^{\rm n}(y)\} \geq 1 - \epsilon$$

— and this is the constraint we do know how to handle — is a sufficient condition for the validity of (3.6.37). Thus, in the case of unimodally and symmetrically distributed $\zeta_\ell$ admitting "Gaussian majorants," we can act, essentially, as if we were in the Gaussian case **A.II**.

It is worth noticing that we can apply the outlined "Gaussian majorization" scheme even in the case when $\zeta_\ell$ are symmetrically and unimodally distributed in $[-1,1]$ (a case that we know how to handle even without the unimodality assumption), and this could be profitable. Indeed, by Example 2.2 (section 2.7.2), in the case in question $\zeta_\ell$ are less diffuse than the random variables $\eta_\ell \sim \mathcal{N}(0,2/\pi)$, and we can again reduce the situation to Gaussian. The advantage of this approach is that the absolute constant factor $\frac{1}{16}$ in the exponent in (3.6.15) is rather small. Therefore replacing (3.6.15) with (3.6.16), even after replacing our original variables $\zeta_\ell$ with their less concentrated "Gaussian majorants" $\eta_\ell$, can lead to better results. To illustrate this point, here is a report on a numerical experiment:

1) We generated $L = 100$ matrices $A_\ell \in \mathbf{S}^{40}$, $\ell = 1,...,L$, such that $\sum_\ell A_\ell^2 \preceq I$, (which clearly implies that $A = I, A_1,...,A_L$ satisfy (3.6.13));
2) We applied the bounded case version of the Randomized $r$ procedure to the matrices $A, A_1,...,A_L$ and the independent random variables $\zeta_\ell$ uniformly distributed on $[-1,1]$, setting $\delta$ and $\epsilon$ to $10^{-10}$;
3) We applied the Gaussian version of the same procedure, with the same $\epsilon$, $\delta$, to the matrices $A, A_1,...,A_L$ and independent $\mathcal{N}(0,2/\pi)$ random variables $\eta_\ell$ in the role of $\zeta_\ell$.

In both 2) and 3), we used the same grid $\omega_k = 0.01 \cdot 10^{0.1k}$, $0 \le k \le 40$.

By the above arguments, both in 2) and in 3) we get, with probability at least $1-10^{-10}$, lower bounds on the largest $\rho$ such that

$$\mathrm{Prob}\{-I \preceq \rho \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq I\} \ge 1 - 10^{-10}.$$

Here are the bounds obtained:

| Bounding scheme | Lower Bound | |
|---|---|---|
| | $N = 1000$ | $N = 10000$ |
| 2) | 0.0489 | 0.0489 |
| 3) | 0.185 | 0.232 |

We see that while we can process the case of uniformly distributed $\zeta_\ell$ "as it is," it is better to process it via Gaussian majorization.

To conclude this section, we present another "Gaussian Majorization" result. Its advantage is that it does not require the random variables $\zeta_\ell$ to be symmetrically or unimodally distributed; what we need, essentially, is just independence plus zero means. We start with some definitions. Let $\mathcal{R}_n$ be the space of Borel probability distributions on $\mathbb{R}^n$ with zero mean. For a random variable $\eta$ taking values in $\mathbb{R}^n$, we denote by $P_\eta$ the corresponding distribution, and we write $\eta \in \mathcal{R}_n$ to express that $P_\eta \in \mathcal{R}_n$. Let also $\mathcal{CF}_n$ be the set of all *convex* functions $f$ on $\mathbb{R}^n$ with linear growth, meaning that there exists $c_f < \infty$ such that $|f(u)| \le c_f(1 + \|u\|_2)$ for all $u$.

**Definition 3.7** *Let $\xi, \eta \in \mathcal{R}_n$. We say that $\eta$ dominates $\xi$ (notation: $\xi \preceq_c \eta$, or $P_\xi \preceq_c P_\eta$, or $\eta \succeq_c \xi$, or $P_\eta \succeq_c P_\xi$) if*

$$\int f(u)dP_\xi(u) \le \int f(u)dP_\eta(u)$$

*for every $f \in \mathcal{CF}_n$.*

Note that in the literature the relation $\succeq_c$ is called "convex dominance." The properties of the relation $\succeq_c$ we need are summarized as follows:

**Proposition 3.3**

1. *$\preceq_c$ is a partial order on $\mathcal{R}_n$.*
2. *If $P_1,...,P_k,Q_1,...,Q_k \in \mathcal{R}_n$, and $P_i \preceq_c Q_i$ for every $i$, then $\sum_i \lambda_i P_i \preceq_c \sum_i \lambda_i Q_i$ for all nonnegative $\lambda_i$ with unit sum.*

3. *If $\xi \in \mathcal{R}_n$ and $t \geq 1$ is deterministic, then $t\xi \succeq_{\rm c} \xi$.*

4. *Let $P_1, Q_1 \in \mathcal{R}_r$, $P_2, Q_2 \in \mathcal{R}_s$ be such that $P_i \preceq_{\rm c} Q_i$, $i = 1, 2$. Then $P_1 \times P_2 \preceq_{\rm c} Q_1 \times Q_2$. In particular, if $\xi_1, ..., \xi_n, \eta_1, ..., \eta_n \in \mathcal{R}_1$ are independent and $\xi_i \preceq_{\rm c} \eta_i$ for every $i$, then $[\xi_1; ...; \xi_n] \preceq_{\rm c} [\eta_1; ...; \eta_n]$.*

5. *If $\xi_1, ..., \xi_k, \eta_1, ..., \eta_k \in \mathcal{R}_n$ are independent random variables, $\xi_i \preceq_{\rm c} \eta_i$ for every $i$, and $S_i \in \mathbb{R}^{m \times n}$ are deterministic matrices, then $\sum_i S_i\xi_i \preceq_{\rm c} \sum_i S_i\eta_i$.*

6. *Let $\xi \in \mathcal{R}_1$ be supported on $[-1, 1]$ and $\eta \sim \mathcal{N}(0, \pi/2)$. Then $\eta \succeq_{\rm c} \xi$.*

7. *If $\xi, \eta$ are symmetrically and unimodally distributed w.r.t. the origin scalar random variables with finite expectations and $\eta \succeq_{\rm m} \xi$ (see section 2.7.2), then $\eta \succeq_{\rm c} \xi$ as well. In particular, if $\xi$ has unimodal w.r.t. 0 distribution and is supported on $[-1, 1]$ and $\eta \sim \mathcal{N}(0, 2/\pi)$, then $\eta \succeq_{\rm c} \xi$ (cf. Example 2.2).*

8. *Assume that $\xi \in \mathcal{R}_n$ is supported in the unit cube $\{u : \|u\|_\infty \leq 1\}$ and is "absolutely symmetrically distributed," meaning that if $J$ is a diagonal matrix with diagonal entries $\pm 1$, then $J\xi$ has the same distribution as $\xi$. Let also $\eta \sim \mathcal{N}(0, (\pi/2)I_n)$. Then $\xi \preceq_{\rm c} \eta$.*

9. *Let $\xi, \eta \in \mathcal{R}_r$, $\xi \sim \mathcal{N}(0, \Sigma)$, $\eta \sim \mathcal{N}(0, \Theta)$ with $\Sigma \preceq \Theta$. Then $\xi \preceq_{\rm c} \eta$.*

The main result here is as follows.

**Theorem 3.16** [Gaussian Majorization [3, Theorem 10.3.3]] *Let $\eta \sim \mathcal{N}(0, I_L)$, and let $\zeta \in \mathcal{R}_L$ be such that $\zeta \preceq_{\rm c} \eta$. Let, further, $Q \subset \mathbb{R}^L$ be a closed convex set such that*

$$\chi \equiv \mathrm{Prob}\{\eta \in Q\} > 1/2.$$

*Then for every $\gamma > 1$, one has*

$$\begin{array}{rl} \mathrm{Prob}\{\zeta \notin \gamma Q\} \leq & \inf\limits_{1 \leq \beta < \gamma} \frac{1}{\gamma - \beta} \int\limits_\beta^\infty \mathrm{Erf}(r\mathrm{ErfInv}(1 - \chi))dr \\ \leq & \inf\limits_{1 \leq \beta < \gamma} \frac{1}{2(\gamma - \beta)} \int\limits_\beta^\infty \exp\{-r^2\mathrm{ErfInv}^2(1 - \chi)/2\}dr, \end{array} \tag{3.6.38}$$

*where* Erf($\cdot$), ErfInv($\cdot$) *are given by* (2.2.6), (2.2.7).

*The assumption $\zeta \preceq_{\rm c} \eta$ is valid, in particular, if $\zeta = [\zeta_1; ...; \zeta_L]$ with independent $\zeta_\ell$ such that $P_{\zeta_\ell} \in \mathcal{R}_1$ and $P_{\zeta_\ell} \preceq_{\rm c} \mathcal{N}(0, 1)$.*

### 3.6.4 Chance Constrained LMIs: Special Cases

We intend to consider two cases where it is easy to justify Conjecture 3.1. While the structural assumptions on the matrices $A, A_1, ..., A_L$ in these two cases seem to be highly restrictive, the results are nevertheless important: they cover the situations arising in randomly perturbed Linear and Conic Quadratic Optimization. We begin with a slight relaxation of Assumptions **A.I–II**:

> **Assumption A.III**: The random perturbations $\zeta_1, ..., \zeta_L$ are independent, zero mean and "of order of 1," meaning that
>
> $$\mathbf{E}\{\exp\{\zeta_\ell^2\}\} \leq \exp\{1\}, \, \ell = 1, ..., L.$$

Note that Assumption **A.III** is implied by **A.I** and is "almost implied" by **A.II**; indeed, $\zeta_\ell \sim \mathcal{N}(0, 1)$ implies that the random variable $\widetilde{\zeta}_\ell = \sqrt{(1 - \mathrm{e}^{-2})/2}\zeta_\ell$ satisfies $\mathbf{E}\{\exp\{\widetilde{\zeta}_\ell^2\}\} \leq \exp\{1\}$.

### The Diagonal Case: Chance Constrained Linear Optimization

**Theorem 3.17** *Let $A, A_1, ..., A_L \in \mathbf{S}^m$ be diagonal matrices satisfying* (3.6.13) *and let the random variables $\zeta_\ell$ satisfy Assumption* **A.III**. *Then, for every $\chi \in (0,1)$, with $\Upsilon = \Upsilon(\chi) \equiv \sqrt{3\ln\left(\frac{2m}{1-\chi}\right)}$ one has*

$$\mathrm{Prob}\{-\Upsilon A \preceq \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq \Upsilon A\} \geq \chi \tag{3.6.39}$$

*(cf.* (3.6.14)*). In the case of $\zeta_\ell \sim \mathcal{N}(0,1)$, relation* (3.6.39) *holds true with $\Upsilon = \Upsilon(\chi) \equiv \sqrt{2\ln\left(\frac{m}{1-\chi}\right)}$.*

**Proof.** It is immediately seen that we lose nothing when assuming that $A \succ 0$ (cf. the proof of Corollary 3.1). With this assumption, passing from diagonal matrices $A, A_\ell$ to the diagonal matrices $B_\ell = A^{-1/2}A_\ell A^{-1/2}$, the statement to be proved reads as follows:

> *If $B_\ell \in \mathbf{S}^m$ are deterministic diagonal matrices such that $\sum_\ell B_\ell^2 \preceq I$ and $\zeta_\ell$ satisfy* **A.III**, *then, for every $\chi \in (0,1)$, one has*
>
> $$\mathrm{Prob}\{\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| \leq \underbrace{\sqrt{3\ln\left(\frac{2m}{1-\chi}\right)}}_{\Upsilon(\chi)}\} \geq \chi. \tag{3.6.40}$$
>
> *When $\zeta_\ell \sim \mathcal{N}(0,1)$, $\ell = 1, ..., L$, the relation remains true with $\Upsilon(\chi)$ reduced to $\sqrt{2\ln(m/(1-\chi))}$.*

The proof of the latter statement is based on the standard argument used in deriving results on large deviations of sums of "light-tail" independent random variables. First we need the following result.

**Lemma 3.5** *Let $\beta_\ell$, $\ell = 1, ..., L$, $\gamma > 0$ be deterministic reals such that $\sum_\ell \beta_\ell^2 \leq 1$. Then*

$$\forall \Upsilon > 0 : \mathrm{Prob}\left\{|\sum_{\ell=1}^{L} \beta_\ell \zeta_\ell| > \Upsilon\right\} \leq 2\exp\{-\Upsilon^2/3\}. \tag{3.6.41}$$

**Proof of Lemma 3.5.** Observe, first, that whenever $\xi$ is a random variable with zero mean such that $\mathbf{E}\{\exp\{\xi^2\}\} \leq \exp\{1\}$, one has

$$\mathbf{E}\{\exp\{\gamma\xi\}\} \leq \exp\{3\gamma^2/4\}. \tag{3.6.42}$$

Indeed, observe that by Holder Inequality the relation $\mathbf{E}\left\{\exp\{\xi^2\}\right\} \leq \exp\{1\}$ implies that $\mathbf{E}\left\{\exp\{s\xi^2\}\right\} \leq \exp\{s\}$ for all $s \in [0,1]$. It is immediately seen that $\exp\{x\} - x \leq \exp\{9x^2/16\}$ for all $x$. Assuming that $9\gamma^2/16 \leq 1$, we therefore have

$$\begin{aligned}\mathbf{E}\left\{\exp\{\gamma\xi\}\right\} &= \mathbf{E}\left\{\exp\{\gamma\xi\} - \gamma\xi\right\} \ [\xi \text{ is with zero mean}]\\ &\leq \mathbf{E}\left\{\exp\{9\gamma^2\xi^2/16\}\right\}\\ &\leq \exp\{9\gamma^2/16\} \ [\text{since } 9\gamma^2/16 \leq 1]\\ &\leq \exp\{3\gamma^2/4\},\end{aligned}$$

as required in (3.6.42). Now let $9\gamma^2/16 \geq 1$. For all $\gamma$ we have $\gamma\xi \leq 3\gamma^2/8 + 2\xi^2/3$, whence

$$\begin{aligned}\mathbf{E}\left\{\exp\{\gamma\xi\}\right\} \leq\ & \exp\{3\gamma^2/8\}\exp\{2\xi^2/3\} \leq \exp\{3\gamma^2/8 + 2/3\}\\ &\leq \exp\{3\gamma^2/4\} \ [\text{since } \gamma^2 \geq 16/9]\end{aligned}$$

We see that (3.6.42) is valid for all $\gamma$.

We now have

$$\begin{array}{l}\mathbf{E}\left\{\exp\{\gamma\sum_{\ell=1}^L \beta_\ell\zeta_\ell\}\right\} = \prod\limits_{\ell=1}^L \mathbf{E}\left\{\exp\{\gamma\beta_\ell\zeta_\ell\}\right\}\ [\zeta_1,...,\zeta_L \text{ are independent}]\\ \leq \prod\limits_{\ell=1}^L \exp\{3\gamma^2\beta_\ell^2/4\}\ [\text{by Lemma}]\\ \leq \exp\{3\gamma^2/4\}\ [\text{since } \sum_\ell \beta_\ell^2 \leq 1].\end{array}$$

We now have

$$\begin{array}{l}\mathrm{Prob}\left\{\sum_{\ell=1}^L \beta_\ell\zeta_\ell > \Upsilon\right\}\\ \leq \min_{\gamma\geq 0}\exp\{-\Upsilon\gamma\}\mathbf{E}\left\{\exp\{\gamma\sum_\ell \beta_\ell\zeta_\ell\}\right\}\ [\text{Tschebyshev Inequality}]\\ \leq \min_{\gamma\geq 0}\exp\{-\Upsilon\gamma + 3\gamma^2/4\}\ [\text{by (3.6.42)}]\\ = \exp\{-\Upsilon^2/3\}.\end{array}$$

Replacing $\zeta_\ell$ with $-\zeta_\ell$, we get that $\mathrm{Prob}\left\{\sum_\ell \beta_\ell\zeta_\ell < -\Upsilon\right\} \leq \exp\{-\Upsilon^2/3\}$ as well, and (3.6.41) follows. □

**Proof of (3.6.39).** Let $s_i$ be the $i$-th diagonal entry in the random diagonal matrix $S = \sum\limits_{\ell=1}^L \zeta_\ell B_\ell$. Taking into account that $B_\ell$ are diagonal with $\sum\limits_\ell B_\ell^2 \preceq I$, we can apply Lemma 3.5 to get the bound

$$\mathrm{Prob}\{|s_i| > \Upsilon\} \leq 2\exp\{-\Upsilon^2/3\};$$

since $\|S\| = \max\limits_{1\leq i\leq m} |s_i|$, (3.6.40) follows.

Refinements in the case of $\zeta_\ell \sim \mathcal{N}(0,1)$ are evident: here the $i$-th diagonal entry $s_i$ in the random diagonal matrix $S = \sum\limits_\ell \zeta_\ell B_\ell$ is $\sim \mathcal{N}(0,\sigma_i^2)$ with $\sigma_i \leq 1$, whence $\mathrm{Prob}\{|s_i| > \Upsilon\} \leq \exp\{-\Upsilon^2/2\}$ and therefore $\mathrm{Prob}\{\|S\| > \Upsilon\} \leq m\exp\{-\Upsilon^2/2\}$, so that $\Upsilon(\chi)$ in (3.6.40) can indeed be reduced to $\sqrt{2\ln(m/(1-\chi))}$. □

The case of chance constrained LMI with diagonal matrices $\mathcal{A}^{\mathrm{n}}(y)$, $\mathcal{A}^\ell(y)$ has an important application — Chance Constrained Linear Optimization. Indeed, consider a randomly perturbed Linear Optimization problem

$$\min_y \left\{c^T y : A_\zeta y \geq b_\zeta\right\} \tag{3.6.43}$$

where $A_\zeta$, $b_\zeta$ are affine in random perturbations $\zeta$:

$$[A_\zeta, b_\zeta] = [A^{\mathrm{n}}, b^{\mathrm{n}}] + \sum_{\ell=1}^L \zeta_\ell [A^\ell, b^\ell];$$

as usual, we have assumed w.l.o.g. that the objective is certain. The chance constrained version of this problem is

$$\min_y \left\{c^T y : \mathrm{Prob}\{A_\zeta y \geq b_\zeta\} \geq 1-\epsilon\right\}. \tag{3.6.44}$$

Setting $\mathcal{A}^{\mathrm{n}}(y) = \mathrm{Diag}\{A^{\mathrm{n}}y - b^{\mathrm{n}}\}$, $\mathcal{A}^\ell(y) = \mathrm{Diag}\{A^\ell y - b^\ell\}$, $\ell = 1,...,L$, we can rewrite (3.6.44) equivalently as the chance constrained semidefinite problem

$$\min_y \left\{c^T y : \mathrm{Prob}\{\mathcal{A}_\zeta(y) \succeq 0\} \geq 1-\epsilon\right\},\ \mathcal{A}_\zeta(y) = \mathcal{A}^{\mathrm{n}}(y) + \sum_\ell \zeta_\ell \mathcal{A}^\ell(y), \tag{3.6.45}$$

and process this problem via the outlined approximation scheme. Note the essential difference between what we are doing now and what was done in lecture 2. There we focused on safe approximation of chance constrained *scalar* linear inequality, here we are speaking about approximating a chance constrained coordinate-wise *vector* inequality. Besides this, our approximation scheme is, in general, "semi-analytic" — it involves simulation and as a result produces a solution that is feasible for the chance constrained problem with probability close to 1, but not with probability 1.

Of course, the safe approximations of chance constraints developed in lecture 2 can be used to process coordinate-wise vector inequalities as well. The natural way to do it is to replace the chance constrained vector inequality in (3.6.44) with a bunch of chance constrained scalar inequalities

$$\mathrm{Prob}\left\{(A_\zeta y - b_\zeta)_i \geq 0\right\} \geq 1-\epsilon_i,\ i=1,...,m \equiv \dim b_\zeta, \tag{3.6.46}$$

where the tolerances $\epsilon_i \geq 0$ satisfy the relation $\sum_i \epsilon_i = \epsilon$. The validity of (3.6.46) clearly is a sufficient condition for the validity of the chance constraint in (3.6.44), so that replacing these constraints with their safe tractable approximations from lecture 2, we end up with a safe tractable approximation of the chance constrained LO problem (3.6.44). A drawback of this approach is in the necessity to "guess" the quantities $\epsilon_i$. The ideal solution would be to treat them as additional decision variables and to optimize the safe approximation in both $y$ and $\epsilon_i$. Unfortunately, all approximation schemes for scalar chance constraints presented in lecture 2 result in approximations that are *not* jointly convex in $y, \{\epsilon_i\}$. As a result, joint optimization in $y, \epsilon_i$ is more wishful thinking than a computationally solid strategy. Seemingly the only simple way to resolve this difficulty is to set all $\epsilon_i$ equal to $\epsilon/m$.

It is instructive to compare the "constraint-by-constraint" safe approximation of a chance constrained LO (3.6.44) given by the results of lecture 2 with our present approximation scheme. To this end, let us focus on the following version of the chance constrained problem:

$$\max_{\rho,y}\left\{\rho : c^T y \leq \tau_*, \mathrm{Prob}\left\{A_{\rho\zeta} y \geq b_{\rho\zeta}\right\} \geq 1-\epsilon\right\} \tag{3.6.47}$$

(cf. (3.6.29)). To make things as simple as possible, we assume also that $\zeta_\ell \sim \mathcal{N}(0,1)$, $\ell = 1, ..., L$.

The "constraint-by-constraint" safe approximation of (3.6.47) is the chance constrained problem

$$\max_{\rho,y}\left\{\rho : c^T y \leq \tau_*, \mathrm{Prob}\left\{(A_{\rho\zeta} y - b_{\rho\zeta})_i \geq 0\right\} \geq 1-\epsilon/m\right\},$$

where $m$ is the number of rows in $A_\zeta$. A chance constraint

$$\mathrm{Prob}\left\{(A_{\rho\zeta} y - b_{\rho\zeta})_i \geq 0\right\} \geq 1-\epsilon/m$$

can be rewritten equivalently as

$$\mathrm{Prob}\{[b^{\mathrm{n}} - A^{\mathrm{n}} y]_i + \rho \sum_{\ell=1}^{L} [b^\ell - A^\ell y]_i \zeta_\ell > 0\} \leq \epsilon/m.$$

Since $\zeta_\ell \sim \mathcal{N}(0,1)$ are independent, this scalar chance constraint is exactly equivalent to

$$[b^{\mathrm{n}} - A^{\mathrm{n}} y]_i + \rho \mathrm{ErfInv}(\epsilon/m)\sqrt{\sum_\ell [b^\ell - A^\ell y]_i^2} \leq 0.$$

The associated safe tractable approximation of the problem of interest (3.6.47) is the conic quadratic program

$$\max_{\rho,y}\left\{\rho : c^T y \leq \tau_*, \mathrm{ErfInv}(\epsilon/m)\sqrt{\sum_\ell [b^\ell - A^\ell y]_i^2} \leq \frac{[A^{\mathrm{n}} y - b^{\mathrm{n}}]_i}{\rho}, 1 \leq i \leq m\right\}. \tag{3.6.48}$$

Now let us apply our new approximation scheme, which treats the chance constrained vector inequality in (3.6.44) "as a whole." To this end, we should solve the problem

$$\min_{\nu,y,\{U_\ell\}}\left\{\nu : \begin{array}{l} c^T y \leq \tau_*, \left[\begin{array}{c|c} U_\ell & \mathrm{Diag}\{A^\ell y - b^\ell\} \\ \hline \mathrm{Diag}\{A^\ell y - b^\ell\} & \mathrm{Diag}\{A^{\mathrm{n}} y - b^{\mathrm{n}}\}\end{array}\right] \succeq 0, \\ \hfill 1 \leq \ell \leq L \\ \sum_\ell U_\ell \preceq \nu \mathrm{Diag}\{A^{\mathrm{n}} y - b^{\mathrm{n}}\},\ c^T y \leq \tau_* \end{array}\right\}, \tag{3.6.49}$$

treat its optimal solution $y_*$ as the $y$ component of the optimal solution to the approximation and then bound from below the feasibility radius $\rho_*(y_*)$ of this solution, (e.g., by applying to $y_*$ the Randomized $r$ procedure). Observe that problem (3.6.49) is nothing but the problem

$$\min_{\nu,y}\left\{\nu : \begin{array}{l}\sum\limits_{\ell=1}^{L}[A^\ell y-b]_i^2/[A^{\rm n}y-b^{\rm n}]_i\leq \nu[A^{\rm n}y-b^{\rm n}]_i, 1\leq i\leq m,\\ A^{\rm n}y-b^{\rm n}\geq 0,\ c^Ty\leq\tau_*\end{array}\right\},$$

where $a^2/0$ is 0 for $a=0$ and is $+\infty$ otherwise. Comparing the latter problem with (3.6.48), we see that

> *Problems* (3.6.49) *and* (3.6.48) *are equivalent to each other, the optimal values being related as*
>
> $$\mathrm{Opt}(3.6.48)=\frac{1}{\mathrm{ErfInv}(\epsilon/m)\sqrt{\mathrm{Opt}(3.6.49)}}.$$
>
> *Thus, the approaches we are comparing result in the same vector of decision variables $y_*$, the only difference being the resulting value of a lower bound on the feasibility radius of $y_*$. With the "constraint-by-constraint" approach originating from lecture 2, this value is the optimal value in* (3.6.48), *while with our new approach, which treats the vector inequality $Ax\geq b$ "as a whole," the feasibility radius is bounded from below via the provable version of Conjecture 3.1 given by Theorem 3.17, or by the Randomized $r$ procedure.*

A natural question is, which one of these approaches results in a less conservative lower bound on the feasibility radius of $y_*$. On the theoretical side of this question, it is easily seen that when the second approach utilizes Theorem 3.17, it results in the same (within an absolute constant factor) value of $\rho$ as the first approach. From the practical perspective, however, it is much more interesting to consider the case where the second approach exploits the Randomized $r$ procedure, since experiments demonstrate that this version is less conservative than the "100%-reliable" one based on Theorem 3.17. Thus, let us focus on comparing the "constraint-by-constraint" safe approximation of (3.6.44), let it be called Approximation I, with Approximation II based on the Randomized $r$ procedure. Numerical experiments show that no one of these two approximations "generically dominates" the other one, so that the best thing is to choose the best — the largest — of the two respective lower bounds.

### The Arrow Case: Chance Constrained Conic Quadratic Optimization

We are about to justify Conjecture 3.1 in the *arrow-type case*, that is, when the matrices $A_\ell\in\mathbf{S}^m$, $\ell=1,...,L$, are of the form

$$A_\ell=[ef_\ell^T+f_\ell e^T]+\lambda_\ell G, \tag{3.6.50}$$

where $e,f_\ell\in\mathbb{R}^m$, $\lambda_\ell\in\mathbb{R}$ and $G\in\mathbf{S}^m$. We encounter this case in the Chance Constrained Conic Quadratic Optimization. Indeed, a Chance Constrained CQI

$$\mathrm{Prob}\{\|A(y)\zeta+b(y)\|_2\leq c^T(y)\zeta+d(y)\}\geq 1-\epsilon, \qquad [A(\cdot):p\times q]$$

can be reformulated equivalently as the chance constrained LMI

$$\mathrm{Prob}\{\left[\begin{array}{c|c}c^T(y)\zeta+d(y) & \zeta^TA^T(y)+b^T(y)\\ \hline A(y)\zeta+b(y) & (c^T(y)\zeta+d(y))I\end{array}\right]\succeq 0\}\geq 1-\epsilon \tag{3.6.51}$$

(see Lemma 3.1). In the notation of (3.6.1), for this LMI we have

$$\mathcal{A}^{\rm n}(y)=\left[\begin{array}{c|c}d(y) & b^T(y)\\ \hline b(y) & d(y)I\end{array}\right],\ \mathcal{A}^\ell(y)=\left[\begin{array}{c|c}c_\ell(y) & a_\ell^T(y)\\ \hline a_\ell(y) & c_\ell(y)I\end{array}\right],$$

where $a_\ell(y)$ in (3.6.50) is $\ell$-th column of $A(y)$. We see that the matrices $\mathcal{A}^\ell(y)$ are arrow-type $(p+1)\times(p+1)$ matrices where $e$ in (3.6.50) is the first basic orth in $\mathbb{R}^{p+1}$, $f_\ell=[0;a_\ell(y)]$ and $G=I_{p+1}$.

Another example is the one arising in the chance constrained Truss Topology Design problem, see section 3.6.2.

The justification of Conjecture 3.1 in the arrow-type case is given by the following

**Theorem 3.18** *Let $m \times m$ matrices $A_1, ..., A_L$ of the form (3.6.50) along with a matrix $A \in \mathbf{S}^m$ satisfy the relation (3.6.13), and $\zeta_\ell$ be independent with zero means and such that $\mathbf{E}\{\zeta_\ell^2\} \le \sigma^2$, $\ell = 1, ..., L$ (under Assumption **A.III**, one can take $\sigma = \sqrt{\exp\{1\} - 1}$). Then, for every $\chi \in (0,1)$, with $\Upsilon = \Upsilon(\chi) \equiv \frac{2\sqrt{2}\sigma}{\sqrt{1-\chi}}$ one has*

$$\mathrm{Prob}\{-\Upsilon A \preceq \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq \Upsilon A\} \ge \chi \tag{3.6.52}$$

*(cf. (3.6.14)). When $\zeta$ satisfies Assumption **A.I**, or $\zeta$ satisfies Assumption **A.II** and $\chi \ge \frac{6}{7}$, relation (3.6.52) is satisfied with $\Upsilon = \Upsilon_{\mathrm{I}}(\chi) \equiv 2 + 4\sqrt{3 \ln \frac{4}{1-\chi}}$ and with $\Upsilon = \Upsilon_{\mathrm{II}}(\chi) \equiv \sqrt{3\left(1 + 3\ln\frac{1}{1-\chi}\right)}$, respectively.*

**Proof.** First of all, when $\zeta_\ell$, $\ell = 1, ..., L$, satisfy Assumption **A.III**, we indeed have $\mathbf{E}\{\zeta_\ell^2\} \le \exp\{1\} - 1$ due to $t^2 \le \exp\{t^2\} - 1$ for all $t$. Further, same as in the proof of Theorem 3.17, it suffices to consider the case when $A \succ 0$ and to prove the following statement:

> *Let $A_\ell$ be of the form of (3.6.50) and such that the matrices $B_\ell = A^{-1/2} A_\ell A^{-1/2}$ satisfy $\sum\limits_\ell B_\ell^2 \preceq I$. Let, further, $\zeta_\ell$ satisfy the premise in Theorem 3.18. Then, for every $\chi \in (0,1)$, one has*
>
> $$\mathrm{Prob}\{\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| \le \frac{2\sqrt{2}\sigma}{\sqrt{1-\chi}}\} \ge \chi. \tag{3.6.53}$$

Observe that $B_\ell$ are also of the arrow-type form (3.6.50):

$$B_\ell = [g h_\ell^T + h_\ell g^T] + \lambda_\ell H \qquad [g = A^{-1/2} e, h_\ell = A^{-1/2} f_\ell, H = A^{-1/2} G A^{-1/2}]$$

Note that w.l.o.g. we can assume that $\|g\|_2 = 1$ and then rotate the coordinates to make $g$ the first basic orth. In this situation, the matrices $B_\ell$ become

$$B_\ell = \left[\begin{array}{c|c} q_\ell & r_\ell^T \\ \hline r_\ell & \lambda_\ell Q \end{array}\right]; \tag{3.6.54}$$

by appropriate scaling of $\lambda_\ell$, we can ensure that $\|Q\| = 1$. We have

$$B_\ell^2 = \left[\begin{array}{c|c} q_\ell^2 + r_\ell^T r_\ell & q_\ell r_\ell^T + \lambda_\ell r_\ell^T Q \\ \hline q_\ell r_\ell + \lambda_\ell Q r_\ell & r_\ell r_\ell^T + \lambda_\ell^2 Q^2 \end{array}\right].$$

We conclude that $\sum\limits_{\ell=1}^{L} B_\ell^2 \preceq I_m$ implies that $\sum\limits_\ell (q_\ell^2 + r_\ell^T r_\ell) \le 1$ and $[\sum\limits_\ell \lambda_\ell^2] Q^2 \preceq I_{m-1}$; since $\|Q^2\| = 1$, we arrive at the relations

$$\begin{array}{ll} (a) & \sum_\ell \lambda_\ell^2 \le 1, \\ (b) & \sum_\ell (q_\ell^2 + r_\ell^T r_\ell) \le 1. \end{array} \tag{3.6.55}$$

Now let $p_\ell = [0; r_\ell] \in \mathbb{R}^m$. We have

$$\begin{array}{rl} & S[\zeta] \equiv \sum_\ell \zeta_\ell B_\ell = [g^T(\underbrace{\sum_\ell \zeta_\ell p_\ell}_{\xi}) + \xi^T g] + \mathrm{Diag}\{\underbrace{\sum_\ell \zeta_\ell q_\ell}_{\theta}, (\underbrace{\sum_\ell \zeta_\ell \lambda_\ell}_{\eta})Q\} \\ \Rightarrow & \|S[\zeta]\| \le \|g\xi^T + \xi g^T\| + \max[|\theta|, |\eta| \|Q\|] = \|\xi\|_2 + \max[|\theta|, |\eta|]. \end{array}$$

Setting

$$\alpha = \sum_\ell r_\ell^T r_\ell, \ \beta = \sum_\ell q_\ell^2,$$

we have $\alpha + \beta \leq 1$ by (3.6.55.$b$). Besides this,

$$\begin{array}{ll}
\mathbf{E}\{\xi^T\xi\} = \sum_{\ell,\ell'} \mathbf{E}\{\zeta_\ell\zeta_{\ell'}\}p_\ell^T p_{\ell'} = \sum_\ell \mathbf{E}\{\zeta_\ell^2\}r_\ell^T r_\ell & [\zeta_\ell \text{ are independent zero mean}]\\
\leq \sigma^2 \sum_\ell r_\ell^T r_\ell = \sigma^2\alpha & \\
\quad \Rightarrow \quad \text{Prob}\{\|\xi\|_2 > t\} \leq \frac{\sigma^2\alpha}{t^2}\ \forall t > 0 & [\text{Tschebyshev Inequality}]\\
\mathbf{E}\{\eta^2\} = \sum_\ell \mathbf{E}\{\zeta_\ell^2\}\lambda_\ell^2 \leq \sigma^2 \sum_\ell \lambda_\ell^2 \leq \sigma^2 & [\ (3.6.55.a)]\\
\quad \Rightarrow \quad \text{Prob}\{|\eta| > t\} \leq \frac{\sigma^2}{t^2}\ \forall t > 0 & [\text{Tschebyshev Inequality}]\\
\mathbf{E}\{\theta^2\} = \sum_\ell \mathbf{E}\{\zeta_\ell^2\}q_\ell^2 \leq \sigma^2\beta & \\
\quad \Rightarrow \quad \text{Prob}\{|\theta| > t\} \leq \frac{\sigma^2\beta}{t^2}\ \forall t > 0 & [\text{Tschebyshev Inequality}].
\end{array}$$

Thus, for every $\Upsilon > 0$ and all $\lambda \in (0,1)$ we have

$$\begin{array}{rcl}
\text{Prob}\{\|S[\zeta]\| > \Upsilon\} & \leq & \text{Prob}\{\|\xi\|_2 + \max[|\theta|,|\eta|] > \Upsilon\} \leq \text{Prob}\{\|\xi\|_2 > \lambda\Upsilon\}\\
& & +\text{Prob}\{|\theta| > (1-\lambda)\Upsilon\} + \text{Prob}\{|\eta| > (1-\lambda)\Upsilon\}\\
& \leq & \frac{\sigma^2}{\Upsilon^2}\left[\frac{\alpha}{\lambda^2} + \frac{\beta+1}{(1-\lambda)^2}\right],
\end{array}$$

whence, due to $\alpha + \beta \leq 1$,

$$\text{Prob}\{\|S[\zeta]\| > \Upsilon\} \leq \frac{\sigma^2}{\Upsilon^2} \max_{\alpha\in[0,1]} \min_{\lambda\in(0,1)} \left[\frac{\alpha}{\lambda^2} + \frac{2-\alpha}{(1-\lambda)^2}\right] = \frac{8\sigma^2}{\Upsilon^2};$$

with $\Upsilon = \Upsilon(\chi)$, this relation implies (3.6.52).

Assume now that $\zeta_\ell$ satisfy Assumption **A.I**. We should prove that here the relation (3.6.52) holds true with $\Upsilon = \Upsilon_{\text{I}}(\chi)$, or, which is the same,

$$\text{Prob}\left\{\|S[\zeta]\| > \Upsilon\right\} \leq 1-\chi,\ S[\zeta] = \sum_\ell \zeta_\ell B_\ell = \left[\begin{array}{c|c} \sum_\ell \zeta_\ell q_\ell & \sum_\ell \zeta_\ell r_\ell^T \\ \hline \sum_\ell \zeta_\ell r_\ell & (\sum_\ell \zeta_\ell \lambda_\ell)Q \end{array}\right]. \tag{3.6.56}$$

Observe that for a symmetric block-matrix $P = \left[\begin{array}{c|c} A & B^T \\ \hline B & C\end{array}\right]$ we have $\|P\| \leq \|\left[\begin{array}{c|c} \|A\| & \|B\| \\ \hline \|B\| & \|C\|\end{array}\right]\|$, and that the norm of a symmetric matrix does not exceed its Frobenius norm, whence

$$\|S[\zeta]\|^2 \leq |\sum_\ell \zeta_\ell q_\ell|^2 + 2\|\sum_\ell \zeta_\ell r_\ell\|_2^2 + |\sum_\ell \zeta_\ell \lambda_\ell|^2 \equiv \alpha[\zeta] \tag{3.6.57}$$

(recall that $\|Q\| = 1$). Let $E_\rho$ be the ellipsoid $E_\rho = \{z : \alpha[z] \leq \rho^2\}$. Observe that $E_\rho$ contains the centered at the origin Euclidean ball of radius $\rho/\sqrt{3}$. Indeed, applying the Cauchy Inequality, we have

$$\alpha[z] \leq \left(\sum_\ell z_\ell^2\right)\left[\sum_\ell q_\ell^2 + 2\sum_\ell \|r_\ell\|_2^2 + \sum_\ell \lambda_\ell^2\right] \leq 3\sum_\ell z_\ell^2$$

(we have used (3.6.55)). Further, $\zeta_\ell$ are independent with zero mean and $\mathbf{E}\{\zeta_\ell^2\} \leq 1$ for every $\ell$; applying the same (3.6.55), we therefore get $\mathbf{E}\{\alpha[\zeta]\} \leq 3$. By the Tschebyshev Inequality, we have

$$\text{Prob}\{\zeta \in E_\rho\} \equiv \text{Prob}\{\alpha[\zeta] \leq \rho^2\} \geq 1 - \frac{3}{\rho^2}.$$

Invoking the Talagrand Inequality (Theorem A.9), we have

$$\rho^2 > 3 \Rightarrow \mathbf{E}\left\{\exp\{\frac{\text{dist}^2_{\|\cdot\|_2}(\zeta, E_\rho)}{16}\}\right\} \leq \frac{1}{\text{Prob}\{\zeta \in E_\rho\}} \leq \frac{\rho^2}{\rho^2 - 3}.$$

On the other hand, if $r > \rho$ and $\alpha[\zeta] > r^2$, then $\zeta \notin (r/\rho)E_\rho$ and therefore $\mathrm{dist}_{\|\cdot\|_2}(\zeta, E_\rho) \geq (r/\rho - 1)\rho/\sqrt{3} = (r-\rho)/\sqrt{3}$ (recall that $E_\rho$ contains the centered at the origin $\|\cdot\|_2$-ball of radius $\rho/\sqrt{3}$). Applying the Tschebyshev Inequality, we get

$$\begin{array}{l} r^2 > \rho^2 > 3 \Rightarrow \mathrm{Prob}\{\alpha[\zeta] > r^2\} \leq \mathbf{E}\left\{\exp\{\frac{\mathrm{dist}^2_{\|\cdot\|_2}(\zeta,E_\rho)}{16}\}\right\}\exp\{-\frac{(r-\rho)^2}{48}\} \\ \leq \frac{\rho^2\exp\{-\frac{(r-\rho)^2}{48}\}}{\rho^2-3}. \end{array}$$

With $\rho = 2$, $r = \Upsilon_{\mathrm{I}}(\chi) = 2 + 4\sqrt{3\ln\frac{4}{1-\chi}}$ this bound implies $\mathrm{Prob}\{\alpha[\zeta] > r^2\} \leq 1-\chi$; recalling that $\sqrt{\alpha[\zeta]}$ is an upper bound on $\|S[\zeta]\|$, we see that (3.6.52) indeed holds true with $\Upsilon = \Upsilon_{\mathrm{I}}(\chi)$.

Now consider the case when $\zeta \sim \mathcal{N}(0, I_L)$. Observe that $\alpha[\zeta]$ is a homogeneous quadratic form of $\zeta$: $\alpha[\zeta] = \zeta^T A\zeta$, $A_{ij} = q_iq_j + 2r_i^Tr_j + \lambda_i\lambda_j$. We see that the matrix $A$ is positive semidefinite, and $\mathrm{Tr}(A) = \sum_i(q_i^2 + \lambda_i^2 + 2\|r_i\|_2^2) \leq 3$. Denoting by $\mu_\ell$ the eigenvalues of $A$, we have $\zeta^TA\zeta = \sum_{\ell=1}^L \mu_\ell\xi_\ell^2$, where $\xi \sim \mathcal{N}(0, I_L)$ is an appropriate rotation of $\zeta$. Now we can use the Bernstein scheme to bound from above $\mathrm{Prob}\{\alpha[\zeta] > \rho^2\}$:

$$\begin{array}{l} \forall(\gamma \geq 0, \max\limits_\ell \gamma\mu_\ell < 1/2): \\ \ln\left(\mathrm{Prob}\{\alpha[\zeta] > \rho^2\}\right) \leq \ln\left(\mathbf{E}\left\{\exp\{\gamma\zeta^TA\zeta\}\right\}\exp\{-\gamma\rho^2\}\right) \\ = \ln\left(\mathbf{E}\left\{\exp\{\gamma\sum_\ell \mu_\ell\xi_\ell^2\}\right\}\right) - \gamma\rho^2 = \sum_\ell \ln\left(\mathbf{E}\left\{\exp\{\gamma\mu_\ell\xi_\ell^2\}\right\}\right) - \gamma\rho^2 \\ = -\frac{1}{2}\sum_\ell \ln(1-2\mu_\ell\gamma) - \gamma\rho^2. \end{array}$$

The concluding expression is a convex and monotone function of $\mu$'s running through the box $\{0 \leq \mu_\ell < \frac{1}{2\gamma}\}$. It follows that when $\gamma < 1/6$, the maximum of the expression over the set $\{\mu_1, ..., \mu_L \geq 0, \sum_\ell \mu_\ell \leq 3\}$ is $-\frac{1}{2}\ln(1-6\gamma) - \gamma\rho^2$. We get

$$0 \leq \gamma < \frac{1}{6} \Rightarrow \ln\left(\mathrm{Prob}\{\alpha[\zeta] > \rho^2\}\right) \leq -\frac{1}{2}\ln(1-6\gamma) - \gamma\rho^2.$$

Optimizing this bound in $\gamma$ and setting $\rho^2 = 3(1+\Delta)$, $\Delta \geq 0$, we get $\mathrm{Prob}\{\alpha[\zeta] > 3(1+\Delta)\} \leq \exp\{-\frac{1}{2}[\Delta - \ln(1+\Delta)]\}$. It follows that if $\chi \in (0,1)$ and $\Delta = \Delta(\chi) \geq 0$ is such that $\Delta - \ln(1+\Delta) = 2\ln\frac{1}{1-\chi}$, then

$$\mathrm{Prob}\{\|S[\zeta]\| > \sqrt{3(1+\Delta)}\} \leq \mathrm{Prob}\{\alpha[\zeta] > 3(1+\Delta)\} \leq 1-\chi.$$

It is easily seen that when $1-\chi \leq \frac{1}{7}$, one has $\Delta(\chi) \leq 3\ln\frac{1}{1-\chi}$, that is, $\mathrm{Prob}\{\|S[\zeta]\| > \sqrt{3\left(1+3\ln\frac{1}{1-\chi}\right)}\} \leq 1-\chi$, which is exactly what was claimed in the case of Gaussian $\zeta$. □

### Application: Recovering Signal from Indirect Noisy Observations

Consider the situation as follows (cf. section 3.2.6): we observe in noise a linear transformation

$$u = As + \rho\xi \tag{3.6.58}$$

of a random signal $s \in \mathbb{R}^n$; here $A$ is a given $m\times n$ matrix, $\xi \sim \mathcal{N}(0, I_m)$ is the noise, (which is independent of $s$), and $\rho \geq 0$ is a (deterministic) noise level. Our goal is to find a linear estimator

$$\widehat{s}(u) = Gu \equiv GAs + \rho G\xi \tag{3.6.59}$$

such that

$$\mathrm{Prob}\{\|\widehat{s}(u) - s\|_2 \leq \tau_*\} \geq 1-\epsilon, \tag{3.6.60}$$

where $\tau_* > 0$ and $\epsilon \ll 1$ are given. Note that the probability in (3.6.60) is taken w.r.t. the joint distribution of $s$ and $\xi$. We assume below that $s \sim \mathcal{N}(0, C)$ with known covariance matrix $C \succ 0$.

Besides this, we assume that $m \geq n$ and $A$ is of rank $n$. When there is no observation noise, we can recover $s$ from $u$ in a linear fashion without any error; it follows that when $\rho > 0$ is small enough, there exists $G$ that makes (3.6.60) valid. Let us find the largest such $\rho$, that is, let us solve the optimization problem

$$\max_{G,\rho} \{\rho : \mathrm{Prob}\{\|(GA - I_n)s + \rho G\xi\|_2 \leq \tau_*\} \geq 1 - \epsilon\}. \tag{3.6.61}$$

Setting $S = C^{1/2}$ and introducing a random vector $\theta \sim \mathcal{N}(0, I_n)$ independent of $\xi$ (so that the random vector $[S^{-1}s; \xi]$ has exactly the same $\mathcal{N}(0, I_{n+m})$ distribution as the vector $\zeta = [\theta; \xi]$), we can rewrite our problem equivalently as

$$\max_{G,\rho} \{\rho : \mathrm{Prob}\{\|H_\rho(G)\zeta\|_2 \leq \tau_*\} \geq 1 - \epsilon\}, \ \ H_\rho(G) = [(GA - I_n)S, \rho G]. \tag{3.6.62}$$

Let $h_\rho^\ell(G)$ be the $\ell$-th column in the matrix $H_\rho(G)$, $\ell = 1, ..., L = m + n$. Invoking Lemma 3.1, our problem is nothing but the chance constrained program

$$\begin{array}{c}\max\limits_{G,\rho} \left\{\rho : \mathrm{Prob}\left\{\sum\limits_{\ell=1}^{L} \zeta_\ell \mathcal{A}_\rho^\ell(G) \preceq \tau_* \mathcal{A}^{\mathrm{n}} \equiv \tau_* I_{n+1}\right\} \geq 1 - \epsilon\right\} \\ \mathcal{A}_\rho^\ell(G) = \left[\begin{array}{c|c} & [h_\rho^\ell(G)]^T \\ \hline h_\rho^\ell(G) & \end{array}\right]. \end{array} \tag{3.6.63}$$

We intend to process the latter problem as follows:

A) We use our "Conjecture-related" approximation scheme to build a nondecreasing continuous function $\Gamma(\rho) \to 0, \rho \to +0$, and matrix-valued function $G_\rho$ (both functions are efficiently computable) such that

$$\mathrm{Prob}\{\|(GA - I_n)s + \rho G\xi\|_2 > \tau_*\} = \mathrm{Prob}\{\sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}_\rho^\ell(G_\rho) \not\preceq \tau_* I_{n+1}\} \leq \Gamma(\rho). \tag{3.6.64}$$

B) We then solve the approximating problem

$$\max_{\rho} \{\rho : \Gamma(\rho) \leq \epsilon\}. \tag{3.6.65}$$

Clearly, a feasible solution $\rho$ to the latter problem, along with the associated matrix $G_\rho$, form a feasible solution to the problem of interest (3.6.63). On the other hand, the approximating problem is efficiently solvable: $\Gamma(\rho)$ is nondecreasing, efficiently computable and $\Gamma(\rho) \to 0$ as $\rho \to +0$, so that the approximating problem can be solved efficiently by bisection. We find a feasible nearly optimal solution $\widehat{\rho}$ to the approximating problem and treat $(\widehat{\rho}, G_{\widehat{\rho}})$ as a suboptimal solution to the problem of interest. By our analysis, this solution is feasible for the latter problem.

**Remark 3.2** In fact, the constraint in (3.6.62) is simpler than a general-type chance constrained conic quadratic inequality — it is a chance constrained Least Squares inequality (the right hand side is affected neither by the decision variables, nor by the noise), and as such it admits a Bernstein-type approximation described in section 2.5.3, see Corollary 2.1. Of course, in the outlined scheme one can use the Bernstein approximation as an alternative to the Conjecture-related approximation.

Now let us look at steps A, B in more details.
**Step A).** We solve the semidefinite program

$$\nu_*(\rho) = \min_{\nu,G} \left\{\nu : \sum_{\ell=1}^{L} (\mathcal{A}_\rho^\ell(G))^2 \preceq \nu I_{n+1}\right\}; \tag{3.6.66}$$

whenever $\rho > 0$, this problem clearly is solvable. Due to the fact that part of the matrices $\mathcal{A}_\rho^\ell(G)$ are independent of $\rho$, and the remaining ones are proportional to $\rho$, the optimal value is a positive continuous and nondecreasing function of $\rho > 0$. Finally, $\nu_*(\rho) \to +0$ as $\rho \to +0$ (look what happens at the point $G$ satisfying the relation $GA = I_n$).

Let $G_\rho$ be an optimal solution to (3.6.66). Setting $A_\ell = \mathcal{A}^\ell_\rho(G_\rho)\nu_*^{-\frac{1}{2}}(\rho)$, $A = I_{n+1}$, the arrow-type matrices $A, A_1, ..., A_L$ satisfy (3.6.13); invoking Theorem 3.18, we conclude that

$$\begin{array}{l}\chi \in [\frac{6}{7}, 1) \Rightarrow \mathrm{Prob}\{-\Upsilon(\chi)\nu_*^{\frac{1}{2}}(\rho)I_{n+1} \preceq \sum\limits_{\ell=1}^{L} \zeta_\ell \mathcal{A}^\ell_\rho(G_\rho) \preceq \Upsilon(\chi)\nu_*^{\frac{1}{2}}(\rho)I_{n+1}\} \\ \geq \chi,\ \Upsilon(\chi) = \sqrt{3\left(1 + 3\ln\frac{1}{1-\chi}\right)}.\end{array}$$

Now let $\chi$ and $\rho$ be such that $\chi \in [6/7, 1)$ and $\Upsilon(\chi)\sqrt{\nu_*(\rho)} \leq \tau_*$. Setting

$$Q = \{z : \|\sum_{\ell=1}^{L} z_\ell \mathcal{A}^\ell_\rho(G_\rho)\| \leq \Upsilon(\chi)\sqrt{\nu_*(\rho)}\},$$

we get a closed convex set such that the random vector $\zeta \sim \mathcal{N}(0, I_{n+m})$ takes its values in $Q$ with probability $\geq \chi > 1/2$. Invoking Theorem A.10 (where we set $\alpha = \tau_*/(\Upsilon(\chi)\sqrt{\nu_*(\rho)})$), we get

$$\begin{array}{l}\mathrm{Prob}\left\{\sum\limits_{\ell=1}^{L} \zeta_\ell \mathcal{A}^\ell_\rho(G_\rho) \not\preceq \tau_* I_{n+1}\right\} \leq \mathrm{Erf}\left(\frac{\tau_*\mathrm{ErfInv}(1-\chi)}{\sqrt{\nu_*(\rho)}\Upsilon(\chi)}\right) \\ = \mathrm{Erf}\left(\frac{\tau_*\mathrm{ErfInv}(1-\chi)}{\sqrt{3\nu_*(\rho)\left[1+3\ln\frac{1}{1-\chi}\right]}}\right).\end{array}$$

Setting

$$\Gamma(\rho) = \inf_\chi \left\{ \mathrm{Erf}\left(\frac{\tau_*\mathrm{ErfInv}(1-\chi)}{\sqrt{3\nu_*(\rho)\left[1 + 3\ln\frac{1}{1-\chi}\right]}}\right) : \begin{array}{l}\chi \in [6/7, 1), \\ 3\nu_*(\rho)\left[1 + 3\ln\frac{1}{1-\chi}\right] \leq \tau_*^2\end{array}\right\} \tag{3.6.67}$$

(if the feasible set of the right hand side optimization problem is empty, then, by definition, $\Gamma(\rho) = 1$), we ensure (3.6.64). Taking into account that $\nu_*(\rho)$ is a nondecreasing continuous function of $\rho > 0$ that tends to 0 as $\rho \to +0$, it is immediately seen that $\Gamma(\rho)$ possesses these properties as well.

**Solving (3.6.66).** Good news is that problem (3.6.66) has a closed form solution. To see this, note that the matrices $\mathcal{A}^\ell_\rho(G)$ are pretty special arrow type matrices: their diagonal entries are zero, so that these $(n+1) \times (n+1)$ matrices are of the form $\left[\begin{array}{c|c} & [h^\ell_\rho(G)]^T \\ \hline h^\ell_\rho(G) & \end{array}\right]$ with $n$-dimensional vectors $h^\ell_\rho(G)$ affinely depending on $G$. Now let us make the following observation:

**Lemma 3.6** *Let $f_\ell \in \mathbb{R}^n$, $\ell = 1, ..., L$, and $\nu \geq 0$. Then*

$$\sum_{\ell=1}^{L} \left[\begin{array}{c|c} & f_\ell^T \\ \hline f_\ell & \end{array}\right]^2 \preceq \nu I_{n+1} \tag{$*$}$$

*if and only if $\sum\limits_\ell f_\ell^T f_\ell \leq \nu$.*

**Proof.** Relation $(*)$ is nothing but

$$\sum_\ell \left[\begin{array}{c|c} f_\ell^T f_\ell & \\ \hline & f_\ell f_\ell^T \end{array}\right] \preceq \nu I_{n+1},$$

so it definitely implies that $\sum\limits_\ell f_\ell^T f_\ell \leq \nu$. To prove the inverse implication, it suffices to verify that the relation $\sum\limits_\ell f_\ell^T f_\ell \leq \nu$ implies that $\sum\limits_\ell f_\ell f_\ell^T \preceq \nu I_n$. This is immediate due to $\mathrm{Tr}(\sum\limits_\ell f_\ell f_\ell^T) = \sum\limits_\ell f_\ell^T f_\ell \leq \nu$, (note that the matrix $\sum_\ell f_\ell f_\ell^T$ is positive semidefinite, and therefore its maximal eigenvalue does not exceed its trace). □

In view of Lemma 3.6, the optimal solution and the optimal value in (3.6.66) are exactly the same as their counterparts in the minimization problem

$$\nu = \min_G \sum_\ell [h_\rho^\ell(G)]^T h_\rho^\ell(G).$$

Thus, (3.6.66) is nothing but the problem

$$\nu_*(\rho) = \min_G \left\{ \mathrm{Tr}((GA - I_n)C(GA - I)^T) + \rho^2 \mathrm{Tr}(GG^T) \right\}. \tag{3.6.68}$$

The objective in this unconstrained problem has a very transparent interpretation: it is the mean squared error of the linear estimator $\widehat{s} = Gu$, the noise intensity being $\rho$. The matrix $G$ minimizing this objective is called the *Wiener filter*; a straightforward computation yields

$$\begin{array}{rcl} G_\rho &=& CA^T(ACA^T + \rho^2 I_m)^{-1}, \\ \nu_*(\rho) &=& \mathrm{Tr}\left((G_\rho A - I_n)C(G_\rho A - I_n)^T + \rho^2 G_\rho G_\rho^T\right). \end{array} \tag{3.6.69}$$

**Remark 3.3** The Wiener filter is one of the oldest and the most basic tools in Signal Processing; it is good news that our approximation scheme recovers this tool, albeit from a different perspective: we were seeking a linear filter that ensures that with probability $1 - \epsilon$ the recovering error does not exceed a given threshold (a problem that seemingly does not admit a closed form solution); it turned out that the *sub*optimal solution yielded by our approximation scheme is the precise solution to a simple classical problem.

**Refinements.** The pair $(\widehat{\rho}, G_{\mathrm{W}} = G_{\widehat{\rho}})$ ("W" stands for "Wiener") obtained via the outlined approximation scheme is feasible for the problem of interest (3.6.63). However, we have all reason to expect that our provably 100%-reliable approach is conservative — exactly because of its 100% reliability. In particular, it is very likely that $\widehat{\rho}$ is a too conservative lower bound on the actual feasibility radius $\rho_*(G_{\mathrm{W}})$ — the largest $\rho$ such that $(\rho, G_{\mathrm{W}})$ is feasible for the chance constrained problem of interest. We can try to improve this lower bound by the Randomized $r$ procedure, e.g., as follows:

Given a confidence parameter $\delta \in (0, 1)$, we run $\nu = 10$ steps of bisection on the segment $\Delta = [\widehat{\rho}, 100\widehat{\rho}]$. At a step $t$ of this process, given the previous localizer $\Delta_{t-1}$ (a segment contained in $\Delta$, with $\Delta_0 = \Delta$), we take as the current trial value $\rho_t$ of $\rho$ the midpoint of $\Delta_{t-1}$ and apply the Randomized $r$ procedure in order to check whether $(\rho_t, G_{\mathrm{W}})$ is feasible for (3.6.63). Specifically, we

- compute the $L = m + n$ vectors $h_{\rho_t}^\ell(G_{\mathrm{W}})$ and the quantity $\mu_t = \sqrt{\sum_{\ell=1}^{m+n} \|h_{\rho_t}^\ell(G_{\mathrm{W}})\|_2^2}$. By Lemma 3.6, we have

  $$\sum_{\ell=1}^{L} \left[\mathcal{A}_{\rho_t}^\ell(G_{\mathrm{W}})\right]^2 \preceq \mu_t^2 I_{n+1},$$

  so that the matrices $A = I_{n+1}$, $A_\ell = \mu_t^{-1}\mathcal{A}_{\rho_t}^\ell(G_{\mathrm{W}})$ satisfy (3.6.13);

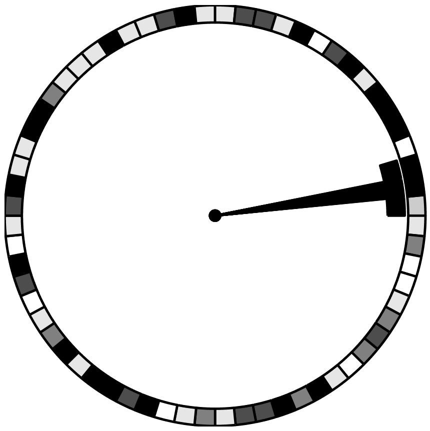
- apply to the matrices $A, A_1, ..., A_L$ the Randomized $r$ procedure with parameters $\epsilon, \delta/\nu$, thus ending up with a random quantity $r_t$ such that "up to probability of bad sampling $\le \delta/\nu$," one has

  $$\mathrm{Prob}\{\zeta : -I_{n+1} \preceq r_t \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq I_{n+1}\} \ge 1 - \epsilon,$$

  or, which is the same,

  $$\mathrm{Prob}\{\zeta : -\frac{\mu_t}{r_t} I_{n+1} \preceq \sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}_\rho^\ell(G_W) \preceq \frac{\mu_t}{r_t} I_{n+1}\} \ge 1 - \epsilon. \tag{3.6.70}$$

  Note that when the latter relation is satisfied and $\frac{\mu_t}{r_t} \le \tau_*$, the pair $(\rho_t, G_{\mathrm{W}})$ is feasible for (3.6.63);



$(K * s)_i = 0.2494s_{i-1} + 0.5012s_i + 0.2494s_{i+1}$

Figure 3.9: A scanner.

- finally, complete the bisection step, namely, check whether $\mu_t/r_t \leq \tau_*$. If it is the case, we take as our new localizer $\Delta_t$ the part of $\Delta_{t-1}$ to the right of $\rho_t$, otherwise $\Delta_t$ is the part of $\Delta_{t-1}$ to the left of $\rho_t$.

After $\nu$ bisection steps are completed, we claim that the left endpoint $\widetilde{\rho}$ of the last localizer $\Delta_\nu$ is a lower bound on $\rho_*(G_{\rm W})$. Observe that this claim is valid, provided that all $\nu$ inequalities (3.6.70) take place, which happens with probability at least $1-\delta$.

**Illustration: Deconvolution.** A rotating scanning head reads random signal $s$ as shown in figure 3.9. The signal registered when the head observes bin $i$, $0 \leq i < n$, is

$$u_i = (As)_i + \rho\xi_i \equiv \sum_{j=-d}^{d} K_j s_{(i-j) \bmod n} + \rho\xi_i,\ 0 \leq i < n,$$

where $r = p \bmod n$, $0 \leq r < n$, is the remainder when dividing $p$ by $n$. The signal $s$ is assumed to be Gaussian with zero mean and known covariance $C_{ij} = \mathbf{E}\{s_i s_j\}$ depending on $(i-j) \bmod n$ only ("stationary periodic discrete-time Gaussian process"). The goal is to find a linear recovery $\widehat{s} = Gu$ and the largest $\rho$ such that

$$\mathrm{Prob}_{[s;\xi]}\left\{\|G(As+\rho\xi) - s\|_2 \leq \tau_*\right\} \geq 1-\epsilon.$$

We intend to process this problem via the outlined approach using two safe approximations of the chance constraint of interest — the Conjecture-related and the Bernstein (see Remark 3.2). The recovery matrices and critical levels of noise as given by these two approximations will be denoted $G_W, \rho_W$ ("W" for "Wiener") and $G_B, \rho_B$ ("B" for "Bernstein"), respectively.

Note that in the case in question one can immediately verify that the matrices $A^TA$ and $C$ commute. Whenever this is the case, the computational burden to compute $G_W$ and $G_B$ reduces dramatically. Indeed, after appropriate rotations of $x$ and $y$ we arrive at the situation where both $A$ and $C$ are diagonal, in which case in both our approximation schemes one loses nothing by restricting $G$ to be diagonal. This significantly reduces the dimensions of the convex problems we need to solve.

In the experiment we use

$$n = 64,\ d = 1,\ \tau_* = 0.1\sqrt{n} = 0.8,\ \epsilon = 1.\text{e-}4;$$

$C$ was set to the unit matrix, (meaning that $s \sim \mathcal{N}(0, I_{64})$), and the convolution kernel $K$ is the one shown in figure 3.9. After $(G_W, \rho_w)$ and $(G_B, \rho_B)$ were computed, we used the Randomized $r$ procedure

| Admissible noise level | Bernstein approximation | Conjecture-related approximation |
|---|---|---|
| Before refinement | 1.92e-4 | 1.50e-4 |
| After refinement ($\delta = 1.e\text{-}6$) | 3.56e-4 | 3.62e-4 |

Table 3.4: Results of deconvolution experiment.

| Noise level | $\mathrm{Prob}\{\|\widehat{s} - s\|_2 > \tau_*\}$ | |
|---|---|---|
| | $G = G_B$ | $G = G_W$ |
| 3.6e-4 | 0 | 0 |
| 7.2e-4 | 6.7e-3 | 6.7e-3 |
| 1.0e-3 | 7.4e-2 | 7.5e-2 |

Table 3.5: Empirical value of $\mathrm{Prob}\{\|\widehat{s} - s\|_2 > 0.8\}$ based on 10,000 simulations.

with $\delta = 1.e\text{-}6$ to refine the critical values of noise for $G_W$ and $G_B$; the refined values of $\rho$ are denoted $\widehat{\rho}_W$ and $\widehat{\rho}_B$, respectively.

The results of the experiments are presented in table 3.4. While $G_B$ and $G_W$ turned out to be close, although not identical, the critical noise levels as yielded by the Conjecture-related and the Bernstein approximations differ by $\approx 30\%$. The refinement increases these critical levels by a factor $\approx 2$ and makes them nearly equal. The resulting critical noise level 3.6e-4 is not too conservative: the simulation results shown in table 3.5 demonstrate that at a twice larger noise level, the probability for the chance constraint to be violated is by far larger than the required 1.e-4.

**Modifications.** We have addressed the Signal Recovery problem (3.6.58), (3.6.59), (3.6.60) in the case when $s \sim \mathcal{N}(0, C)$ is random, the noise is independent of $s$ and the probability in (3.6.60) is taken w.r.t. the joint distribution of $\xi$ and $s$. Next we want to investigate two other versions of the problem.

**Recovering a uniformly distributed signal.** Assume that the signal $s$ is

(a) uniformly distributed in the unit box $\{s \in \mathbb{R}^n : \|s\|_\infty \leq 1\}$,

or

(b) uniformly distributed on the vertices of the unit box

and is independent of $\xi$. Same as above, our goal is to ensure the validity of (3.6.60) with as large $\rho$ as possible. To this end, let us use Gaussian Majorization. Specifically, in the case of (a), let $\widetilde{s} \sim \mathcal{N}(0, (2/\pi)I)$. As it was explained in section 3.6.3, the condition

$$\mathrm{Prob}\{\|(GA - I)\widetilde{s} + \rho G\xi\|_2 \leq \tau_*\} \geq 1 - \epsilon$$

is sufficient for the validity of (3.6.60). Thus, we can use the Gaussian case procedure presented in section 3.6.4 with the matrix $(2/\pi)I$ in the role of $C$; an estimator that is good in this case will be at least as good in the case of the signal $s$.

In case of (b), we can act similarly, utilizing Theorem 3.16. Specifically, let $\widetilde{s} \sim \mathcal{N}(0, (\pi/2)I)$ be independent of $\xi$. Consider the parametric problem

$$\nu(\rho) \equiv \min_G \left\{ \frac{\pi}{2} \mathrm{Tr}\left((GA - I)(GA - I)^T\right) + \rho^2 \mathrm{Tr}(GG^T) \right\}, \tag{3.6.71}$$

$\rho \geq 0$ being the parameter (cf. (3.6.68) and take into account that the latter problem is equivalent to (3.6.66)), and let $G_\rho$ be an optimal solution to this problem. The same reasoning as on p. 168 shows that

$$6/7 \leq \chi < 1 \Rightarrow \mathrm{Prob}\{(\widetilde{s}, \xi) : \|(G_\rho A - I)\widetilde{s} + \rho G_\rho \xi\|_2 \leq \Upsilon(\chi)\nu_*^{1/2}(\rho)\} \geq \chi,$$

$$\Upsilon(\chi) = \sqrt{3\left(1 + 3\ln \tfrac{1}{1-\chi}\right)}.$$

Applying Theorem 3.16 to the convex set $Q = \{(z,x) : \|(G_\rho A - I)z + \rho G_\rho x\|_2 \leq \Upsilon(\chi)\nu_*^{1/2}(\rho)$ and the random vectors $[s;\xi]$, $[\widetilde{s};\xi]$, we conclude that

$$\forall \begin{pmatrix} \chi\in[6/7,1) \\ \gamma>1 \end{pmatrix} : \mathrm{Prob}\{(s,\xi) : \|(G_\rho A - I)s + \rho G_\rho \xi\|_2 > \gamma\Upsilon(\chi)\nu_*^{1/2}(\rho)\}$$
$$\leq \min_{\beta\in[1,\gamma)} \frac{1}{\gamma-\beta}\int_\beta^\infty \mathrm{Erf}(r\mathrm{ErfInv}(1-\chi))dr.$$

We conclude that setting

$$\widetilde{\Gamma}(\rho) = \inf_{\chi,\gamma,\beta}\left\{\frac{1}{\gamma-\beta}\int_\beta^\infty \mathrm{Erf}(r\mathrm{ErfInv}(1-\chi))dr : \begin{array}{l} 6/7\leq\chi<1,\gamma>1 \\ 1\leq\beta<\gamma \\ \gamma\Upsilon(\chi)\nu_*^{1/2}(\rho)\leq\tau_* \end{array}\right\}$$
$$\left[\Upsilon(\chi) = \sqrt{3\left(1+3\ln\frac{1}{1-\chi}\right)}\right]$$

($\widetilde{\Gamma}(\rho) = 1$ when the right hand side problem is infeasible), one has

$$\mathrm{Prob}\{(s,\xi) : \|(G_\rho A - I)s + \rho G_\rho\xi\|_2 > \tau_*\} \leq \widetilde{\Gamma}(\rho)$$

(cf. p. 168). It is easily seen that $\widetilde{\Gamma}(\cdot)$ is a continuous nondecreasing function of $\rho > 0$ such that $\widetilde{\Gamma}(\rho) \to 0$ as $\rho \to +0$, and we end up with the following safe approximation of the Signal Recovery problem:

$$\max_\rho \left\{\rho : \widetilde{\Gamma}(\rho) \leq \epsilon\right\}$$

(cf. (3.6.65)).

Note that in the above "Gaussian majorization" scheme we could use the Bernstein approximation, based on Corollary 2.1, of the chance constraint $\mathrm{Prob}\{\|(GA-I)\widetilde{s} + \rho G\xi\|_2 \leq \tau_*\} \geq 1-\epsilon$ instead of the Conjecture-related approximation.

**The case of deterministic uncertain signal.** Up to now, signal $s$ was considered as random and independent of $\xi$, and the probability in (3.6.60) was taken w.r.t. the joint distribution of $s$ and $\xi$; as a result, certain "rare" realizations of the signal can be recovered very poorly. Our current goal is to understand what happens when we replace the specification (3.6.60) with

$$\begin{array}{l}\forall (s\in\mathcal{S}) : \\ \mathrm{Prob}\{\xi : \|Gu - s\|_2 \leq \tau_*\} \equiv \mathrm{Prob}\{\xi : \|(GA-I)s + \rho G\xi\|_2 \leq \tau_*\} \geq 1-\epsilon,\end{array} \tag{3.6.72}$$

where $\mathcal{S} \subset \mathbb{R}^n$ is a given compact set.

Our starting point is the following observation:

**Lemma 3.7** *Let $G$, $\rho \geq 0$ be such that*

$$\Theta \equiv \frac{\tau_*^2}{\max\limits_{s\in\mathcal{S}} s^T(GA-I)^T(GA-I)s + \rho^2\mathrm{Tr}(G^TG)} \geq 1. \tag{3.6.73}$$

*Then for every $s \in \mathcal{S}$ one has*

$$\mathrm{Prob}_{\zeta\sim\mathcal{N}(0,I)}\left\{\|(GA-I)s + \rho G\zeta\|_2 > \tau_*\right\} \leq \exp\left\{-\frac{(\Theta-1)^2}{4(\Theta+1)}\right\}. \tag{3.6.74}$$

**Proof.** There is nothing to prove when $\Theta = 1$, so that let $\Theta > 1$. Let us fix $s \in \mathcal{S}$ and let $g = (GA-I)s$, $W = \rho^2 G^T G$, $w = \rho G^T g$. We have

$$\begin{aligned}&\mathrm{Prob}\{\|(GA-I)s + \rho G\zeta\|_2 > \tau_*\} = \mathrm{Prob}\left\{\|g + \rho G\zeta\|_2^2 > \tau_*^2\right\} \\ &= \mathrm{Prob}\left\{\zeta^T[\rho^2 G^T G]\zeta + 2\zeta^T\rho G^T g > \tau_*^2 - g^T g\right\} \\ &= \mathrm{Prob}\left\{\zeta^T W\zeta + 2\zeta^T w > \tau_*^2 - g^T g\right\}.\end{aligned} \tag{3.6.75}$$

Denoting by $\lambda$ the vector of eigenvalues of $W$, we can assume w.l.o.g. that $\lambda \neq 0$, since otherwise $W = 0$, $w = 0$ and thus the left hand side in (3.6.75) is 0 (note that $\tau_*^2 - g^T g > 0$ due to (3.6.73) and since $s \in \mathcal{S}$), and thus (3.6.74) is trivially true. Setting

$$\Omega = \frac{\tau_*^2 - g^T g}{\sqrt{\lambda^T\lambda + w^T w}}$$

and invoking Proposition 2.3, we arrive at

$$\begin{array}{l} \mathrm{Prob}\{\|(GA-I)s + \rho G\zeta\|_2 > \tau_*\} \leq \exp\left\{-\frac{\Omega^2\sqrt{\lambda^T\lambda+w^Tw}}{4\left[2\sqrt{\lambda^T\lambda+w^Tw}+\|\lambda\|_\infty\Omega\right]}\right\} \\ = \exp\left\{-\frac{[\tau_*^2-g^Tg]^2}{4[2[\lambda^T\lambda+w^Tw]+\|\lambda\|_\infty[\tau_*^2-g^Tg]]}\right\} \\ = \exp\left\{-\frac{[\tau_*^2-g^Tg]^2}{4[2[\lambda^T\lambda+g^T[\rho^2GG^T]g]+\|\lambda\|_\infty[\tau_*^2-g^Tg]]}\right\} \\ \leq \exp\left\{-\frac{[\tau_*^2-g^Tg]^2}{4\|\lambda\|_\infty[2[\|\lambda\|_1+g^Tg]+[\tau_*^2-g^Tg]]}\right\}, \end{array} \tag{3.6.76}$$

where the concluding inequality is due to $\rho^2 GG^T \preceq \|\lambda\|_\infty I$ and $\lambda^T\lambda \leq \|\lambda\|_\infty\|\lambda\|_1$. Further, setting $\alpha = g^T g$, $\beta = \mathrm{Tr}(\rho^2 G^T G)$ and $\gamma = \alpha + \beta$, observe that $\beta = \|\lambda\|_1 \geq \|\lambda\|_\infty$ and $\tau_*^2 \geq \Theta\gamma \geq \gamma$ by (3.6.73). It follows that

$$\frac{[\tau_*^2 - g^T g]^2}{4\|\lambda\|_\infty\left[2[\|\lambda\|_1 + g^T g] + [\tau_*^2 - g^T g]\right]} \geq \frac{(\tau_*^2 - \gamma + \beta)^2}{4\beta(\tau_*^2 + \gamma + \beta)} \geq \frac{(\tau_*^2 - \gamma)^2}{4\gamma(\tau_*^2 + \gamma)},$$

where the concluding inequality is readily given by the relations $\tau_*^2 \geq \gamma \geq \beta > 0$. Thus, (3.6.76) implies that

$$\mathrm{Prob}\{\|(GA-I)s + \rho G\zeta\|_2 > \tau_*\} \leq \exp\left\{-\frac{(\tau_*^2 - \gamma)^2}{4\gamma(\tau_*^2 + \gamma)}\right\} \leq \exp\left\{-\frac{(\Theta - 1)^2}{4(\Theta + 1)}\right\}. \qquad \square$$

Lemma 3.7 suggests a safe approximation of the problem of interest as follows. Let $\Theta(\epsilon) > 1$ be given by

$$\exp\{-\frac{(\Theta-1)^2}{4(\Theta+1)}\} = \epsilon \qquad [\Rightarrow \Theta(\epsilon) = (4 + o(1))\ln(1/\epsilon) \text{ as } \epsilon \to +0]$$

and let

$$\phi(G) = \max_{s\in\mathcal{S}} s^T(GA-I)^T(GA-I)s, \tag{3.6.77}$$

(this function clearly is convex). By Lemma 3.7, the optimization problem

$$\max_{\rho,G}\left\{\rho : \phi(G) + \rho^2\mathrm{Tr}(G^T G) \leq \gamma_* \equiv \Theta^{-1}(\epsilon)\tau_*^2\right\} \tag{3.6.78}$$

is a safe approximation of the problem of interest. Applying bisection in $\rho$, we can reduce this problem to a "short series" of convex feasibility problems of the form

$$\text{find } G\text{: } \phi(G) + \rho^2\mathrm{Tr}(G^T G) \leq \gamma_*. \tag{3.6.79}$$

Whether the latter problems are or are not computationally tractable depends on whether the function $\phi(G)$ is so, which happens if and only if we can efficiently optimize positive semidefinite quadratic forms $s^T Q s$ over $\mathcal{S}$.

**Example 3.8** Let $\mathcal{S}$ be an ellipsoid centered at the origin:

$$\mathcal{S} = \{s = Hv : v^T v \leq 1\}$$

In this case, it is easy to compute $\phi(G)$ — this function is semidefinite representable:

$$\begin{array}{l}\phi(G)\leq t \Leftrightarrow \max\limits_{s\in\mathcal{S}} s^T(GA-I)^T(GA-I)s\leq t\\ \Leftrightarrow \max\limits_{v:\|v\|_2\leq 1} v^T(H^T(GA-I)^T(GA-I)Hv\leq t\\ \Leftrightarrow \lambda_{\max}(H^T(GA-I)^T(GA-I)H)\leq t\\ \Leftrightarrow tI-H^T(GA-I)^T(GA-I)H\succeq 0 \Leftrightarrow \left[\begin{array}{c|c} tI & H^T(GA-I)^T\\ \hline (GA-I)H & I\end{array}\right]\succeq 0,\end{array}$$

where the concluding $\Leftrightarrow$ is given by the Schur Complement Lemma. Consequently, (3.6.79) is the efficiently solvable convex feasibility problem

$$\text{Find } G,t:\quad t+\rho^2\mathrm{Tr}(G^TG)\leq\gamma_*,\ \left[\begin{array}{c|c} tI & H^T(GA-I)^T\\ \hline (GA-I)H & I\end{array}\right]\succeq 0.$$

Example 3.8 allows us to see the dramatic difference between the case where we are interested in "highly reliable with high probability" recovery of a *random* signal and "highly reliable" recovery of *every* realization of uncertain signal. Specifically, assume that $G,\rho$ are such that (3.6.60) is satisfied with $s\sim\mathcal{N}(0,I_n)$. Note that when $n$ is large, $s$ is nearly uniformly distributed over the sphere $\mathcal{S}$ of radius $\sqrt{n}$ (indeed, $s^Ts=\sum_i s_i^2$, and by the Law of Large Numbers, for $\delta>0$ the probability of the event $\{\|s\|_2\notin[(1-\delta)\sqrt{n},(1+\delta)\sqrt{n}]\}$ goes to 0 as $n\to\infty$, in fact exponentially fast. Also, the direction $s/\|s\|_2$ of $s$ is uniformly distributed on the unit sphere). Thus, the recovery in question is, essentially, a highly reliable recovery of random signal uniformly distributed over the above sphere $\mathcal{S}$. Could we expect the recovery to "nearly satisfy" (3.6.72), that is, to be reasonably good in the worst case over the signals from $\mathcal{S}$? The answer is negative when $n$ is large. Indeed, a *sufficient* condition for (3.6.60) to be satisfied is

$$\mathrm{Tr}((GA-I)^T(GA-I))+\rho^2\mathrm{Tr}(G^TG)\leq\frac{\tau_*^2}{O(1)\ln(1/\epsilon)}\qquad(*)$$

with appropriately chosen absolute constant $O(1)$. A *necessary* condition for (3.6.72) to be satisfied is

$$n\lambda_{\max}((GA-I)^T(GA-I))+\rho^2\mathrm{Tr}(G^TG)\leq O(1)\tau_*^2.\qquad(**)$$

Since the trace of the $n\times n$ matrix $Q=(GA-I)^T(GA-I)$ can be nearly $n$ times less than $n\lambda_{\max}(Q)$, the validity of $(*)$ *by far* does not imply the validity of $(**)$. To be more rigorous, consider the case when $\rho=0$ and $GA-I=\mathrm{Diag}\{1,0,...,0\}$. In this case, the $\|\cdot\|_2$-norm of the recovering error, in the case of $s\sim\mathcal{N}(0,I_n)$, is just $|s_1|$, and $\mathrm{Prob}\{|s_1|>\tau_*\}\leq\epsilon$ provided that $\tau_*\geq\sqrt{2\ln(2/\epsilon)}$, in particular, when $\tau_*=\sqrt{2\ln(2/\epsilon)}$. At the same time, when $s=\sqrt{n}[1;0;...;0]\in\mathcal{S}$, the norm of the recovering error is $\sqrt{n}$, which, for large $n$, is incomparably larger than the above $\tau_*$.

**Example 3.9** Here we consider the case where $\phi(G)$ cannot be computed efficiently, specifically, the case where $\mathcal{S}$ is the unit box $B_n=\{s\in\mathbb{R}^n:\|s\|_\infty\leq 1\}$ (or the set $V_n$ of vertices of this box). Indeed, it is known that for a general-type positive definite quadratic form $s^TQs$, computing its maximum over the unit box is NP-hard, even when instead of the precise value of the maximum its 4%-accurate approximation is sought. In situations like this we could replace $\phi(G)$ in the above scheme by its efficiently computable upper bound $\widehat{\phi}(G)$. To get such a bound in the case when $\mathcal{S}$ is the unit box, we can use the following wonderful result:

**Nesterov's $\frac{\pi}{2}$ Theorem** [76] *Let $A\in\mathbf{S}^n_+$. Then the efficiently computable quantity*

$$\mathrm{SDP}(A)=\min_{\lambda\in\mathbb{R}^n}\left\{\sum_i\lambda_i:\mathrm{Diag}\{\lambda\}\succeq A\right\}$$

*is an upper bound, tight within the factor $\frac{\pi}{2}$, on the quantity*

$$\mathrm{Opt}(A)=\max_{s\in B_n}s^TAs.$$

Assuming that $\mathcal{S}$ is $B_n$ (or $V_n$), Nesterov's $\frac{\pi}{2}$ Theorem provides us with an efficiently computable and tight, within the factor $\frac{\pi}{2}$, upper bound

$$\widehat{\phi}(G) = \min_{\lambda}\left\{\sum_i \lambda_i : \left[\begin{array}{c|c} \mathrm{Diag}(\lambda) & (GA-I)^T \\ \hline GA-I & I \end{array}\right] \succeq 0\right\}$$

on $\phi(G)$. Replacing $\phi(\cdot)$ by its upper bound, we pass from the intractable problems (3.6.79) to their tractable approximations

$$\text{find } G, \lambda\text{:} \quad \sum_i \lambda_i + \rho^2 \mathrm{Tr}(G^T G) \leq \gamma_*, \left[\begin{array}{c|c} \mathrm{Diag}(\lambda) & (GA-I)^T \\ \hline GA-I & I \end{array}\right] \succeq 0; \tag{3.6.80}$$

we then apply bisection in $\rho$ to rapidly approximate the largest $\rho = \rho_*$, along with the associated $G = G_*$, for which problems (3.6.80) are solvable, thus getting a feasible solution to the problem of interest.

## 3.7 Exercises

**Exercise 3.1** Consider a semi-infinite conic constraint

$$\forall (\zeta \in \rho\mathcal{Z}) : a_0[x] + \sum_{\ell=1}^{L} \zeta_i a_\ell[x] \in \mathbf{Q} \tag{$C_{\mathcal{Z}}[\rho]$}$$

Assume that for certain $\vartheta$ and some closed convex set $\mathcal{Z}_*$, $0 \in \mathcal{Z}_*$, the constraint $(C_{\mathcal{Z}_*}[\cdot])$ admits a safe tractable approximation tight within the factor $\vartheta$. Now let $\mathcal{Z}$ be a closed convex set that can be approximated, up to a factor $\lambda$, by $\mathcal{Z}_*$, meaning that for certain $\gamma > 0$ we have

$$\gamma\mathcal{Z}_* \subset \mathcal{Z} \subset (\lambda\gamma)\mathcal{Z}_*.$$

Prove that $(C_{\mathcal{Z}}[\cdot])$ admits a safe tractable approximation, tight within the factor $\lambda\vartheta$.

**Exercise 3.2** Let $\vartheta \geq 1$ be given, and consider the semi-infinite conic constraint $(C_{\mathcal{Z}}[\cdot])$ "as a function of $\mathcal{Z}$," meaning that $a_\ell[\cdot]$, $0 \leq \ell \leq L$, and $\mathbf{Q}$ are once and forever fixed. In what follows, $\mathcal{Z}$ always is a solid (convex compact set with a nonempty interior) symmetric w.r.t. 0.

Assume that whenever $\mathcal{Z}$ is an ellipsoid centered at the origin, $(C_{\mathcal{Z}}[\cdot])$ admits a safe tractable approximation tight within factor $\vartheta$ (as it is the case for $\vartheta = 1$ when $\mathbf{Q}$ is the Lorentz cone, see section 3.2.5).

1. Prove that when $\mathcal{Z}$ is the intersection of $M$ centered at the origin ellipsoids:

   $$\mathcal{Z} = \{\zeta : \zeta^T Q_i \zeta \leq 1,\, i = 1, ..., M\} \qquad [Q_i \succeq 0, \textstyle\sum_i Q_i \succ 0]$$

   $(C_{\mathcal{Z}}[\cdot])$ admits a safe tractable approximation tight within the factor $\sqrt{M}\vartheta$.

2. Prove that if $\mathcal{Z} = \{\zeta : \|\zeta\|_\infty \leq 1\}$, then $(C_{\mathcal{Z}}[\cdot])$ admits a safe tractable approximation tight within the factor $\vartheta\sqrt{\dim \zeta}$.

3. Assume that $\mathcal{Z}$ is the intersection of $M$ ellipsoids not necessarily centered at the origin. Prove that then $(C_{\mathcal{Z}}[\cdot])$ admits a safe tractable approximation tight within a factor $\sqrt{2M}\vartheta$.

**Exercise 3.3** Consider the situation as follows (cf. section 3.2.6). We are given an observation

$$y = Ax + b \in \mathbb{R}^m$$

of unknown signal $x \in \mathbb{R}^n$. The matrix $B \equiv [A; b]$ is not known exactly; all we know is that $B \in \mathcal{B} = \{B = B_{\rm n} + L^T \Delta R : \Delta \in \mathbb{R}^{p\times q}, \|\Delta\|_{2,2} \leq \rho\}$. Build an estimate $v$ of the vector $Qx$, where $Q$ is a given $k \times n$ matrix, that minimizes the worst-case, over all possible true values of $x$, $\|\cdot\|_2$ estimation error.

**Exercise 3.4** Consider an uncertain Least Squares inequality

$$\|A(\eta)x + b(\eta)\|_2 \leq \tau, \ \ \eta \in \rho\mathcal{Z}$$

where $\mathcal{Z}$, $0 \in \mathrm{int}\mathcal{Z}$, is a symmetric w.r.t. the origin convex compact set that is the intersection of $J > 1$ ellipsoids not necessarily centered at the origin:

$$\mathcal{Z} = \{\eta : (\eta - a_j)^T Q_j (\eta - a_j) \leq 1, \, 1 \leq j \leq J\} \qquad [Q_j \succeq 0, \textstyle\sum_j Q_j \succ 0]$$

Prove that the RC of the uncertain inequality in question admits a safe tractable approximation tight within the factor $O(1)\sqrt{\ln J}$ (cf. Theorem 3.9).

**Exercise 3.5** [Robust Linear Estimation, see [44]] Let a signal $v \in \mathbb{R}^n$ be observed according to

$$y = Av + \xi,$$

where $A$ is an $m \times n$ matrix, known up to "unstructured norm-bounded perturbation":

$$A \in \mathcal{A} = \{A = A_{\rm n} + L^T \Delta R : \Delta \in \mathbb{R}^{p\times q}, \|\Delta\|_{2,2} \leq \rho\},$$

and $\xi$ is a zero mean random noise with a known covariance matrix $\Sigma$. Our a priori information on $v$ is that

$$v \in V = \{v : v^T Q v \leq 1\},$$

where $Q \succ 0$. We are looking for a linear estimate

$$\widehat{v} = Gy$$

with the smallest possible worst-case mean squared error

$$\mathrm{EstErr} = \sup_{v\in V, A\in\mathcal{A}} \left(\mathbf{E}\left\{\|G[Av+\xi] - v\|_2^2\right\}\right)^{1/2}$$

(cf. section 3.2.6).

1) Reformulate the problem of building the optimal estimate equivalently as the RC of uncertain semidefinite program with unstructured norm-bounded uncertainty and reduce this RC to an explicit semidefinite program.

2) Assume that $m = n$, $\Sigma = \sigma^2 I_n$, and the matrices $A_{\rm n}^T A_{\rm n}$ and $Q$ commute, so that $A_{\rm n} = V\mathrm{Diag}\{a\}U^T$ and $Q = U\mathrm{Diag}\{q\}U^T$ for certain orthogonal matrices $U, V$ and certain vectors $a \geq 0$, $q > 0$. Let, further, $\mathcal{A} = \{A_{\rm n} + \Delta : \|\Delta\|_{2,2} \leq \rho\}$. Prove that in the situation in question we lose nothing when looking for $G$ in the form of

$$G = U\mathrm{Diag}\{g\}V^T,$$

and build an explicit convex optimization program with just two variables specifying the optimal choice of $G$.

**Exercise 3.6**

1) Let $p, q \in \mathbb{R}^n$ and $\lambda > 0$. Prove that $\lambda pp^T + \frac{1}{\lambda}qq^T \succeq \pm[pq^T + qp^T]$.

2) Let $p, q$ be as in 1) with $p, q \neq 0$, and let $Y \in \mathbf{S}^n$ be such that $Y \succeq \pm[pq^T + qp^T]$. Prove that there exists $\lambda > 0$ such that $Y \succeq \lambda pp^T + \frac{1}{\lambda}qq^T$.

3) Consider the semi-infinite LMI of the following specific form:

$$\forall(\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq 1) : \mathcal{A}_{\rm n}(x) + \rho \sum_{\ell=1}^{L} \zeta_\ell \left[L_\ell^T(x)R_\ell + R_\ell^T L_\ell(x)\right] \succeq 0, \tag{3.7.1}$$

where $L_\ell^T(x), R_\ell^T \in \mathbb{R}^n$, $R_\ell \neq 0$ and $L_\ell(x)$ are affine in $x$, as is the case in Lyapunov Stability Analysis/Synthesis under interval uncertainty (3.5.7) with $\mu = 1$.

Prove that the safe tractable approximation, tight within the factor $\pi/2$, of (3.7.1), that is, the system of LMIs

$$\begin{array}{l} Y_\ell \succeq \pm\left[L_\ell^T(x)R_\ell + R_\ell^T L_\ell(x)\right],\ 1 \leq \ell \leq L \\ \mathcal{A}_{\rm n}(x) - \rho\sum_{\ell=1}^L Y_\ell \succeq 0 \end{array} \tag{3.7.2}$$

in $x$ and in matrix variables $Y_1, ..., Y_L$ is equivalent to the LMI

$$\left[\begin{array}{c|cccc} \mathcal{A}_{\rm n}(x) - \rho\sum_{\ell=1}^L \lambda_\ell R_\ell^T R_\ell & L_1^T(x) & L_2^T(x) & \cdots & L_L^T(x) \\ \hline L_1(x) & \lambda_1/\rho & & & \\ L_2(x) & & \lambda_2/\rho & & \\ \vdots & & & \ddots & \\ L_L(x) & & & & \lambda_L/\rho \end{array}\right] \succeq 0 \tag{3.7.3}$$

in $x$ and real variables $\lambda_1..., \lambda_L$. Here the equivalence means that $x$ can be extended to a feasible solution of (3.7.2) if and only if it can be extended to a feasible solution of (3.7.3).

**Exercise 3.7** Consider the Signal Processing problem as follows. We are given uncertainty-affected observations

$$y = Av + \xi$$

of a signal $v$ known to belong to a set $V$. Uncertainty "sits" in the "measurement error" $\xi$, known to belong to a given set $\Xi$, and in $A$ — all we know is that $A \in \mathcal{A}$. We assume that $V$ and $\Xi$ are intersections of ellipsoids centered at the origin:

$$V = \{v \in \mathbb{R}^n : v^T P_i v \leq 1,\ 1 \leq i \leq I\},\ [P_i \succeq 0, \textstyle\sum_i P_i \succ 0]$$

$$\Xi = \{\xi \in \mathbb{R}^m : \xi^T Q_j \xi \leq \rho_\xi^2,\ 1 \leq j \leq J\},\ [Q_j \succeq 0, \textstyle\sum_j Q_j \succeq 0]$$

and $\mathcal{A}$ is given by structured norm-bounded perturbations:

$$\mathcal{A} = \{A = A_{\rm n} + \sum_{\ell=1}^{L} L_\ell^T \Delta_\ell R_\ell, \Delta_\ell \in \mathbb{R}^{p_\ell \times q_\ell}, \|\Delta_\ell\|_{2,2} \leq \rho_A\}.$$

We are interested to build a linear estimate $\widehat{v} = Gy$ of $v$ via $y$. The $\|\cdot\|_2$ error of such an estimate at a particular $v$ is

$$\|Gy - v\|_2 = \|G[Av + \xi] - v\|_2 = \|(GA - I)v + G\xi\|_2,$$

and we want to build $G$ that minimizes the worst, over all $v, A, \xi$ compatible with our a priori information, estimation error

$$\max_{\xi\in\Xi, v\in V, A\in\mathcal{A}} \|(GA - I)v + G\xi\|_2.$$

Build a safe tractable approximation of this problem that seems reasonably tight when $\rho_\xi$ and $\rho_A$ are small.