Scenario Approximations of Chance Constraints

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Summary. We consider an optimization problem of minimization of a linear function subject to the chance constraint $\text{Prob}\{G(x,\xi) \in C\} \geq 1-\varepsilon$, where C is a convex set, $G(x,\xi)$ is bi-affine mapping and ξ is a vector of random perturbations with known distribution. When C is multi-dimensional and ε is small, like 10^{-6} or 10^{-10}, this problem is, generically, a problem of minimizing under a nonconvex and difficult to compute constraint and as such is computationally intractable. We investigate the potential of conceptually simple *scenario approximation* of the chance constraint. That is, approximation of the form $G(x,\eta^j) \in C$, $j = 1,...,N$, where $\{\eta^j\}_{j=1}^N$ is a sample drawn from a properly chosen trial distribution. The emphasis is on the situation where the solution to the approximation should, with probability at least $1-\delta$, be feasible for the problem of interest, while the sample size N should be polynomial in the size of this problem and in $\ln(1/\varepsilon)$, $\ln(1/\delta)$.

Key words: Convex optimization, stochastic uncertainty, chance constraints, Monte Carlo sampling, importance sampling.

1 Introduction

Consider the following optimization problem

$$\operatorname*{Min}_{x\in\mathbb{R}^n} f(x) \text{ subject to } G(x,\xi) \in C, \tag{1}$$

where $C \subset \mathbb{R}^m$ is a closed convex set and $f(x)$ is a real valued function. We assume that the constraint mapping $G : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^m$ depends on uncertain parameters represented by vector ξ which can vary in a set $\Xi \subset \mathbb{R}^d$. Of course, for a fixed $\xi \in \Xi$, the constraint $G(x,\xi) \in C$ means existence of $z \in C$ such that $G(x,\xi) = z$. In particular, suppose that the set C is given in the form

$$C := \{z : z = Wy - w,\ y \in \mathbb{R}^\ell,\ w \in \mathbb{R}^m_+\}, \tag{2}$$

where W is a given matrix. Then the constraint $G(x, \xi) \in C$ means that the system $Wy \geq G(x, \xi)$ has a feasible solution $y = y(\xi)$. Given x and ξ, we refer to the problem of finding $y \in \mathbb{R}^\ell$ satisfying $Wy \geq G(x, \xi)$ as the second stage feasibility problem.

We didn't specify yet for what values of the uncertain parameters the corresponding constraints should be satisfied. One way of dealing with this is to require the constraints to hold for *every* possible realization $\xi \in \Xi$. If we view ξ as a random vector with a (known) probability distribution having support[3] Ξ, this requires the second stage feasibility problem to be solvable (feasible) with probability one. In many situations this may be too conservative, and a more realistic requirement is to ensure feasibility of the second stage problem with probability close to one, say at least with probability $1 - \varepsilon$. When ε is really small, like $\varepsilon = 10^{-6}$ or $\varepsilon = 10^{-12}$, for all practical purposes confidence $1 - \varepsilon$ is as good as confidence 1. At the same time, it is well known that passing from $\varepsilon = 0$ to a positive ε, even as small as 10^{-12}, may improve significantly the optimal value in the corresponding two-stage problem.

The chance constraints version of problem (1) involves constraints of the form

$$\text{Prob}\big\{G(x, \xi) \in C\big\} \geq 1 - \varepsilon. \tag{3}$$

Chance constrained problems were studied extensively in the stochastic programming literature (see, e.g., [9] and references therein). We call $\varepsilon > 0$ the *confidence parameter* of chance constraint (3), and every x satisfying (3) as an $(1 - \varepsilon)$-*confident* solution to (1). Our goal is to describe the set X_ε of $(1 - \varepsilon)$-confident solutions in a "computationally meaningful" way allowing for subsequent optimization of a given objective over this set. Unless stated otherwise we assume that the constraint mapping is linear in ξ and has the form

$$G(x, \xi) := A_0(x) + \sigma \sum_{i=1}^{d} \xi_i A_i(x), \tag{4}$$

where $\sigma \geq 0$ is a coefficient, representing the perturbation level of the problem, and $A_i : \mathbb{R}^n \to \mathbb{R}^m$, $i = 0, ..., d$, are given affine mappings. Of course, the coefficient σ can be absorbed into the perturbation vector ξ. However, in the sequel we use techniques which involve change of the perturbation level of the data. Sometimes we use notation $G_\sigma(x, \xi)$ for the right hand side of (4) in order to emphasize its dependence on the perturbation level of the problem.

Example 1. Suppose that we want to design a communication network with p terminal nodes and n arcs. The topology of the network is given, and all we need to specify is vector x of capacities of the arcs; $c^T x$ is the cost of the network to be minimized. The load d in the would-be network (that is, the amounts of data d_{rs}, $r, s = 1, ..., p$, to be transmitted from terminal node r to terminal node s per unit time) is uncertain and is modeled as $d_{rs} =$

[3] The support of the probability distribution of random vector ξ is the smallest closed set $\Xi \subset \mathbb{R}^d$ such that the probability of the event $\{\xi \in \Xi\}$ is equal to one.

$d^*_{rs}+\xi_{rs}$, where d^* is the nominal demand and $\xi = \{\xi_{rs}\}$ is a vector of random perturbations which is supposed to vary in a given set Ξ. The network can carry load d if the associated multicommodity flow problem (to assign arcs γ with flows $y^\gamma_{rs} \geq 0$ – amounts of data with origin at r and destination at s passing through γ – obeying the standard flow conservation constraints with "boundary conditions" d and the capacity bounds $\sum_{r,s} y^\gamma_{rs} \leq x_\gamma$) is solvable. This requirement can be formulated as existence of vector y such that $Wy \geq G(x,d)$, where W is a matrix and $G(x,d)$ is an affine function of x and the load d, associated with the considered network. When the design specifications require "absolute reliability" of the network, i.e., it should be capable to carry every realization of random load, the network design problem can be modeled as problem (1) with the requirement that the corresponding constraints $G(x, d^*+\xi) \in C$ should be satisfied for every $\xi \in \Xi$. This, however, can lead to a decision which is too conservative for practical purposes.

As an illustration, consider the simplest case of the network design problem, where p "customer nodes" are linked by arcs of infinite capacity with a central node ("server") c, which, in turn is linked by an arc (with capacity x to be specified) with "ground node" g, and all data to be transmitted are those from the customer nodes to the ground one; in fact, we are speaking about p jobs sharing a common server with performance x. Suppose that the loads d_r created by jobs r, $r = 1, ..., p$, are independent random variables with, say, uniform distributions in the respective segments $[d^*_r(1-\sigma), d^*_r(1+\sigma)]$, where $\sigma \in (0,1)$ is a given parameter. Then the "absolutely reliable" optimal solution clearly is

$$x^* = \sum_{r=1}^{p} d^*_r(1+\sigma).$$

At the same time, it can be shown[4] that for $\tau \geq 0$,

$$\text{Prob}\left\{\sum_r d_r > \sum_r d^*_r + \tau\sigma\sqrt{\sum_r (d^*_r)^2}\right\} \leq e^{-\tau^2/2}.$$

It follows that whenever $\varepsilon \in (0,1)$ and for $D := \sum_{r=1}^{p} d^*_r$, the solution

$$x(\varepsilon) = D + \sigma\sqrt{2\ln(1/\varepsilon)}\sqrt{\sum_r (d^*_r)^2}$$

is $(1-\varepsilon)$-confident. The cost of this solution is by the factor

$$\kappa = \frac{1+\sigma}{1+\sigma\sqrt{2\ln(1/\varepsilon)}\left(\sum_r (d^*_r)^2\right)^{1/2} D^{-1}}$$

is less than the cost of the absolutely reliable solution. For example, with $\varepsilon = 10^{-9}$, $p = 1000$ and all d^*_r, $r = 1, ..., p$, equal to each other, we get κ as large as 1.66; reducing ε to 10^{-12}, we still get $\kappa = 1.62$.

[4] This follows from the following inequality due to Hoeffding: if $X_1, ..., X_n$ are independent random variables such that $a_i \leq X_i \leq b_i$, $i = 1, ..., n$, then for $t \geq 0$,

$$\text{Prob}\left\{\sum_{i=1}^{n}(X_i - \mathbb{E}[X_i]) > t\right\} \leq \exp\left\{\frac{-2t^2}{\sum_{i=1}^{n}(b_i-a_i)^2}\right\}.$$

The difference between the absolutely reliable and $(1-\varepsilon)$-confident solutions will be even more dramatic if we assume that d_r are normally distributed independent random variables. Then the corresponding random vector d is supported on the whole space and hence the demand cannot be satisfied with probability one for any value of x, while for any $\varepsilon > 0$, there exists a finite $(1-\varepsilon)$-confident solution.

It is important to point out that "computationally meaningful" *precise* description of the solution set X_ε of (3) seems to be intractable, except for few simple particular cases. Indeed, clearly a necessary condition for existence of a "computationally meaningful" description of the set X_ε is the possibility to solve efficiently the associated problem for a fixed first stage decision vector: "given x, check whether $x \in X_\varepsilon$". To the best of our knowledge, the only generic case where the function

$$\phi(x) := \mathrm{Prob}\big\{G(x,\xi) \in C\big\} \tag{5}$$

can be efficiently computed analytically is the case where ξ has a normal distribution and C is a segment in $\mathbb{R}$, which is pretty restrictive. Computing $\phi(x)$ is known to be NP-hard already in the situation as simple as the one where ξ is uniformly distributed in a box and C is a polytope.

Of course, there is always a possibility to evaluate $\phi(x)$ by Monte Carlo simulation, provided that C is computationally tractable which basically, means that we can check efficiently whether a given point belongs to C. Straightforward simulation, however, requires sample sizes of order ε^{-1} and becomes therefore impractical for ε like 10^{-8} or 10^{-12}. We are not aware of generic cases where this difficulty[5] can be avoided.

Aside from difficulties with efficient computation of $\phi(x)$, there is another severe problem: the set X_ε typically is nonconvex. The only generic exception we know of, is again the case of randomly perturbed linear constraint, where C is a segment, with ξ having a normal distribution. Nonconvexity of X_ε makes our ultimate goal (to optimize efficiently over X_ε) highly problematic.

In view of the outlined difficulties, we pass from the announced goal to its relaxed version, where we are looking for "tractable approximations" of chance constraint (3). Specifically, we are looking for *sufficient* conditions for the validity of (3), conditions which should be both efficiently verifiable and define a convex set in the space of design variables. The corresponding rationale is clear; we want to stay at the safe side, this is why we are looking for *sufficient* conditions for the validity of (3), and we want to be able to optimize efficiently objectives (at least simple ones) under these conditions.

[5] It should be stressed that the difficulties with Monte Carlo estimation of $\mathrm{Prob}\,\{\xi \in Q_x\}$, where $Q_x := \{\xi : G(x,\xi) \notin C\}$, come from nonconvexity of Q_x rather than from the fact that we are interested in rare events. Indeed, at least for uniformly distributed ξ, advanced Monte Carlo techniques allow for polynomial time estimation of the quantity $\mathrm{Prob}\{\xi \in Q\}$ with every fixed relative accuracy, provided that Q is convex, [4, 5].

This is why the conditions should be efficiently verifiable and define convex feasible sets.

There are two major avenues for building tractable approximations of chance constraints. The first is to consider one by one interesting generic randomly perturbed constraints (linear, conic quadratic, semidefinite, etc.) and to look for specific tractable approximations of their chance counterparts. This approach is easy to implement for linear constraints with $m = 1$ and $C := \mathbb{R}_+$. Then the constraint $G(x, \xi) \in C$ is equivalent to $a^T x + \xi^T A(x) \le b$, with $A(x)$ being affine in x. Assuming that we know an upper bound V on the covariance matrix of ξ, so that $\mathbb{E}\{(h^T\xi)^2\} \le h^T V h$ for every vector h, a natural "safe version" of the random constraint in question is

$$a^T x + \gamma \sqrt{A^T(x) V A(x)} \le b, \tag{6}$$

where $\gamma = \gamma(\varepsilon)$ is a "safety parameter" which should satisfy the condition

$$\mathrm{Prob}\big\{\xi : h^T\xi > \gamma\sqrt{h^T V h}\big\} \le \varepsilon \ \text{ for any } h \in \mathbb{R}^d.$$

An appropriate value of γ can be derived from the standard results on probabilities of large deviations for scalar random variables. For example, for the case when ξ has "light[6] tail", it suffices to take $\gamma(\varepsilon) = 2\sqrt{1 + \ln(\varepsilon^{-1})}$.

Results of the outlined type can be obtained for randomly perturbed conic quadratic[7] constraints $\|Ax - b\| \le \tau$ (here $C := \{(y, t) : t \ge \|y\|\}$ is the Lorentz cone), as well as for randomly perturbed semidefinite constraints (C is the semidefinite cone in the space of matrices), see [8]. However, the outlined approach has severe limitations: it hardly could handle the case when C possesses complicated geometry. For example, using "safe version" (6) of a single randomly perturbed linear inequality, one can easily build an approximation of the chance constraint corresponding to the case when C is a polyhedral set given by a list of linear inequalities. At the same time, it seems hopeless to implement the approach in question in the case of a simple two-stage stochastic program, where we need a safe version of the constraint $G(x, \xi) \in C$ with the set C given in the form (2). Here the set C, although polyhedral, is *not* given by an explicit list of linear inequalities (such a list can be exponentially long), which makes the aforementioned tools completely inapplicable.

The second avenue of building tractable approximations of chance constraints is the *scenario approach* based on Monte Carlo simulation. Specifically, given the probability distribution $\mathbf{P}$ of random data vector ξ and level of perturbations σ, we choose somehow a "trial" distribution $\mathbf{F}$ (which does not need to be the same as $\mathbf{P}$). Consequently, we generate a sample $\eta^1, ..., \eta^N$ of N realizations, called *scenarios*, of ξ drawn from the distribution $\mathbf{F}$, and treat the system of constraints

[6] Specifically, $\mathbb{E}\left[\exp\{\frac{(h^T\xi)^2}{4h^T V h}\}\right] \le \exp\{1\}$ for every $h \in \mathbb{R}^d$, as in the case where ξ has normal distribution with zero mean and covariance matrix V

[7] Unless stated otherwise, $\|z\| := (z^T z)^{1/2}$ denotes the Euclidean norm.

$$G(x, \eta^j) \in C,\ \ j = 1, ..., N, \tag{7}$$

as an approximation of chance constraint (3). This is the approach we investigate in this paper.

The rationale behind this scenario based approach is as follows. First of all, (7) is of the same level of "computational tractability" as the *unperturbed* constraint, so that (7) is computationally tractable, provided that C is so and that the number of scenarios N is reasonable. Thus, all we should understand is what can be achieved with a reasonable N. For the time being, let us forget about optimization with respect to x, fix $x = \bar{x}$ and let us ask ourselves what are the relations between the predicates "$\bar{x}$ satisfies (3)" and "$\bar{x}$ satisfies (7)". Recall that the random sample $\{\eta^j\}_{j=1}^N$ is drawn from the trial distribution $\mathbf{F}$. We assume in the remainder of this section the following.

The trial distribution $\mathbf{F}$ *is the distribution of* $s\xi$*, where* $s \geq 1$ *is fixed and* $\mathbf{P}$ *is the probability distribution of random vector* ξ*.*

Because of (4) we have that $G_\sigma(\bar{x}, \xi) \in C$ iff $\xi \in Q_{\bar{x},\sigma}$, where

$$Q_{\bar{x},\sigma} := \left\{ z \in \mathbb{R}^d : \sigma \textstyle\sum_{i=1}^d z_i A_i(\bar{x}) \in C - A_0(\bar{x}) \right\}. \tag{8}$$

Note that the set $Q_{\bar{x},\sigma}$ is closed and convex along with C, and for any $s > 0$,

$$s^{-1} Q_{\bar{x},\sigma} = \{\xi : G_{s\sigma}(\bar{x}, \xi) \in C\}.$$

Now, in "good cases" $\mathbf{P}$ possesses the following "concentration" property.

(!) For every convex set $Q \subset \mathbb{R}^d$ with $\mathbf{P}(Q)$ not too small, e.g., $\mathbf{P}(Q) \geq 0.9$, the mass $\mathbf{P}(sQ)$ of s-fold enlargement of Q rapidly approaches 1 as s grows. That is, if Q is convex and $\mathbf{P}(Q) \geq 0.9$, then there exists $\kappa > 0$ such that for $s \geq 1$ it holds that

$$\mathbf{P}\big(\{\xi \notin sQ\}\big) \leq e^{-\kappa s^2}. \tag{9}$$

(we shall see that, for example, in the case of normal distribution, estimate (9) holds true with $\kappa = 0.82$).

Assuming that the above property (!) holds, and given small $\varepsilon > 0$, let us set[8]

$$s := \sqrt{\kappa^{-1}\ln(\varepsilon^{-1})} \ \text{ and } \ N := \text{ceil}\big[\ln(\delta)/\ln(0.9\big] \approx \text{ceil}\big[10\ln(\delta^{-1})\big], \tag{10}$$

where $\delta > 0$ is a small reliability parameter, say, $\delta = \varepsilon$. Now, if $\bar{x}$ satisfies the constraint

$$\mathbf{P}\big(\{\xi : G_{s\sigma}(\bar{x}, \xi) \notin C\}\big) \leq \varepsilon, \tag{11}$$

that is, $\bar{x}$ satisfies the strengthened version of (3) obtained by replacing the original level of perturbations σ with $s\sigma$, then the probability to get a sample $\{\eta^j\}_{j=1}^N$ such that $\bar{x}$ does *not* satisfy (7) is at most

[8] Notation ceil$[a]$ stands for the smallest integer which is greater than or equal to $a \in \mathbb{R}$.

$$\sum_{j=1}^{N} \text{Prob}\big(\{G(\bar{x},\eta^j) \notin C\}\big) \le \varepsilon N = O(1)\varepsilon \ln(\delta^{-1}),$$

where the constant $O(1)$ is slightly bigger than $[\ln(0.9^{-1})]^{-1} = 9.5$. For $\delta = \varepsilon$, say, this probability is nearly of order ε.

Let $Q := s^{-1}Q_{\bar{x},\sigma}$, and hence

$$\mathbf{P}(\{\xi \in Q\}) = \mathbf{P}(\{s\xi \in Q_{\bar{x},\sigma}\}) = \text{Prob}\big(\{G(\bar{x},\eta^j) \in C\}\big).$$

Consequently, if $\mathbf{P}(\{\xi \in Q\}) < 0.9$, then the probability p of getting a sample $\{\eta^j\}_{j=1}^N$ for which $\bar{x}$ satisfies (7) , is the probability to get N successes in N Bernoulli trials with success probability for a single experiment less than 0.9. That is, $p \le 0.9^N$, and by (10) we obtain $p \le \delta$. For small $\delta = \varepsilon$, such an event is highly unlikely. And if $\mathbf{P}(\{\xi \in Q\}) \ge 0.9$, then by using (9) and because of (10) we have

$$\mathbf{P}(\{\xi \notin Q_{\bar{x},\sigma}\}) = \mathbf{P}(\{\xi \notin sQ\}) \le e^{-\kappa s^2} = \varepsilon.$$

That is, $\bar{x}$ satisfies the chance constraint (3).

We can summarize the above discussion as follows.

(!!) If $\bar{x}$ satisfies the chance constraint (3) with a moderately increased level of perturbations (by factor $s = \sqrt{O(\ln(\varepsilon^{-1}))}$), then it is highly unlikely that $\bar{x}$ does not satisfy (7) (probability of that event is less than $O(1)\varepsilon\ln(\varepsilon^{-1})$). If $\bar{x}$ does satisfy (7), then it is highly unlikely that $\bar{x}$ is infeasible for (3) at the original level of perturbations (probability of that event is then less than $\delta = \varepsilon$). Note that the sample size which ensures this conclusion is just of order $O(1)\ln(\varepsilon^{-1})$.

The approach we follow is closely related the *importance sampling* method, where one samples from a properly chosen artificial distribution rather than from the actual one in order to make the rare event in question "more frequent". The difference with the traditional importance sampling scheme is that the latter is aimed at estimating the expected value of a given functional and uses change of the probability measure in order to reduce the variance of the estimator. In contrast to this, we do not try to estimate the quantity of interest (which in our context is $\text{Prob}\{\xi \notin Q\}$, where Q is a given convex set) because of evident hopeless of the estimation problem. Indeed, we are interested in multidimensional case and dimension independent constructions and results, while the traditional importance sampling is heavily affected by the "curse of dimensionality". For example, the distributions of two proportional to each other with coefficient 2 normally distributed vectors ξ and η of dimension 200 are "nearly singular"with respect to each other: one can find two *nonintersecting* sets U, V in $\mathbb{R}^{200}$ such that $\text{Prob}\{\xi \notin U\} = \text{Prob}\{\eta \notin V\} < 1.2\cdot 10^{-11}$. Given this fact, it seems ridiculous to *estimate* a quantity related to one of these distributions via a sample drawn from the other one. What could be done (and what we intend to do) is to use the sample of realizations of the larger random vector η to make conclusions of the type "if all elements of a random sample of size $N = 10,000$ of η belong to a given convex set Q, then, up to chance of "bad sampling" as small as 10^{-6}, the probability for the smaller vector ξ to take value outside Q is at most $4.6 \cdot 10^{-9}$. Another difference between what we are doing and the usual results on importance sampling is in the fact that in our context the convexity of Q is crucial for the statements (and the proofs), while in the traditional importance sampling it plays no significant role.

Scenario approach is widely used in Stochastic Optimization. We may refer to [10], and references therein, for a discussion of the Monte Carlo sampling approach to solving two-stage stochastic programming problems of the generic form

$$\operatorname*{Min}_{x \in X} \mathbb{E}\left[F(x,\xi)\right], \tag{12}$$

where $F(x,\xi)$ is the optimal value of the corresponding second stage problem. That theory presents moderate upper bounds on the number of scenarios required to solve the problem within a given accuracy and confidence. However, all results of this type known to us postulate from the very beginning that $F(x,\xi)$ is finite valued with probability one, i.e., that the problem has a relatively complete recourse.

As far as problems with chance constraints of the form (3) are concerned, seemingly the only possibility to convert such a problem into one with simple (relatively complete) recourse is to penalize violations of constraints. That is, to approximate the problem of minimization of $f(x) := c^T x$ subject to $Ax \geq b$ and chance constraint (3), by the problem

$$\operatorname*{Min}_{x} c^T x + \gamma \mathbb{E}\left[F(x,\xi)\right] \;\; \text{s.t. } Ax \geq b, \tag{13}$$

where $F(x,\xi) := \inf_{t \geq 0, y} \left\{ t : Wy + te \geq G(x,\xi) \right\}$, e is vector of ones and $\gamma > 0$ is a penalty parameter. The difficulty, however, is that in order to solve (12) within a fixed absolute accuracy in terms of the objective, the number of scenarios N should be of order of the maximal, over $x \in X$, variance of $\gamma F(x,\xi)$. For problem (13), that means N should be of order of γ^2; in turn, the penalty parameter γ should be inverse proportional to the required confidence parameter ε, and we arrive at the same difficulty as in the case of straightforward Monte Carlo simulation: the necessity to work with prohibitively large samples of scenarios when high level of confidence is required.

To the best of our knowledge, the most recent and advanced results on chance versions of randomly perturbed convex programs are those of Calafiore and Campi [3]. These elegant and general results state that whatever are the distributions of random perturbations (perhaps entering nonlinearly into the objective and the constraints) affecting a convex program with n decision variables, for

$$N \geq 2\left[n\varepsilon^{-1}\ln(\varepsilon^{-1}) + \varepsilon^{-1}\ln(\delta^{-1}) + n\right], \tag{14}$$

an N-scenario sample is sufficient to solve the problem within confidence $1-\varepsilon$ with reliability $1-\delta$. That is, for (fixed) $\varepsilon > 0$, $\delta > 0$ and sample size N satisfying (14), the probablity that an optimal solution of the associated sample average problem satisfies the corresponding chance constraint condition (3) is at least $1-\delta$. Here again everything is fine except for the fact that the sample size is proportional to $\varepsilon^{-1}\ln(\varepsilon^{-1})$, which makes the approach impractical when high level of confidence is required.

The rest of the paper is organized as follows. In Section 2, we develop our techniques as applied to the *analysis* problem, as in the motivating discussion above. Note that validity of our scheme for the analysis problem does not yield automatically its validity for the synthesis one, where one optimizes a given objective over the feasible set[9] of (7). Applications of the methodology in the synthesis context form the subject of Section 3. Technical proofs are relegated to Appendix.

We use the following notation: $\mathbb{E}_{\mathbf{P}}\{\cdot\}$ stands for the expectation with respect to a probability distribution $\mathbf{P}$ on $\mathbb{R}^n$ (we skip index $\mathbf{P}$, when the distribution is clear from the context). By default, all probability distributions are Borel ones with finite first moments. For $\lambda \in \mathbb{R}$, a distribution $\mathbf{P}$ on $\mathbb{R}^d$ and $\xi \sim \mathbf{P}$, we denote by $\mathbf{P}^{(\lambda)}$ the distribution of random vector $\lambda\xi$. Finally, in the sequel, "symmetric" for sets and distributions always means "symmetric with respect to the origin". Unless stated otherwise all considered norms on $\mathbb{R}^d$ are Euclidean norms.

2 The Analysis problem

In this section, the assumption that the mappings $A_i(\cdot)$, $i = 1, ..., d$, are affine plays no role and is discarded. Recall that the Analysis version of (3) is to check, given $\bar{x}$, σ, $\varepsilon > 0$, and (perhaps, partial) information on the distribution $\mathbf{P}$ of ξ, whether $\mathbf{P}\big(\{\xi : G_\sigma(\bar{x}, \xi) \notin C\}\big) \le \varepsilon$. Consider the set $Q := Q_{\bar{x},\sigma}$, where $Q_{\bar{x},\sigma}$ is defined in (8). Recall that Q is closed and convex. The Analysis problem can be formulated as to check whether the relation

$$\mathbf{P}\big(\{\xi : \xi \notin Q\}\big) \le \varepsilon \tag{15}$$

holds true. The scenario approach, presented in section 1, results in the following generic test:

(T) *Given confidence parameter* $\varepsilon \in (0,1)$, *reliability parameter* $\delta \in (0,1)$, *and information on (zero mean) distribution* $\mathbf{P}$ *on* $\mathbb{R}^d$, *we act as follows:*
(i) We specify a trial distribution $\mathbf{F}$ *on* $\mathbb{R}^d$ *along with integers* $N > 0$ *(sample size) and* $K \ge 0$ *(acceptance level), where* $K < N$.

[9] Indeed, our motivating discussion implies only that every *fixed* point $\bar{x}$ which does not satisfy (3) is highly unlikely to be feasible for (7) – the probability of the corresponding "pathological" sample $\{\eta^j\}$ is as small as δ. This, however, does not exclude the possibility that a point x which depends on the sample, e.g., the point which optimizes a given objective over the feasible set of (7) – is not that unlikely to violate (3).

(ii) We generate a sample $\{\eta^j\}_{j=1}^N$, drawn from trial distribution $\mathbf{F}$, *and check whether at most K of the N sample elements violate the condition*[10]

$$\eta^j \in Q. \tag{16}$$

If it is the case, we claim that (15) *is satisfied ("acceptance conclusion"), otherwise we make no conclusion at all.*

We are about to analyze this test, with emphasis on the following major questions:

A. How to specify the "parameters" of the test, that is, trial distribution $\mathbf{F}$, sample size N and acceptance level K, in a way which ensures the validity of the acceptance with reliability at least $1-\delta$, so that the probability of false acceptance (i.e., generating a sample $\{\eta^j\}$ which results in the acceptance conclusion in the case when (15) is false) is less than δ.

B. What is the "resolution" of the test (for specified parameters)? Here "resolution" is defined as a factor $r = r(\varepsilon, \delta) \geq 1$ such that whenever $\mathbf{P}(\{\xi \in Q_{\bar{x}, r\sigma}\}) \leq \varepsilon$ (that is, $\bar{x}$ satisfies (3) with the level of perturbations increased by factor r), the probability of *not* getting the acceptance conclusion is at most δ.

2.1 Majorization

In section 1 we focused on scenario approach in the case when the scenario perturbations are multiples of the "true" ones. In fact we can avoid this restriction; all we need is the assumption that the trial distribution *majorizes* the actual distribution of perturbations in the following sense.

Definition 1. Let $\mathbf{F}$, $\mathbf{P}$ be probability distributions on $\mathbb{R}^d$. It is said that $\mathbf{F}$ majorizes[11] $\mathbf{P}$ (written $\mathbf{F} \succeq \mathbf{P}$) if for every convex lower semicontinous function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ one has $\mathbb{E}_{\mathbf{F}}[f] \geq \mathbb{E}_{\mathbf{P}}[f]$, provided that these expectations are well defined.

It is well known that the above majorization is a partial order (e.g., [7]). Some other basic properties of majorization are summarized in the following proposition.

Proposition 1. *The following statements hold.*
(i) $\mathbf{F} \succeq \mathbf{P}$ *if and only if* $\mathbb{E}_{\mathbf{F}}[f] \geq \mathbb{E}_{\mathbf{P}}[f]$ *for every convex function f with linear growth (that is, a real valued convex function f such that $|f(x)| \leq O(\|x\|)$ as $\|x\| \to \infty$).*
(ii) *Distribution of the sum $\xi + \eta$ of two independent random vectors $\xi, \eta \in \mathbb{R}^d$*

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[10] Recall that $\eta^j \in Q$ is equivalent to $G(\bar{x}, \eta^j) \in C$.

[11] In the literature on stochastic orderings the relation " $\succeq$ " is called the *convex order*, [7].

majorizes the distributions of ξ*, provided that* $\mathbb{E}[\eta] = 0$.
(iii) *If* $\mathbf{F} \succeq \mathbf{P}$ *and* $\mathbf{F}' \succeq \mathbf{P}'$*, then* $\lambda\mathbf{F} + (1-\lambda)\mathbf{F}' \succeq \lambda\mathbf{P} + (1-\lambda)\mathbf{P}'$ *whenever* $\lambda \in [0,1]$.
(iv) *Let* $\mathbf{F} \succeq \mathbf{P}$ *be distributions on* $\mathbb{R}^p \times \mathbb{R}^q$*, and* $\tilde{\mathbf{F}}, \tilde{\mathbf{P}}$ *be the associated marginal distributions on* $\mathbb{R}^p$*. Then* $\tilde{\mathbf{F}} \succeq \tilde{\mathbf{P}}$.
(v) *If* $\mathbf{F}, \mathbf{P}$ *are distributions on* $\mathbb{R}^p$ *and* $\mathbf{F}', \mathbf{P}'$ *are distributions on* $\mathbb{R}^q$*, then the distribution* $\mathbf{F} \times \mathbf{F}'$ *on* $\mathbb{R}^{p+q}$ *majorizes the distribution* $\mathbf{P} \times \mathbf{P}'$ *if and only if both* $\mathbf{F} \succeq \mathbf{P}$ *and* $\mathbf{F}' \succeq \mathbf{P}'$.
(vi) *Let* ξ*,* η *be random vectors in* $\mathbb{R}^d$ *and* A *be an* $m \times d$ *matrix. If the distribution of* ξ *majorizes the one of* η*, then the distribution of* $A\xi$ *majorizes the one of* $A\eta$.
(vii) *For symmetric distributions* $\mathbf{F}, \mathbf{P}$*, it holds that* $\mathbf{F} \succeq \mathbf{P}$ *if and only if* $\mathbb{E}_{\mathbf{F}}\{f\} \geq \mathbb{E}_{\mathbf{P}}\{f\}$ *for all even convex functions* f *with linear growth such that* $f(0) = 0$.
(viii) *For* $\alpha \geq 1$ *and symmetrically distributed random vector* ξ*, the distribution of* $\alpha\xi$ *majorizes the distribution of* ξ.

Proof. (i) This is evident, since every lower semicontinuous convex function on $\mathbb{R}^d$ is pointwise limit of a nondecreasing sequence of finite convex functions with linear growth.

(ii) For a real valued convex f we have

$$\mathbb{E}_{\xi+\eta}[f(\eta+\xi)] = \mathbb{E}_\xi\{\mathbb{E}_\eta[f(\eta+\xi)]\} \geq \mathbb{E}_\xi\{f(\mathbb{E}_\eta[\eta+\xi])\} = \mathbb{E}_\xi[f(\xi)],$$

where the inequality follows by the Jensen inequality.

(iii), (iv), (vi) and (vii) are evident, and (viii) is readily given by (vii).

(v) Assuming $\mathbf{F} \succeq \mathbf{P}$ and $\mathbf{F}' \succeq \mathbf{P}'$, for a convex function $f(u,u')$ with linear growth ($u \in \mathbb{R}^p$, $u' \in \mathbb{R}^q$) we have

$$\begin{array}{l}\int f(u,u')\mathbf{F}(du)\mathbf{F}'(du') = \int \left\{\int f(u,u')\mathbf{F}'(du')\right\} \mathbf{F}(du) \geq \\ \int\{\int f(u,u')\mathbf{P}'(du')\}\mathbf{F}(du) \geq \int\{\int f(u,u')\mathbf{P}'(du')\}\mathbf{P}(du) = \\ \int f(u,u')\mathbf{P}(du)\mathbf{P}'(du')\end{array}$$

(we have used the fact that $\int f(u,u')\mathbf{P}'(du')$ is a convex function of u with linear growth). We see that $\mathbf{F} \times \mathbf{F}' \succeq \mathbf{P} \times \mathbf{P}'$. The inverse implication $\mathbf{F} \times \mathbf{F}' \succeq \mathbf{P} \times \mathbf{P}' \Rightarrow \{\mathbf{F} \succeq \mathbf{P} \;\&\; \mathbf{F}' \succeq \mathbf{P}'\}$ is readily given by (iv). ∎

Let us also make the following simple observation:

Proposition 2. *Let* $\mathbf{F}$ *and* $\mathbf{P}$ *be symmetric distributions on* $\mathbb{R}$ *such that* $\mathbf{P}$ *is supported on* $[-a,a]$ *and the first absolute moment of* $\mathbf{F}$ *is* $\geq a$*. Then* $\mathbf{F} \succeq \mathbf{P}$*. In particular, we have:*

(i) *the distribution of the random variable taking values* ± 1 *with probabilities* $1/2$ *majorizes every symmetric distribution supported in* $[-1,1]$*;*
(ii) *the normal distribution* $N(0, \pi/2)$ *majorizes every symmetric distribution supported in* $[-1,1]$.

Proof. Given symmetric probability distribution $\mathbf{P}$ supported on $[-a, a]$ and a symmetric distribution $\mathbf{F}$ with the first absolute moment $\geq a$, we should prove that for every convex function f with linear growth on $\mathbb{R}$ it holds that $\mathbb{E}_{\mathbf{P}}[f] \leq \mathbb{E}_{\mathbf{F}}[f]$. Replacing $f(x)$ with $(f(x) + f(-x))/2 + c$, which does not affect the quantities to be compared, we reduce the situation to the one where f is even convex function with $f(0) = 0$. The left hand side of the inequality to be proven is linear in $\mathbf{P}$, thus, it suffices to prove the inequality for a weakly dense subset of the set of extreme points in the space of symmetric probability distributions on $[-a, a]$, e.g., for distributions assigning masses $1/2$ to points $\pm\alpha$ with $\alpha \in (0, a]$. Thus, we should prove that if f is nondecreasing finite convex function on the ray $\mathbb{R}_+ := \{x : x \geq 0\}$ such that $f(0) = 0$ and $\alpha \in (0, a]$, then $f(\alpha) \leq 2\int_0^\infty f(x)\mathbf{F}(dx)$. When proving this fact, we can assume without loss of generality that $\mathbf{F}$ possesses continuous density $p(x)$. Since f is convex, nondecreasing and nonnegative on $\mathbb{R}_+$ and $f(0) = 0$, for $x \geq 0$ we have $f(x) \geq g(x) := \max[0, f(\alpha) + f'(\alpha)(x - \alpha)]$, and $g(0) = 0$, so that $g(x) = c(x - \beta)_+$ for certain $c \geq 0$ and $\beta \in [0, \alpha]$. Replacing f with g, we do not affect the left hand side of the inequality to be proven and can only decrease the right hand side of it. Thus, it suffices to consider the case when $f(x) = (x - \beta)_+$ for certain $\beta \in [0, \alpha]$. The difference

$$h(\beta) = f(\alpha) - 2\int_0^\infty f(x)\mathbf{F}(dx) = \alpha - \beta - 2\int_\beta^\infty (x - \beta)p(x)dx,$$

which we should prove is nonpositive for $\beta \in [0, \alpha]$, is nonincreasing in β. Indeed, $h'(\beta) = -1 + 2\int_\beta^\infty p(x)dx \leq 0$. Consequently,

$$h(\beta) \leq h(0) = \alpha - 2\int_0^\infty xp(x)dx = \alpha - \int |x|\mathbf{F}(dx) \leq 0$$

due to $\alpha \leq a \leq \int |x|\mathbf{F}(dx)$. ■

Corollary 1. *Let $\mathbf{P}$ be a probability distribution on d-dimensional unit*[12] *cube $\{z \in \mathbb{R}^d : \|z\|_\infty \leq 1\}$ which is "sign-symmetric", that is, if $\xi \sim \mathbf{P}$ and E is a diagonal matrix with diagonal entries ±1, then $E\xi \sim \mathbf{P}$. Let, further, $\mathbf{U}$ be the uniform distributions on the vertices of the unit cube, and*[13] *let $\mathbf{F} \sim N(0, \frac{\pi}{2}I_d)$. Then $\mathbf{P} \preceq \mathbf{U} \preceq \mathbf{F}$.*

Proof. Without loss of generality we can assume that $\mathbf{P}$ has density. The restriction of $\mathbf{P}$ on the nonnegative orthant is a weak limit of convex combinations of masses $\mathbf{P}(\mathbb{R}^d_+) = 2^{-d}$ sitting at points from the intersection of the unit cube and $\mathbb{R}^d_+$, Consequently, $\mathbf{P}$ itself is a weak limit of uniform distributions on the vertices of boxes of the form $\{x : |x_i| \leq a_i \leq 1, i = 1, ..., d\}$, that is, limit of direct products $\mathbf{U}_a$ of uniform distributions sitting at the points $\pm a_i$.

[12] The norm $\|z\|_\infty$ is the max-norm, i.e., $\|z\|_\infty := \max\{|z_1|, ..., |z_d|\}$.

[13] By $N(\mu, \Sigma)$ we denote normal distribution with mean μ and covariance matrix Σ, and by I_d we denote the $d \times d$ unit matrix.

By Proposition 1(iii), in order to prove that $\mathbf{P} \preceq \mathbf{U}$ it suffices to verify that $\mathbf{U} \succeq \mathbf{U}_a$ for all a with $0 \le a_i \le 1$. By Proposition 1(v), to prove the latter fact it suffices to verify that the uniform distribution on $\{-1;1\}$ majorizes uniform distribution on $\{-a;a\}$ for every $a \in [0,1]$, which indeed is the case by Proposition 2. To prove that $\mathbf{F} \succeq \mathbf{U}$, by Proposition 1(v) it suffices to verify that the $N(0,\frac{\pi}{2})$-distribution on the axis majorizes the uniform distribution on $\{-1;1\}$, which again is stated by Proposition 2. ∎

Another observation of the same type as in Proposition 2 is as follows.

Proposition 3. *The uniform distribution on $[-a,a]$ majorizes every symmetric unimodal distribution $\mathbf{P}$ on the segment (that is, distribution with density which is noincreasing function of $|x|$ and vanishes for $|x| > a$) and is majorized by normal distribution $N(0,\sigma^2)$ with $\sigma = \frac{\sqrt{2\pi}}{4} \approx 0.6267$.*

Proof. The first statement is evident. To prove the second statement is the same as to prove that the uniform distribution on $[-a,a]$ with $a = 4/\sqrt{2\pi}$ is majorized by the standard normal distribution $N(0,1)$. To this end, same as in the proof of Proposition 2, it suffices to verify that

$$\int_0^a a^{-1} f(x)dx \le \frac{2}{\sqrt{2\pi}} \int_0^\infty f(x)\exp\{-x^2/2\}dx$$

for every real valued nondecreasing convex function $f(x)$ on $[0,\infty]$ such that $f(0) = 0$. Functions of this type clearly can be approximated by linear combinations, with nonnegative coefficients, of functions of the form $(x-\beta)_+$, with $\beta \ge 0$. Thus, it suffices to prove the inequality in question for $f(x) = (x-\beta)_+$, which is straightforward. ∎

2.2 Concentration

Let us consider the following “concentration” property.

Definition 2. Let $\bar\theta \in [\frac{1}{2},1)$ and $\psi(\theta,\gamma)$ be a function of $\theta \in (\bar\theta,1]$ and $\gamma \ge 1$ which is convex, nondecreasing and nonconstant as a function of $\gamma \in [1,\infty)$. We say that a probability distribution $\mathbf{F}$ on $\mathbb{R}^d$ possesses $(\bar\theta,\psi)$-concentration property (notation: $\mathbf{F} \in \mathcal{C}(\bar\theta,\psi)$), if for every closed convex set $Q \subset \mathbb{R}^d$ one has

$$\mathbf{F}(Q) \ge \theta > \bar\theta \text{ and } \gamma \ge 1 \quad \Rightarrow \quad \mathbf{F}(\{x \notin \gamma Q\}) \le \exp\{-\psi(\theta,\gamma)\}.$$

If the above implication is valid under additional assumption that Q is symmetric, we say that $\mathbf{F}$ possesses symmetric $(\bar\theta,\psi)$-concentration property (notation: $\mathbf{F} \in \mathcal{SC}(\bar\theta,\psi)$).

Distributions with such concentration properties admit a certain calculus summarized in the following proposition.

Proposition 4. *The following statements hold.*
(i) *A symmetric distribution which possesses a symmetric concentration property possesses concentration property as well: if* $\mathbf{F} \in \mathcal{SC}(\bar{\theta}, \psi)$ *is symmetric, then* $\mathbf{F} \in \mathcal{C}(\widehat{\theta}, \widehat{\psi})$ *with* $\widehat{\theta} := (1+\bar{\theta})/2$ *and* $\widehat{\psi}(\theta, \gamma) := \psi(2\theta - 1, \gamma)$.
(ii) *Let* $\xi \sim \mathbf{F}$ *be a random vector in* $\mathbb{R}^d$, A *be an* $m \times d$ *matrix and* $\mathbf{F}^{(A)}$ *be the distribution of* $A\xi$. *Then* $\mathbf{F} \in \mathcal{C}(\bar{\theta}, \psi)$ *implies that* $\mathbf{F}^{(A)} \in \mathcal{C}(\bar{\theta}, \psi)$.
(iii) *Let* $\mathbf{F} \in \mathcal{C}(\bar{\theta}, \psi)$ *be a distribution on* $\mathbb{R}^p \times \mathbb{R}^q$, *and* $\tilde{\mathbf{F}}$ *be the associated marginal distribution on* $\mathbb{R}^p$. *Then* $\tilde{\mathbf{F}} \in \mathcal{C}(\bar{\theta}, \psi)$.
(iv) *Let* ξ^i, $i = 1, ..., p$, *be independent random vectors in* $\mathbb{R}^d$ *with symmetric distributions* $\mathbf{F}_1, ..., \mathbf{F}_p$, *such that* $\mathbf{F}_i \in \mathcal{C}(\bar{\theta}, \psi)$, $i = 1, ..., p$. *Then the distribution* $\mathbf{F}$ *of* $\eta = \xi^1 + ... + \xi^p$ *belongs to* $\mathcal{C}(\widehat{\theta}, \widehat{\psi})$ *with* $\widehat{\theta} := 2\bar{\theta} - 1$ *and* $\widehat{\psi}(\theta, \cdot)$ *given by the convex hull*[14] *of the function*

$$\varphi(\gamma) := \begin{cases} \ln\left(\frac{1}{1-\theta}\right), & 1 \le \gamma < p, \\ \max\left\{\ln\left(\frac{1}{1-\theta}\right), \psi(2\theta - 1, \gamma/p) - \ln p\right\}, & \gamma \ge p, \end{cases} \tag{17}$$

where $\gamma \in [1, \infty)$ *and* $\theta > \widehat{\theta}$.
(v) *Let* $\mathbf{F}_i \in \mathcal{C}(\bar{\theta}, \psi)$ *be distributions on* $\mathbb{R}^{m_i}$, $i = 1, ..., p$, *and assume that all* $\mathbf{F}_i$ *are symmetric. Then* $\mathbf{F}_1 \times ... \times \mathbf{F}_p \in \mathcal{C}(\widehat{\theta}, \widehat{\psi})$ *with* $\widehat{\theta}$ *and* $\widehat{\psi}$ *exactly as in* (iv).

Moreover, statements (ii) – (v) *remain valid if the class* $\mathcal{C}(\bar{\theta}, \psi)$ *in the premises and in the conclusions is replaced with* $\mathcal{SC}(\bar{\theta}, \psi)$.

Proof. (i) Let $\mathbf{F}$ satisfy the premise of (i), and let Q be a closed convex set such that $\mathbf{F}(Q) \ge \theta > \widehat{\theta}$. By symmetry of $\mathbf{F}$, we have $\mathbf{F}(Q \cap (-Q)) \ge 2\theta - 1 > \bar{\theta}$, and hence

$$\mathbf{F}(\{\xi \notin \gamma Q\}) \le \mathbf{F}(\{\xi \notin \gamma(Q \cap (-Q))\}) \le \exp\{-\psi(2\theta - 1, \gamma)\}.$$

The statements (ii) and (iii) are evident.

(iv) Let Q be a closed convex set such that $\theta := \mathbf{F}(Q) > \widehat{\theta}$. We claim that then

$$\mathbf{F}_i(\{x \in Q\}) \ge 2\theta - 1 > \bar{\theta}, \quad i = 1, ..., p. \tag{18}$$

Indeed, let us fix i, and let ζ be the sum of all ξ^j except for ξ^i, so that $\eta = \zeta + \xi^i$, ζ and ξ^i are independent and ζ is symmetrically distributed. Observe that conditional, given the value u of ξ^i, probability for ζ to be outside Q is at least $1/2$, provided that $u \notin Q$. Indeed, when $u \notin Q$, there exists a closed half-space Π_u containing u which does not intersect Q (recall that Q is closed and convex); since ζ is symmetrically distributed, $u + \zeta \in \Pi_u$ with probability at least $1/2$, as claimed. From our observation it follows that if $\xi^i \notin Q$ with probability s, then $\eta \notin Q$ with probability at least $s/2$; the latter probability is at most $1 - \theta$, whence $s \le 2 - 2\theta$, and (18) follows.

[14] The convex hull of a function φ is the largest convex function majorized by φ.

Assuming $\gamma \geq p$ and $\mathrm{Prob}\{\xi^1 + ... + \xi^p \in Q\} \geq \theta > \widehat{\theta}$, we have

$$\mathrm{Prob}\{\xi^1 + ... + \xi^p \notin \gamma Q\} \leq \sum_{i=1}^{p} \mathrm{Prob}\{\xi^i \notin (\gamma/p)Q\} \leq p\exp\{-\psi(2\theta - 1, \gamma/p)\},$$

where the concluding inequality is given by (18) and the inclusions $\mathbf{F}_i \in \mathcal{C}(\bar{\theta}, \psi)$. Now, the distribution of η is symmetric, so that $\mathbf{F}(\{\eta \in Q\}) > \widehat{\theta} \geq 1/2$ implies that Q intersect $-Q$, that is, that $0 \in Q$. Due to the latter inclusion, for $\gamma \geq 1$ one has $\mathbf{F}(\{\eta \in \gamma Q\}) \geq \mathbf{F}(\{\eta \in Q\}) \geq \theta$. Thus,

$$\mathbf{F}(\{\eta \notin \gamma Q\}) \leq \begin{cases} 1 - \theta, & 1 \leq \gamma \leq p, \\ p\exp\{-\psi(2\theta - 1, \gamma/p\}, & \gamma \geq p, \end{cases}$$

and (iv) follows.

(v) Let $\xi^i \sim \mathbf{F}_i$ be independent, $i = 1, ..., p$, and let $\bar{\mathbf{F}}_i$ be the distribution of the $(m_1 + ... + m_p)$-dimensional random vector

$$\zeta^i = (0_{m_1+...+m_{i-1}}, \xi^i, 0_{m_{i+1}+...+m_p}).$$

Clearly, $\bar{\mathbf{F}}_i \in \mathcal{C}(\bar{\theta}, \psi)$ due to similar inclusion for $\mathbf{F}_i$. It remains to note that $\sum_i \zeta^i \sim \mathbf{F}_1 \times ... \times \mathbf{F}_p$ and to use (iv). ∎

We intend now to present a number of concrete distributions possessing the concentration property.

Example 1: Normal distribution.

Consider the cumulative distribution function $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\{-z^2/2\}dz$ of the standard normal distribution and let[15] $\phi(\theta) := \Phi^{-1}(\theta)$ for $\theta \in (0, 1)$.

Theorem 1. *Let B be a closed convex set in $\mathbb{R}^d$. Then the following holds.*
(i) *If $\eta \sim N(0, I_d)$ and $\mathrm{Prob}\{\eta \in B\} \geq \theta > \frac{1}{2}$, then for $\alpha \in (0, 1)$:*

$$\mathrm{Prob}\{\alpha\eta \in B\} \geq 1 - \exp\left\{-\frac{\phi^2(\theta)}{2\alpha^2}\right\}. \tag{19}$$

(ii) *If $\zeta \sim N(0, \Sigma)$ and $\mathrm{Prob}\{\zeta \notin B\} \equiv 1 - \theta < \frac{1}{2}$, then for $\gamma \geq 1$:*

$$\mathrm{Prob}\{\zeta \notin \gamma B\} \leq \min\left\{1 - \theta, \exp\left(-\tfrac{1}{2}\phi^2(\theta)\gamma^2\right)\right\}. \tag{20}$$

In other words, a zero mean normal distribution on $\mathbb{R}^d$ belongs to $\mathcal{C}(\frac{1}{2}, \psi)$ with

$$\psi(\theta, \gamma) := \max\left\{\ln[(1 - \theta)^{-1}], \tfrac{1}{2}\phi^2(\theta)\gamma^2\right\}.$$

Proof. Our proof of this result is based on the following result due to Borell [1]:

[15] The inverse function $\phi(\theta) := \Phi^{-1}(\theta)$ is defined by the equation $\Phi(\phi(\theta)) = \theta$.

(!!!) For $\eta \sim N(0, I_d)$, every $\gamma > 0$, $\varepsilon \geq 0$ and every closed set $X \subset \mathbb{R}^d$ such that $\text{Prob}\{\eta \in X\} \geq \gamma$, one has

$$\text{Prob}\left\{\text{dist}(\eta, X) > \varepsilon\right\} \leq 1 - \Phi(\phi(\gamma) + \varepsilon), \tag{21}$$

where $\text{dist}(a, X) := \inf_{x \in X} \|a - x\|$.

Now let η, ζ be independent $N(0, I_d)$ random vectors, and let

$$p(\alpha) = \text{Prob}\{\alpha\eta \notin B\}.$$

We have that $\alpha\eta + \sqrt{1-\alpha^2}\,\zeta \sim N(0, I_d)$, and hence

$$\text{Prob}\{\text{dist}(\alpha\eta + \sqrt{1-\alpha^2}\zeta, B) > t\} \leq 1 - \Phi(\phi(\theta) + t) \tag{22}$$

by (21). On the other hand, let $\alpha\eta \notin B$, and let $e = e(\eta)$ be a vector such that $\|e\| = 1$ and $e^T[\alpha\eta] > \max\limits_{x \in B} e^T x$. If ζ is such that $\sqrt{1-\alpha^2} e^T\zeta > t$, then $\text{dist}(\alpha\eta + \sqrt{1-\alpha^2}\zeta, B) > t$, and hence if $\alpha\eta \notin B$, then

$$\text{Prob}\left\{\zeta : \text{dist}(\alpha\eta + \sqrt{1-\alpha^2}\zeta, B) > t\right\} \geq 1 - \Phi(t/\sqrt{1-\alpha^2}).$$

Whence for all $t \geq 0$ such that $\delta(t) := \phi(\theta) + t - t/\sqrt{1-\alpha^2} \geq 0$ one has

$$\begin{aligned} p(\alpha)[1 - \Phi(t/\sqrt{1-\alpha^2})] &\leq \text{Prob}\left\{\text{dist}(\alpha\eta + \sqrt{1-\alpha^2}\zeta, B) > t\right\} \\ &\leq 1 - \Phi(\phi(\theta) + t). \end{aligned}$$

It follows that

$$\begin{aligned} p(\alpha) &\leq \frac{1-\Phi(\phi(\theta)+t)}{1-\Phi(t/\sqrt{1-\alpha^2})} = \frac{\int\limits_{t/\sqrt{1-\alpha^2}}^{\infty} \exp\{-(s+\delta(t))^2/2\}ds}{\int\limits_{t/\sqrt{1-\alpha^2}}^{\infty} \exp\{-s^2/2\}ds} \\ &= \frac{\int\limits_{t/\sqrt{1-\alpha^2}}^{\infty} \exp\{-s^2/2 - s\delta(t) - \delta^2(t)/2\}ds}{\int\limits_{t/\sqrt{1-\alpha^2}}^{\infty} \exp\{-s^2/2\}ds} \leq \exp\{-t\delta(t)/\sqrt{1-\alpha^2} - \delta^2(t)/2\}. \end{aligned}$$

Setting in the resulting inequality $t = \frac{\phi(\theta)(1-\alpha^2)}{\alpha^2}$, we get

$$p(\alpha) \leq \exp\left\{-\frac{\phi^2(\theta)}{2\alpha^2}\right\}. \qquad \blacksquare$$

Example 2: Uniform distribution on the vertices of a cube.

We start with the following known fact (which is the Talagrand Inequality in its extended form given in [6]):

Theorem 2. *Let* $(E_t, \|\cdot\|_{E_t})$ *be finite-dimensional normed spaces,* $t = 1, ..., d$, F *be the direct product of* $E_1, ..., E_d$ *equipped with the norm* $\|(x^1, ..., x^d)\|_F := \sqrt{\sum_{t=1}^d \|x^t\|_{E_t}^2}$, $\mathbf{F}_t$ *be Borel probability distributions on the unit balls of* E_t *and* $\mathbf{F}$ *be the product of these distributions. Given a closed convex set* $A \subset F$, *let* $\mathrm{dist}(x, A) = \min_{y \in A} \|x - y\|_F$. *Then*

$$\mathbb{E}_{\mathbf{F}}\left[\exp\left\{\tfrac{1}{16}\mathrm{dist}^2(x, A)\right\}\right] \le \tfrac{1}{\mathbf{F}(A)}. \tag{23}$$

This result immediately implies the following.

Theorem 3. *Let* $\mathbf{P}$ *be the uniform distribution on the vertices of the unit cube* $\left\{x \in \mathbb{R}^d : \|x\|_\infty \le 1\right\}$. *Then* $\mathbf{P} \in \mathcal{SC}(\bar{\theta}, \psi)$ *with the parameters given by:*

$$\begin{array}{l} \bar{\theta} = \frac{1+\exp\{-\pi^2/8\}}{2} \approx 0.6456, \\ \rho(\theta) = \sup\limits_{\omega \in (0, \pi/2]} \left\{\omega^{-1}\mathrm{acos}\left(\frac{1+\exp\{-\omega^2/2\}-\theta}{\theta}\right) : 1 + \exp\{-\omega^2/2\} < 2\theta\right\}, \\ \psi(\theta, \gamma) = \max\left\{\ln\frac{1}{1-\theta}, \ln\frac{\theta}{1-\theta} + \frac{\rho^2(\theta)(\gamma-1)^2}{16}\right\}. \end{array} \tag{24}$$

In order to prove this result we need the following lemma.

Lemma 1. *Let* ξ_j *be independent random variables taking values* ± 1 *with probabilities* $1/2$ *and let* $\zeta := \sum_{j=1}^d a_j \xi_j$ *with* $\|a\| = 1$. *Then for every* $\rho \in [0, 1]$ *and every* $\omega \in [0, \pi/2]$ *one has*

$$\mathrm{Prob}\{|\zeta| \le \rho\}\cos(\rho\omega) - \mathrm{Prob}\{|\zeta| > \rho\} \le \cos^d\left(\tfrac{\omega}{\sqrt{d}}\right) \le \exp\{-\omega^2/2\}.$$

In particular, if

$$\theta := \mathrm{Prob}\{|\zeta| \le \rho\} > \bar{\theta} := \frac{1 + \exp\{-\pi^2/8\}}{2}, \tag{25}$$

then $\rho \ge \rho(\theta)$, *where* $\rho(\theta)$ *is defined in* (24).

Proof. For $\omega \in [0, \pi/2]$ we have

$$\mathbb{E}\{\exp\{i\zeta\omega\}\} = \prod_j \mathbb{E}\{\exp\{ia_j\xi_j\omega\}\} = \prod_j \cos(a_j\omega).$$

Observe that the function $f(s) = \ln\cos(\sqrt{s})$ is concave on $[0, (\pi/2)^2]$. Indeed, $f'(s) = -\operatorname{tg}(\sqrt{s})\frac{1}{2\sqrt{s}}$ and

$$f''(s) = -\tfrac{1}{\cos^2(\sqrt{s})}\tfrac{1}{4s} + \operatorname{tg}(\sqrt{s})\tfrac{1}{4s\sqrt{s}} = -\tfrac{1}{4s^2\cos^2(s)}\left[\sqrt{s} - \sin(\sqrt{s})\cos(\sqrt{s})\right] \le 0.$$

Consequently, for $0 \le \omega \le \pi/2$ we have

$$\sum_j \ln(\cos(a_j\omega)) = \sum_j f(a_j^2\omega^2) \le \max_{\substack{0 \le s_j \le (\pi/2)^2 \\ \sum_j s_j = \omega^2}} \sum_j f(s_j) = df(\omega^2/d) \le \exp\{-\omega^2/2\},$$

and we see that

$$0 \le \omega \le \tfrac{\pi}{2} \Rightarrow \mathbb{E}\{\exp\{i\zeta\omega\}\} \le \cos^d(\tfrac{\omega}{\sqrt{d}}) \le \exp\{-\omega^2/2\}.$$

On the other hand, ζ is symmetrically distributed, and therefore for $0 \le \rho \le 1$ and $\omega \in [0, \pi/2]$ we have, setting $\mu := \text{Prob}\{|\zeta| \le \rho\}$:

$$\mathbb{E}\left\{\exp\{i\omega\zeta\}\right\} \ge \mu\cos(\rho\omega) - (1-\mu),$$

and we arrive at the announced result. ∎

Proof of Theorem 3. Let Q be a symmetric closed convex set in $\mathbb{R}^d$ such that

$$\text{Prob}\{\xi \in Q\} \ge \theta > \bar{\theta}.$$

We claim that then Q contains the centered at the origin Euclidean ball of the radius $\rho(\theta)$. Indeed, otherwise Q would be contained in the strip $\Pi = \{x : |a^T x| \le c\}$ with $c < \rho(\theta)$ and $\|a\| = 1$. Setting $\zeta = a^T\xi$, we get

$$\text{Prob}\{|\zeta| \le c\} = \text{Prob}\{\xi \in \Pi\} \ge \text{Prob}\{\xi \in Q\} \ge \theta,$$

whence by Lemma 1, $c \ge \rho(\theta)$, which is a contradiction.

For $s \ge 1$ from $x \notin sQ$ it follows that the set $x + (s-1)Q$ does not intersect Q; since this set contains the $\|\cdot\|$-ball centered at x of the radius $(s-1)\rho(\theta)$, the Euclidean distance $d_Q(x) := \text{dist}(x, Q)$, from x to Q, is at least $(s-1)\rho(\theta)$. At the same time, by Talagrand Inequality we have

$$\mathbb{E}\left[\exp\left\{\tfrac{d_Q^2(\xi)}{16}\right\}\right] \le \frac{1}{\text{Prob}\{\xi \in Q\}} \le \frac{1}{\theta}.$$

On the other hand, when $\gamma \ge 1$ we have, by the above arguments,

$$\mathbb{E}\left[\exp\left\{\tfrac{d_Q^2(\xi)}{16}\right\}\right] \ge \text{Prob}\left\{\xi \in Q\right\} + \exp\left\{\tfrac{(\gamma-1)^2\rho^2(\theta)}{16}\right\}\text{Prob}\left\{\xi \notin \gamma Q\right\},$$

whence if $\gamma \ge 1$, then

$$\text{Prob}\left\{\xi \notin \gamma Q\right\} \le \tfrac{1-\theta}{\theta}\exp\left\{-\tfrac{(\gamma-1)^2\rho^2(\theta)}{16}\right\},$$

and of course

$$\gamma \ge 1 \Rightarrow \text{Prob}\left\{\xi \notin \gamma Q\right\} \le 1 - \theta,$$

and the result follows. ∎

Example 3: Uniform distribution on the cube.

This example is similar to the previous one.

Theorem 4. *Let* $\mathbf{P}$ *be the uniform distribution on the unit cube* $\{x \in \mathbb{R}^d : \|x\|_\infty \le 1\}$. *Then* $\mathbf{P} \in \mathcal{SC}(\bar{\theta}, \psi)$ *with the parameters given by:*

$$\begin{array}{l} \bar{\theta} = \frac{1+\exp\{-\pi^2/24\}}{2} \approx 0.8314, \\ \rho(\theta) = \sup\limits_{\omega\in(0,\pi/2]}\left\{\omega^{-1}\text{acos}\left(\frac{1+\exp\{-\omega^2/6\}-\theta}{\theta}\right) : 1+\exp\{-\omega^2/6\} < 2\theta\right\}, \\ \psi(\theta,\gamma) = \max\left\{\ln\left(\frac{1}{1-\theta}\right), \ln\left(\frac{\theta}{1-\theta}\right) + \frac{\rho^2(\theta)(\gamma-1)^2}{16}\right\}. \end{array} \quad (26)$$

We have the following analog of Lemma 1.

Lemma 2. *Let ξ_j be independent random variables uniformly distributed in $[-1,1]$ and $\zeta = \sum_{j=1}^d a_j\xi_j$ with $\|a\| = 1$. Then for every $\rho \in [0,1]$ and every $\omega \in [0,\pi/2]$ one has*

$$\text{Prob}\left\{|\zeta| \le \rho\right\}\cos(\rho\omega) - \text{Prob}\left\{|\zeta| > \rho\right\} \le \left(\frac{\sin(\omega d^{-1/2})}{\omega d^{-1/2}}\right)^d \le \exp\left\{-\omega^2/6\right\}. \tag{27}$$

In particular, if

$$\theta := \text{Prob}\left\{|\zeta| \le \rho\right\} > \bar\theta := \frac{1+\exp\{-\pi^2/24\}}{2}, \tag{28}$$

then $\rho \ge \rho(\theta)$, where $\rho(\theta)$ is defined in (26).

Proof. For $\omega \in [0,\pi/2]$ we have

$$\mathbb{E}\{\exp\{i\zeta\omega\}\} = \prod_j \mathbb{E}\{\exp\{ia_j\xi_j\omega\}\} = \prod_j \tfrac{\sin(a_j\omega)}{a_j\omega}.$$

Observe that the function $f(s) = \ln(\sin(\sqrt{s})) - \frac{1}{2}\ln s$ is concave on $[0,(\pi/2)^2]$. Indeed, $f'(s) = \text{ctg}(\sqrt{s})\frac{1}{2\sqrt{s}} - \frac{1}{2s}$ and

$$f''(s) = -\tfrac{1}{\sin^2(\sqrt{s})}\tfrac{1}{4s} - \text{ctg}(\sqrt{s})\tfrac{1}{4s\sqrt{s}} + \tfrac{1}{2s^2} = \tfrac{h(\sqrt{s})}{4s^2\sin^2(\sqrt{s})},$$

where

$$h(r) = 2\sin^2(r) - r\sin(r)\cos(r) - r^2 = 1-\cos(2r) - (r/2)*sin(2r) - r^2.$$

We have $h(0) = 0$, $h'(r) = (3/2)*\sin(2r) - r\cos(2r) - 2r$, so that $h'(0) = 0$, $h''(r) = 2\cos(2r) + r\sin(2r) - 2$, so that $h''(0) = 0$, and finally $h'''(r) = -3\sin(2r) + 2r\cos(2r)$, so that $h'''(0) = 0$ and $h'''(r) \le 0$, $0 \le r \le \pi/2$, due to $\text{tg}(u) \ge u$ for $0 \le u < \pi/2$. It follows that $h(\cdot) \le 0$ on $[0,\pi/2]$, as claimed.

From log-concavity of f on $[0,(\pi/2)^2]$, same as in the proof of Lemma 1, we get the first inequality in (27); the second is straightforward. ■

The remaining steps in the proof of Theorem 4 are completely similar to those for Example 2.

Remark 1. Examples 2 and 3 admit natural extensions. Specifically, let ξ be a random vector in $\mathbb{R}^d$ with independent symmetrically distributed on $[-1,1]$ coordinates ξ_i, and let the distributions $\mathbf{P}_i$ of ξ_i be "not too concentrated at the origin", e.g., are such that: (i) $\mathbb{E}\{\xi_i^2\} \ge \alpha^2 > 0$, $i = 1,...,d$, or (ii) $\mathbf{P}_i$ possesses density which is bounded by $1/\alpha$, $i = 1,...,d$. Let $\mathbf{P}$ be the distribution of ξ. Then $\xi \in \mathcal{C}(\bar\theta, a(\theta) + b(\theta)\gamma^2)$ with $\bar\theta$ and $a(\cdot)$, $b(\cdot) > 0$ depending solely on α. The proof is completely similar to those in Examples 1 and 2.

Remark 2. We have proven that the uniform distributions on the vertices of the unit cube $\{\|x\|_\infty \le 1\}$ and on the entire cube possess symmetric concentration property. In fact they possess as well the general concentration property with slightly "spoiled" $\bar\theta$, $\psi(\cdot,\cdot)$ due to Proposition 4(i).

2.3 Main result

Proposition 5. *Let* $\mathbf{F}$, $\mathbf{P}$ *be probability distributions on* $\mathbb{R}^d$ *such that* $\mathbf{P} \preceq \mathbf{F}$, $\mathbf{F}$ *is symmetric and* $\mathbf{F} \in \mathcal{C}(\bar{\theta}, \psi)$. *Let, further,* Q *be a closed convex set in* $\mathbb{R}^d$ *such that*

$$\mathbf{F}(Q) \geq \theta > \bar{\vartheta} \qquad (29)$$

and let $p_Q(x)$ *be the Minkowski function*[16] *of* Q. *Then for every convex continuous and nondecreasing function* $\Psi : \mathbb{R}_+ \to \mathbb{R}$ *one has*

$$\mathbb{E}_{\mathbf{P}}\left[\Psi(p_Q(\xi))\right] \leq \left(\theta + e^{-\psi(\theta,1)}\right)\Psi(1) + \int_1^\infty \Psi'(\gamma) e^{-\psi(\theta,\gamma)} d\gamma. \qquad (30)$$

If the assumption $\mathbf{F} \in \mathcal{C}(\bar{\theta}, \psi)$ *is weakened to* $\mathbf{F} \in \mathcal{SC}(\bar{\theta}, \psi)$, *then the conclusion remains valid under the additional assumption that* Q *is symmetric.*

Proof. Let $f(x) := \Psi(p_Q(x))$, so that f is a convex lower semicontinuous function on $\mathbb{R}^d$, and let

$$P(\gamma) := \mathbf{F}(\{x \notin \gamma Q\}) = \mathbf{F}(\{x : p_Q(x) > \gamma\}),$$

so that

$$\gamma \geq 1 \Rightarrow P(\gamma) \leq S(\gamma) := \exp\{-\psi(\theta, \gamma)\}.$$

We have that $\mathbb{E}_{\mathbf{P}}\{f\} \leq \mathbb{E}_{\mathbf{F}}\{f\}$, since $\mathbf{P} \preceq \mathbf{F}$, and

$$\begin{aligned}\mathbb{E}_{\mathbf{F}}\{f\} &\leq \Psi(1)\mathbf{F}(Q) - \int_1^\infty \Psi(\gamma) dP(\gamma) \leq \theta\Psi(1) + \Psi(1)P(1) + \int_1^\infty \Psi'(\gamma)P(\gamma)d\gamma \\ &\leq (\theta + S(1))\Psi(1) + \int_1^\infty \Psi'(\gamma)S(\gamma)d\gamma,\end{aligned}$$

as claimed. ∎

Theorem 5. *Let* $\mathbf{F}$, $\mathbf{P}$ *be probability distributions on* $\mathbb{R}^d$ *such that* $\mathbf{P} \preceq \mathbf{F}$, $\mathbf{F}$ *is symmetric and* $\mathbf{F} \in \mathcal{C}(\bar{\theta}, \psi)$. *Let, further,* Q *be a closed convex set in* $\mathbb{R}^d$ *such that*

$$\mathbf{F}(Q) \geq \theta > \bar{\theta} \qquad (31)$$

and let $p_Q(x)$ *be the Minkowski function of* Q. *Then for every* $s > 1$ *one has*

$$\mathbf{P}(\{x : x \notin sQ\}) \leq \mathrm{Err}(s, \theta) := \inf_{1 \leq \beta < s} \tfrac{1}{s-\beta} \int_\beta^\infty \exp\{-\psi(\theta, \gamma)\} d\gamma. \qquad (32)$$

In particular, if $\psi(\theta, \gamma) \geq a(\theta) + b(\theta)\gamma^2/2$ *with* $b(\theta) > 0$, *then*

$$\mathrm{Err}(s, \theta) \leq \frac{2\exp\{-a(\theta) - b(\theta)(s+1)^2/8\}}{b(\theta)(s^2 - 1)}. \qquad (33)$$

If the assumption $\mathbf{F} \in \mathcal{C}(\bar{\theta}, \psi)$ *is weakened to* $\mathbf{F} \in \mathcal{SC}(\bar{\theta}, \psi)$, *then the conclusion remains valid under the additional assumption that* Q *is symmetric.*

---

[16] Minkowski function is defined as $p_Q(x) := \inf\{t : t^{-1}x \in Q,\ t > 0\}$. Under our premise, $0 \in Q$ due to symmetry of $\mathbf{F}$ and $\mathbf{F}(Q) > \bar{\theta} > 1/2$. Consequently, $p_Q(\cdot)$ is a lower semicontinous convex function with values in $\mathbb{R} \cup \{+\infty\}$.

Proof. Let $\beta \in [1, s)$, and let $\Psi(\gamma) = \frac{(\gamma-\beta)_+}{s-\beta}$. Applying (30), we get

$$\begin{aligned}\mathbf{P}(\{x : x \notin sQ\}) = \mathbf{P}(\{x : x \notin sQ\})\Psi(s) \le \mathbb{E}_{\mathbf{P}}\{\Psi \circ p_Q\} \\ \le \tfrac{1}{s-\beta} \int_\beta^\infty \exp\{-\psi(\theta, \gamma)\} d\gamma.\end{aligned}$$

Since this relation holds true for every $\beta \in [1, s)$, (32) follows.

Now let θ be such that $\psi(\theta, \gamma) \ge a + b\gamma^2/2$ for all $\gamma \ge 1$, where $b > 0$. Then (32) implies that

$$\begin{aligned}&\mathbf{P}(\{x : x \notin sQ\}) \le \left[\tfrac{1}{s-\beta} \int_\beta^\infty \exp\{-a - b\gamma^2/2\} d\gamma\right]\Bigg|_{\beta=\frac{1+s}{2}} \\ &= \tfrac{2\exp\{-a\}}{s-1} \int_{\frac{1+s}{2}}^\infty \exp\{-b\gamma^2/2\} d\gamma \le \tfrac{2\exp\{-a\}}{s-1} \int_{\frac{1+s}{2}}^\infty \tfrac{\gamma}{\frac{1+s}{2}} \exp\{-b\gamma^2/2\} d\gamma \\ &= \tfrac{2\exp\{-a-b(s+1)^2/8\}}{b(s^2-1)}.\end{aligned}$$

■

2.4 Putting blocks together

Now we are ready to address the questions A and B posed at the beginning of this Section.

Setup for the test

Theorem 5 suggests the following *Basic Setup* for test (T):
Input: Closed convex set $B \subset \mathbb{R}^d$, zero mean distribution $\mathbf{P}$ on $\mathbb{R}^d$, confidence parameter $\varepsilon \in (0, 1)$, reliability parameter $\delta \in (0, 1)$. The goal is to justify the hypothesis

$$\mathbf{P}(\{\xi \notin B\}) \le \varepsilon. \tag{34}$$

Choosing "pre-trial" distribution. We choose a *symmetric* "pre-trial" distribution $\bar{\mathbf{F}}$ on $\mathbb{R}^d$ in such a way that

I(1) $\bar{\mathbf{F}} \succeq \mathbf{P}$;
I(2) $\bar{\mathbf{F}}$ possesses the Concentration property: $\bar{\mathbf{F}} \in \mathcal{C}(\bar{\theta}, \psi)$ with known $\bar{\theta}$ and ψ.

After $\bar{\mathbf{F}}$ is chosen, we compute the associated "error function" (cf. (32))

$$\mathrm{Err}(s, \theta) = \inf_{1 \le \beta < s} \frac{1}{s-\beta} \int_\beta^\infty \exp\{-\psi(\theta, \gamma)\} d\gamma. \tag{35}$$

Choosing trial distribution, sample size and acceptance level. We choose somehow design parameters $\theta \in (\bar{\theta}, 1)$ and $s > 1$ ("amplification") such that

$$\mathrm{Err}(s,\theta) \le \varepsilon \tag{36}$$

and specify the trial distribution $\mathbf{F}$ as $\bar{\mathbf{F}}^{(s)}$. We further specify sample size N and acceptance level K in such a way that the probability to get at least $N-K$ successes in N Bernoulli experiments with probability θ of success in a single experiment is at most δ:

$$\sum_{r=0}^{K} \binom{N}{r} \theta^{N-r}(1-\theta)^r \le \delta. \tag{37}$$

For example, one can set

$$K := 0,\ \ N := \mathrm{ceil}\left[\frac{\ln(\delta)}{\ln(\theta)}\right]. \tag{38}$$

Theorem 6. *With the outlined setup, the probability of false acceptance for the resulting test* (T) *is* $\le \delta$.

Proof. Let $Q = s^{-1}B$. Assume first that $\bar{\mathbf{F}}(Q) \ge \theta$. Applying (32), we get

$$\mathbf{P}(\{\xi \notin B\}) = \mathbf{P}(\{\xi \notin sQ\}) \le \mathrm{Err}(s,\theta) \le \varepsilon,$$

that is, in the case in question false acceptance is impossible. Now consider the case of $\bar{\mathbf{F}}(Q) < \theta$, or, which is the same, $\mathbf{F}(B) < \theta$. In this case, by (37), the probability to make acceptance conclusion is at most δ. ∎

Remark 3. The outlined reasoning demonstrates that when B is symmetric, Theorem 6 remains valid when the requirement $\bar{\mathbf{F}} \in \mathcal{C}(\bar{\theta},\psi)$ is weakened to $\bar{\mathbf{F}} \in \mathcal{SC}(\bar{\theta},\psi)$. The same is true for Theorem 8 below.

Resolution

Let us try to understand how conservative is our test. The answer is easy when the trial distribution coincides with the actual one.

Theorem 7. *Let* $\mathbf{P}$ *be symmetric and possess the concentration property:* $\mathbf{P} \in \mathcal{C}(\bar{\theta},\psi)$*, so that the choice* $\bar{\mathbf{F}} = \mathbf{P}$ *satisfies* I(1) *and* I(2) *(from the Basic Setup), and let* N, K, θ, s *be the parameters given by the Basic Setup for this choice of pre-trial distribution. Let* $\theta_* := \mathbf{P}(s^{-1}B)$.

Then the probability for (T) *not to make the acceptance conclusion is at most*

$$\delta_* = 1 - \sum_{r=0}^{K} \theta_*^{N-r}(1-\theta_*)^r.$$

When B *is symmetric, the conclusion remains valid when the assumption* $\mathbf{P} \in \mathcal{C}(\bar{\theta},\psi)$ *is weakened to* $\mathbf{P} \in \mathcal{SC}(\bar{\theta},\psi)$.

Proof. The statement is, essentially a tautology: since $\mathbf{F} = \bar{\mathbf{F}}^{(s)} = \mathbf{P}^{(s)}$, we have $\mathbf{F}(B) = \mathbf{P}(s^{-1}B) = \theta_*$, and the probability for (T) not to make the acceptance conclusion is exactly δ_*. ∎

In terms of Question B, Theorem 7 states that the resolution of (T) is not worse than s, provided that

$$1 - \sum_{r=0}^{K} (1-\varepsilon)^{N-r}\varepsilon^r \leq \delta. \tag{39}$$

When the setup parameters N, K are chosen according to (38), that is, $K = 0$, $N = \text{ceil}\left[\frac{\ln(\delta)}{\ln(\theta)}\right]$, condition (39) becomes $1 - (1-\varepsilon)^N \leq \delta$, which is for sure true when $2\varepsilon \ln(1/\delta) \leq \delta \ln(1/\theta)$.

Situation with resolution in the case when the trial distribution is not a scaling $\mathbf{P}^{(s)}$ of the actual one is much more complicated, and its detailed investigation goes beyond the scope of this paper. Here we restrict ourselves to demonstration of a phenomenon which can occur in the general case. Let $\mathbf{P}$ be the uniform distribution on the vertices of the unit d-dimensional cube B, and $\mathbf{F}$ be normal distribution $N(0, \frac{\pi}{2} I_d)$, so that $\mathbf{F} \succeq \mathbf{P}$ by Proposition 2. We have $\mathbf{P}(B) = 1$, while a typical realization of $\mathbf{F}$ is outside the box $\frac{1}{2}\pi\kappa\sqrt{2\ln d}\, B$, $\kappa < 1$ with probability tending to 1 as $d \to \infty$, provided that $\kappa < 1$. It follows that in the situation in question the resolution of (T) is dimension-dependent and deteriorates, although pretty slow, as dimension grows.

Homogenization

We are about to present a slight modification of test (T) – the *homogenized* analysis test (HT) which is better suited for many applications. This test is as follows:

Input: Closed convex set $B \subset \mathbb{R}^d$, zero mean distribution $\mathbf{P}$ on $\mathbb{R}^d$, scale parameter $\bar{\sigma} > 0$, reliability parameter $\delta \in (0,1)$. The goal is to get upper bounds for the probabilities

$$\mathbf{P}(\{s^{-1}\bar{\sigma}\xi \notin B\}), \ \text{ for } s > 1. \tag{40}$$

Setup: *Trial distribution:* We choose a symmetric distribution $\mathbf{F}$ on $\mathbb{R}^d$ such that $\mathbf{F} \succeq \mathbf{P}$ and $\mathbf{F} \in \mathcal{C}(\bar{\theta}, \psi)$ with known $\bar{\theta}$ and ψ, and compute the corresponding function $\text{Err}(\cdot,\cdot)$ according to (35).

Sample size and acceptance level: We choose somehow $\theta \in (\bar{\theta}, 1)$, sample size N and acceptance level K satisfying (37).

Execution: We generate N-element sample $\{\eta^j\}_{j=1}^N$ from the trial distribution and check whether

$$\text{Card}(\{j \leq N : \bar{\sigma}\eta^j \notin B\}) \leq K.$$

If it is the case, we say that (HT) is successful, and claim that

$$\mathbf{P}(\{s^{-1}\bar{\sigma}\xi \notin B\}) \le \mathrm{Err}(s,\theta), \quad \text{for all } s > 1, \tag{41}$$

otherwise we say that (HT) is unsuccessful.
The analogy of Theorem 6 for (HT) is as follows:

Theorem 8. *With the outlined setup, bounds* (41)*, if yielded by* (HT)*, are valid with reliability at least* $1-\delta$*. Equivalently: in the case when not all of the bounds are valid, the probability for* (HT) *to be successful is at most* δ*.*

Indeed, in the case when $\mathbf{F}(\{\eta : \bar{\sigma}^{-1}\eta \in B\}) \ge \theta$, bounds (41) are valid by (32), and in the case when $\mathbf{F}(\{\eta : \bar{\sigma}^{-1}\eta \in B\}) < \theta$, the probability of successful termination is $\le \delta$ by (37).

The difference between (T) and (HT) is clear. The goal of (T) is to justify the hypothesis that $\xi \sim \mathbf{P}$ takes its value outside a given convex set B with probability at most ε. The goal of (HT) is to bound from above the probability for ξ to take value outside of set $s\bar{\sigma}^{-1}B$ as a function of $s > 1$. This second goal is slightly easier than the first one, in the sense that now a *single* sample allows to build bounds for the indicated probabilities simultaneously for all $s > 1$.

2.5 Numerical illustration

Here we illustrate our constructions, by a numerical example.

The situation.

We consider a discrete time linear dynamical system

$$z(t+1) = Az(t), \quad A = \frac{1}{203}\begin{bmatrix} 39 & 69 & 41 & -11 & 69 & 84 \\ 56 & -38 & -92 & 82 & 28 & 57 \\ -85 & -40 & -98 & -41 & 72 & -78 \\ 61 & 86 & -83 & -43 & -31 & 38 \\ -5 & -96 & 51 & -96 & 66 & -77 \\ 54 & 2 & 21 & 27 & 34 & 57 \end{bmatrix} \tag{S}$$

Recall that a necessary and sufficient stability condition "all trajectories converge to 0 as $t \to \infty$" for a system of the form (S) is the existence of a *Lyapunov stability certificate* – a matrix $X \succ 0$ and $\gamma \in [0,1)$ satisfying the relation

$$\begin{bmatrix} \gamma^2 X & A^T X \\ XA & X \end{bmatrix} \succeq 0. \tag{42}$$

System (S) is stable; as the corresponding certificate, one can take

$$X = \bar{X} = \begin{bmatrix} 1954 & 199 & 170 & 136 & 35 & 191 \\ 199 & 1861 & -30 & -136 & 222 & 137 \\ 170 & -30 & 1656 & 17 & -370 & -35 \\ 136 & -136 & 17 & 1779 & 296 & 112 \\ 35 & 222 & -370 & 296 & 1416 & 25 \\ 191 & 137 & -35 & 112 & 25 & 2179 \end{bmatrix},$$

and $\gamma = \bar{\gamma} = 0.95$. The question we are interested in is: Assume that entries in A are subject to random perturbations

$$A_{ij} \mapsto A_{ij}(1 + \sigma\xi_{ij}), \tag{43}$$

where ξ_{ij} are independent random perturbations uniformly distributed on $[-1, 1]$. How large could be the level of perturbations σ in order for $(\bar{X}, \gamma = 0.9999)$ to remain the Lyapunov stability certificate for the perturbed matrix with probability at least $1 - \varepsilon$, with ε like 10^{-8} or 10^{-12}?

For fixed X and γ, (42) is a Linear Matrix Inequality in A, so that the question we have posed can be reformulated as the question of how large could be σ under the restriction that

$$\mathbf{P}(\sigma^{-1}Q) \leq \varepsilon, \tag{44}$$

where $\mathbf{P}$ is the distribution of random 6×6 matrix with independent entries uniformly distributed in $[-1, 1]$ and Q is the closed convex set[17]

$$Q = \left\{ \xi \in \mathbb{R}^{6\times 6} : \begin{bmatrix} & -[A \cdot \xi]^T \bar{X} \\ -\bar{X}[A \cdot \xi] & \end{bmatrix} \preceq \begin{bmatrix} \gamma^2 \bar{X} & A^T \bar{X} \\ \bar{X} A & \bar{X} \end{bmatrix} \right\}. \tag{45}$$

In order to answer this question, we use the (HT) test and act as follows.

(a) As the trial distribution $\mathbf{F}$, we use the zero mean normal distribution with covariance matrix $\frac{\pi}{8} I_{36}$ which, by Proposition 3, majorizes the uniform distribution $\mathbf{P}$.

At the first glance, the choice of normal distribution in the role of $\mathbf{F}$ seems strange – the actual distribution itself possesses Concentration property, so that it would be natural to choose $\bar{\mathbf{F}} = \mathbf{P}$. Unfortunately, function ψ for the uniform distribution (see Theorem 4 and Remark 2), although of the same type as its normal-distribution counterpart (see Theorem 1), leads to more conservative estimates because of worse constant factors; this explains our choice of the trial distribution.

[17] By $A \cdot B$ we denote the componentwise product of two matrices, i.e., $[A \cdot B]_{ij} = A_{ij} B_{ij}$. This is called Hadamard product by some authors. The notation "$\preceq$" stands for the standard partial order in the space $\mathbf{S}^m$ of symmetric $m \times m$ matrices: $A \succeq B$ ($A \succ B$) if and only if $A - B$ is positive semidefinite (positive definite). Thus, "$\succeq$" ("$\preceq$") stand for two different relations, namely majorization as defined in Definition 1, and the partial order induced by the semidefinite cone. What indeed " $\succeq$ " means, will be clear from the context.

(b) We run a "pilot" 1000-element simulation in order to get a rough safe guess $\bar{\sigma}$ of what is the level of perturbations in question. Specifically, we generate a 1000-element sample drawn from $\mathbf{F}$, for every element η of the sample compute the largest σ such that $\eta \in \sigma^{-1}Q$, and then take the minimum, over all elements of the sample, of the resulting quantities, thus obtaining the largest level of perturbations which is compatible with our sample. This level is slightly larger than 0.064, and we set $\bar{\sigma} = 0.064$.

(c) Finally, we run test (HT) itself. First, we specify the sample size N as 1000 and the acceptance level K as 0. Then we compute the largest θ satisfying (38) with reliability parameter $\delta = 10^{-6}$, that is, $\theta = \exp\{\frac{\ln(\delta)}{N}\} = 10^{-0.006} \approx 0.9863$. Second, we build 1000-element sample, drawn from $\mathbf{F}$, and check whether all elements η of the sample satisfy the inclusion $\bar{\sigma}\eta \in Q$, which indeed is the case. According to Theorem 8, the latter fact allows to claim, with reliability at least $1 - \delta$ (that is, with chances to make a wrong claim at most $\delta = 10^{-6}$), that for every $s > 1$ one has

$$\mathbf{P}(s^{-1}\bar{\sigma}\xi \notin Q) \le \mathrm{Err}(s, \theta) = \mathrm{Err}(s, 0.9863)$$

with $\mathrm{Err}(\cdot, \cdot)$ given by (35) (where ψ is as in Theorem 1). In other words, up to probability of bad sampling as small as 10^{-6}, we can be sure that for every $s > 1$, at the level of perturbations $s^{-1}\bar{\sigma} = 0.064s^{-1}$ the probability for $(\bar{X}, 0.9999)$ to remain Lyapunov stability certificate for the perturbed matrix is at least $1 - \mathrm{Err}(s, \theta)$. From the data in Table 1 we see that moderate reduction in

σ	0.0580	0.0456	0.0355	0.0290	0.0246	0.0228	0.0206
$p(\sigma) \le$	0.3560	0.0890	0.0331	0.0039	2.9e-4	6.9e-5	6.3e-6
σ	0.0188	0.0177	0.0168	0.0156	0.0148	0.0136	0.0128
$p(\sigma) \le$	4.6e-7	6.9e-9	9.4e-9	3.8e-10	4.0e-11	3.0e-13	5.9e-15

Table 1. $p(\sigma)$: probability of a perturbation (43) for which $(\bar{X}, 0.9999)$ fails to be a Lyapunov stability certificate.

level of perturbations ρ ensures dramatic decrease in the probability ε of "large deviations" (cf. (33)).

A natural question is how conservative are our bounds. The experiment says that as far as the levels of perturbations are concerned, the bounds are accurate up to moderate constant factor. Indeed, according to our table, perturbation level $\sigma = 0.0128$ corresponds to confidence as high as $1 - \varepsilon$ with $\varepsilon = 5.9\text{e-}15$; simulation demonstrates that 10 times larger perturbations result in confidence as low as $1 - \varepsilon$ with $\varepsilon = 1.6 \cdot 10^{-2}$.

3 The Synthesis problem

We now address the problem of optimizing under chance constraints

$$\operatorname*{Min}_{x\in X} \ c^T x \ \text{ subject to } \ \text{Prob}\{G_\sigma(x,\xi)\in C\} \geq 1-\varepsilon, \tag{46}$$

with $G_\sigma(x,\xi)$ defined in (4) and $\xi \sim \mathbf{P}$. We assume that $C \subset \mathbb{R}^m$ is a closed convex set and X is a compact convex set. As about the distribution $\mathbf{P}$ of perturbations, we assume in the sequel that it is symmetric. In this case, our chance constraint is essentially the same as the symmeterized constraint

$$\text{Prob}\{G_\sigma(x,\xi)\in C \text{ and } G_\sigma(x,-\xi)\in C\} \geq 1-\varepsilon.$$

Indeed, the validity of the symmeterized constraint implies the validity of the original one, and the validity of the original constraint, with ε replaced by $\varepsilon/2$, implies the validity of the symmeterized one. In our context of really small ε the difference between confidence $1-\varepsilon$ and confidence $1-\varepsilon/2$ plays no role, and by reasons to be explained later we prefer to switch from the original form of the chance constraint to its symmeterized form. Thus, from now on our problem of interest is

$$\operatorname*{Min}_{x\in X} \ c^T x \ \text{ subject to } \ \text{Prob}\{G_\sigma(x,\pm\xi)\in C\} \geq 1-\varepsilon. \tag{47}$$

We denote by $\text{Opt}(\sigma,\varepsilon)$ the optimal value of the above problem (47).

Finally, we assume that the corresponding "scenario counterparts" problems of the form

$$\operatorname*{Min}_{x\in X} \ c^T x \ \text{ subject to } \ G_\sigma(x,\pm\eta^j)\in C, \ j=1,...,N, \tag{48}$$

can be processed efficiently, which definitely is the case when the set C is computationally tractable (recall that the mappings $A_i(\cdot)$ are affine).

As it was mentioned in the Introduction section, we focus on the case when problem of interest (47), as it is, is too difficult for numerical processing. Our goal is to use scenario counterpart of (47) with randomly chosen scenarios η^j, $j=1,...,N$, in order to get a suboptimal solution $\widehat{x}$ to the problem of interest, in a way which ensures that:

• [*Reliability*] The resulting solution, if any, should be feasible for (47) with reliability at least $1-\delta$: the probability to generate a "bad" scenario sample – such that $\widehat{x}$ is well defined and is *not* feasible for (47) – should be $\leq \delta$ for a given $\delta \in (0,1)$;

• [*Polynomiality*] The sample size N should be "moderate" – polynomial in the sizes of the data describing (47) and in $\ln(\varepsilon^{-1})$, $\ln(\delta^{-1})$.

Under these *sine qua non* requirements, we are interested in tight scenario approximations. In our context, it is natural to quantify tightness as follows (cf. the definition of resolution):

> *A scenario-based approximation scheme satisfying 1) for given δ, ε, is tight within factor $\kappa=\kappa(\varepsilon,\delta)\geq 1$, if whenever (47) possesses a solution $\bar{x}$ which remains feasible after the uncertainty level is increased by factor κ, the scheme, with probability at least $1-\delta$, is productive ($\widehat{x}$ is well-defined) and ensures that $c^T\widehat{x} \leq c^T\bar{x}$.*

Informally speaking, κ-tight scenario approximation with probability at least $1-2\delta$ is "in-between" the problem of interest (47) and similar problem with κ times larger uncertainty level: up to probability of bad sampling $\leq 2\delta$, the scheme yields an approximate solution which is feasible for the problem of interest and results in the value of the objective not worse than $\mathrm{Opt}(\kappa\sigma, \varepsilon)$.

We are about to present several approximation schemes aimed ad achieving the outlined goals.

3.1 Naive approximation

The conceptually simplest way to build a scenario-base approximation scheme for (47) is to apply the Analysis test (T) as developed in Section 2, with setup as stated in Section 2.4. It is convenient to make two conventions as follows:

– from now on, we allow for the pre-trial distribution $\bar{\mathbf{F}}$ to possess the symmetric concentration property. By Remark 3, this extension of the family of trial distributions we can use[18] keeps intact the conclusion of Theorem 6, provided that the Analysis test is applied to a closed convex and symmetric sets B, which will always be the case in the sequel.

– the parameters N, K of the test are those given by (38), that is, $K = 0$ and $N = \mathrm{ceil}\left[\frac{\ln(\delta)}{\ln(\theta)}\right]$.

Observe that setup of (T) – the pre-trial distribution $\bar{\mathbf{F}}$ and the quantities θ, s, N as defined in Section 2.4 – depends solely on the distribution $\mathbf{P}$ of perturbations and required reliability and confidence parameters δ, ε and is completely independent of the (symmetric) convex set B the test is applied to. It follows, in particular, that a single setup fits all sets from the family

$$B_{x,\sigma} := \left\{\xi \in \mathbb{R}^d : G_\sigma(x, \pm\xi) \in C\right\}, \quad x \in X,\ \sigma > 0.$$

Note that all sets from this family are convex, closed and symmetric.

A straightforward approximation scheme for (47) based on the Analysis test as applied to the sets $B_{x,\sigma}$ would be as follows.

Naive approximation scheme: *With setup parameters $\bar{\mathbf{F}}, \theta, s, N$ as described above, we build a sample $\{\eta^j\}_{j=1}^N$ from distribution $\mathbf{F} = \bar{\mathbf{F}}^{(s)}$ and approximate problem (47) by its scenario counterpart*

$$\operatorname*{Min}_{x \in X} c^T x \ \text{ subject to } G_\sigma(x, \pm\eta^j) \in C,\ j = 1, ..., N. \tag{49}$$

If problem (49) is feasible and therefore solvable (X was assumed to be compact), we take, as $\widehat{x}$, an optimal solution to the problem, otherwise $\widehat{x}$ is undefined (the sample is non-productive).

By Theorem 6 and Remark 3, every *fixed in advance* point $\bar{x}$ which happens to be feasible for (49), with reliability at least $1-\delta$ is feasible for (47).

[18] The desire to allow for this extension is the reason for requiring the symmetry of $\mathbf{P}$ and passing to the symmeterized form of the chance constraint.

Moreover, in view of Theorem 7 and subsequent discussion, our approximation scheme is tight within factor s, provided that $\bar{\mathbf{F}} = \mathbf{P}$ and

$$2\varepsilon \ln(1/\delta) \le \delta \ln(1/\theta). \tag{50}$$

Unfortunately, these good news about the naive scheme cannot overweight is crucial drawback: *we have no reasons to believe that the scheme satisfies the crucial for us the Reliability requirement*. Indeed, the resulting approximate solution $\widehat{x}$ depends on the sample, which makes Theorem 6 inapplicable to $\widehat{x}$.

The outlined severe drawback of the naive approximation scheme is not just a theoretical possibility. Indeed, assume that $X := \{x \in \mathbb{R}^d : \|x\| \le 100d^{1/2}\}$, vector c in (47) has unit length and the chance constraint in question is $\mathrm{Prob}\{-1 \le \xi^T x \le 1\} \ge 1-\varepsilon$, where $\xi \sim \mathbf{P} = N(0, I_d)$. Note that all our constructions and bounds are not explicitly affected by the dimension of ξ or by the size of X. In particular, when applied to the normal distribution $\mathbf{P} = \bar{\mathbf{F}}$ and given ε and δ, they yield sample size N which is independent of $d = \dim \xi$. For large d, therefore, we will get $2N < d$. In this situation, as it is immediately seen, with probability approaching 1 as $d \to \infty$ there will exist a unit vector x (depending on sample $\{\eta^j\}$) orthogonal to all elements of the sample and such that $e^T x \le -0.1d^{-1/2}$. For such an x, the vector $100d^{1/2}x$ will clearly be feasible for (49), whence the optimal value in this problem is ≤ -10. But then every optimal solution to (49), in particular, $\widehat{x}$, is of norm at least 10. Thus, the typical values of $\xi^T \widehat{x} \sim N(0, \|\widehat{x}\|^2)$ are significantly larger than 1, and $\widehat{x}$, with probability approaching 1 as d grows, will be very far from satisfying the chance constraint...

There is an easy way to cure, to some extent, the naive scheme. Specifically, when $\widehat{x}$ is well defined, we generate a new N-element sample from the trial distribution and subject $\widehat{x}$ to our Analysis test. In the case of acceptance conclusion, we treat $\widehat{x}$ as the approximate solution to (47) yielded by the modified approximation scheme, otherwise no approximate solution is yielded. This modification makes the naive scheme $(1-\delta)$-reliable, however, at the price of losing tightness. Specifically, let $\bar{\mathbf{F}} = \mathbf{P}$ and (50) hold true. In this case, as we have seen, the naive scheme is tight within factor s, while there are no reasons for the modified scheme to share this property.

Numerical illustration.

To illustrate the modified naive scheme, consider dynamical system (S) from Section 2.5 and pose the following question: What is the largest level of perturbations $\bar{\sigma}$ for which all, up to probability $\varepsilon << 1$, perturbations of A admit a common Lyapunov stability certificate (X, γ) with $\gamma = 0.9999$ and the condition number of X not exceeding 10^5? Mathematically speaking, we are interested to solve the optimization problem

$$\begin{aligned}&\operatorname*{Max}_{\sigma,X}\ \sigma\ \text{ subject to } I\preceq X\preceq\alpha I\ \text{ and}\\&\qquad\text{Prob}\left\{\xi:\pm\sigma\begin{bmatrix}&(A\cdot\xi)^TX\\X(A\cdot\xi)&\end{bmatrix}\preceq\begin{bmatrix}\gamma^2X&A^TX\\XA&X\end{bmatrix}\right\}\geq 1-\varepsilon,\end{aligned}\tag{51}$$

where $\gamma = 0.9999$, $\alpha = 10^5$ and ξ is a 6×6 random matrix with independent entries uniformly distributed in $[-1,1]$, and $A\cdot\xi$ denotes the Hadamard (i.e., componentwise) product of matrices A and ξ.

Note that this problem is not exactly in the form of (47) – in the latter setting, the level of perturbations σ is fixed, and in (51) it becomes the variable to be optimized. Of course, we could apply bisection in σ in order to reduce (51) to a small series of feasibility problems of the form (47), but on a closest inspection these troubles are completely redundant. Indeed, when applying our methodology to the feasibility problem with a given σ, we were supposed to draw a sample of perturbations $\{s\eta^j\}_{j=1}^N$, with η^j being drawn from pre-trial distribution $\bar{\mathbf{F}}$, with amplification s determined by θ, σ and ε, and then check whether the resulting scenario counterpart of our feasibility problem, that is, the program

$$\begin{aligned}&\text{Find X such that } I\preceq X\preceq\alpha I\ \text{ and}\\&\qquad\pm s\sigma\begin{bmatrix}&(A\cdot\eta^j)^TX\\X(A\cdot\eta^j)&\end{bmatrix}\preceq\begin{bmatrix}\gamma^2X&A^TX\\XA&X\end{bmatrix},\ j=1,...,N,\end{aligned}\tag{52}$$

is or is not feasible. But the answer to this question, given $\{\eta^j\}$, depends solely on the product of $s\sigma$, so that in fact the outlined bisection is equivalent to solving a single problem

$$\begin{aligned}&\operatorname*{Min}_{\sigma,X}\ \sigma\ \text{subject to } I\preceq X\preceq\alpha I\ \text{ and}\\&\qquad\pm s\sigma\begin{bmatrix}&(A\cdot\eta^j)^TX\\X(A\cdot\eta^j)&\end{bmatrix}\preceq\begin{bmatrix}\gamma^2X&A^TX\\XA&X\end{bmatrix},\ j=1,...,N,\end{aligned}\tag{53}$$

with η^j drawn from the pre-trial distribution. The latter problem is quasiconvex and therefore can be efficiently solved. After its solution σ_*, X_* is found, we can apply Analysis test to check whether indeed $(X_*,\gamma = 0.9999)$ remains, with probability at least $1-\varepsilon$, a Lyapunov stability certificate for random perturbations of A at the perturbation level σ_*.

In our experiment, we followed the outlined approach, with the only difference that at the concluding step we used the homogenized Analysis test rather than the basic one. Specifically, we acted as follows:

(a) As in Section 2.5, we chose $N(0,\frac{\pi}{8}I_{36})$ as our pre-trial distribution $\bar{\mathbf{F}}$ and set the sample size N to 1000, which is the size given by (38) for $\delta = 10^{-6}$ and $\theta = 0.9863$.

(b) We drew $N = 1000$-element sample from $\bar{\mathbf{F}}$ and solved resulting problem (53), thus getting $\sigma_*\approx 0.0909$ and certain X_*.

(c) Our concluding step was to bound from below, for small values of ε, the perturbation levels for which $(X = X_*,\gamma = 0.9999)$ is, with probability

$\geq 1-\varepsilon$, a stability certificate for a perturbation of A. This task is completely similar to the one considered in Section 2.5, and we acted exactly as explained there. The numerical results are presented in Table 2. Comparing the data in Tables 1 and 2, we see that optimization in X results, for every value of ε, in "safe" perturbation levels twice as large as those before optimization. To feel the difference, note that at the perturbation level 0.0290 Table 1 guarantees preserving (certificate for) stability with confidence as poor as $1-0.0039$; Table 2 states that even at bit larger perturbation level 0.0297, stability is preserved with confidence as high as $1-4\cdot 10^{-11}$, reliability of both claims being at least 0.999999.

σ	0.116	0.0912	0.0709	0.0580	0.0491	0.0456	0.0412
$p(\sigma)\leq$	0.3560	0.0890	0.0331	0.0039	2.9e-4	6.9e-5	6.3e-6
σ	0.0412	0.0375	0.0355	0.0336	0.0297	0.0272	0.0255
$p(\sigma)\leq$	4.6e-7	6.9e-9	9.4e-9	3.8e-10	4.0e-11	3.0e-13	5.9e-15

Table 2. $p(\sigma)$: probability of a perturbation (43) for which $(X_*, 0.9999)$ fails to be a Lyapunov stability certificate.

3.2 Iterative approximation

As we have seen, the naive approximation scheme has severe drawbacks: without modification, the scheme possesses certain tightness properties, but can be unreliable; modification recovers reliability, but "kills" tightness. We are about to present an *iterative* approximation scheme which is reliable *and* has reasonable tightness properties. In the sequel, we sketch the scheme, skipping straightforward and boring details.

Preliminaries: polynomial time black-box convex optimization.

Consider a situation as follows. We are given:

(a) a convex compact set $X\subset\mathbb{R}^n$ with nonempty interior, which is contained in the centered at the origin Euclidean ball of a known radius R and is equipped with *Separation Oracle* $\mathcal{S}_Q$ – a routine which, given an input point $x\in\mathbb{R}^n$, reports whether $x\in X$, and if it is not the case, returns a *separator* – a linear inequality which is satisfied everywhere on X and is violated at x,

(b) a linear objective c^Tx to be minimized,

(c) access to a "wizard" working as follows. The wizard has in its disposal a once for ever fixed set $\mathcal{L}$ of linear inequalities with n variables; when invoked, it picks an inequality from this set and returns it to us. For the time being, we make absolutely no assumptions on how this inequality is chosen: wizard's choice can be randomized, can depend on past choices, etc.,

(d) positive parameters r ("feasibility margin") and ω (desired accuracy). In Convex Programming, there are methods (e.g., the Ellipsoid algorithm) capable to optimize (what precisely, it will become clear in a moment) in the outlined environment, specifically, as follows. The method generates, one after another, a predetermined number M of *search points* $x_t \in \mathbb{R}^n$, $t = 1, ..., M$. At step $t \geq 1$, the method already has in its disposal point x_{t-1} ($x_0 = 0$) and builds a vector e_t and the next search point x_t, namely, as follows:

• [generating e_t] We call the Separation oracle, x_{t-1} being the input. If the oracle reports that $x_{t-1} \notin X$, we call x_{t-1} *non-productive* and specify e_t as the gradient of the separator returned by the oracle. If $x_{t-1} \in X$, we make a predetermined number N of calls to the wizard and add the N linear inequalities returned by the wizard at step t to the collection of inequalities returned at the previous steps, thus getting a list of Nt linear inequalities. We then check whether x_{t-1} satisfies all these Nt inequalities. If there is an inequality in the list which is violated at x_{t-1}, we qualify x_{t-1} as non-productive and specify e_t as the gradient of the violated inequality. Finally, if $x_{t-1} \in X$ satisfies all inequalities returned so far by the wizard, we qualify x_{t-1} as productive and set $e_t = c$.

• [generating x_t] Given x_{t-1}, e_t and information coming from the previous steps (for the Ellipsoid method, the latter is summarized in a single $n \times n$ matrix B_{t-1}), we build x_t. How x_t is built, it depends on the method in question; the only issue which matters in our context is that the arithmetic cost of generating x_t should be polynomial in n (for the Ellipsoid method, the cost of building x_t *and* updating $B_{t-1} \mapsto B_t$ is just $O(1)n^2$ operations).

After all M search points are built, we treat the best (with the smallest value of $c^T x$) of the *productive* search points as the resulting approximate solution $\widehat{x}$; if no productive search points were generated, the result is undefined.

Now, upon termination, we have in our disposal a list $\mathcal{I}$ of NM linear inequalities $\ell(x) \leq 0$ which came from the wizard; these inequalities define the convex compact set

$$X^{\mathcal{I}} = \{x \in X : \ell(x) \leq 0,\ \ell \in \mathcal{I}\}.$$

The convex optimization algorithms we are speaking about ensure the following property

(P): *With properly chosen and polynomial in* n *and* $\ln\left(\frac{nR}{r} \cdot \frac{R\|c\|}{\omega}\right)$ *number of steps* $M = M(n, R, r, \omega)$ *(for the Ellipsoid method,* $M = 2n^2 \ln\left(\frac{nR^2\|c\|}{r\omega} + 2\right)$*), the following is true: whenever the set* $X^{\mathcal{I}}$ *contains Euclidean ball of radius* r, $\widehat{x}$ *is well defined and*

$$c^T \widehat{x} \leq \min_{x \in X^{\mathcal{I}}} c^T x + \omega. \tag{54}$$

Now we can finally explain what is the optimization problem we were solving: this is the problem $\min_{x\in X^{\mathcal{I}}} c^T x$ *defined in course of the solution process*[19].

Iterative approximation scheme.

We are ready to present an *iterative approximation scheme* for solving (47). Assume that the domain X of (47) is contained in the centered at the origin ball of known radius R and that both X and C are equipped with Separation Oracles. Given required confidence and reliability parameters ε, δ, let us choose trial distribution $\mathbf{F}$, θ, s and sample size N exactly in the same fashion as for naive scheme. Besides this, let us choose an optimization algorithm possessing property (P); for the sake of definiteness, let it be the Ellipsoid method. Finally, let us choose a small positive r and specify the number M of steps of the method according to (P), that is,

$$M = O(1)n^2 \ln\left(\frac{nR}{r} \cdot \frac{R\|c\|}{\omega} + 2\right),$$

where ω is the accuracy within which we want to solve (47). Now let us run the Ellipsoid method, mimicking the wizard as follows:

> The linear inequalities returned by the wizard at step t are uniquely defined by the search point x_{t-1} and a realization η^τ of a random vector $\eta \sim \mathbf{F}$; here τ counts the calls to the wizard, *and* $\eta^1, \eta^2, \ldots$ *are independent of each other*. Given x_{t-1} and η^τ, the wizard computes the points $y_\pm = G_\sigma(x_{t-1}, \pm\eta^\tau)$ and calls the Separation Oracle for C to check whether both these points belong to C. If it is the case, the wizard returns a trivial – identically true – inequality $\ell(x) \equiv 0^T x \leq 0$. If at least one of the points, say, y_+, does *not* belong to C, the wizard acts as follows. Let $e(u) \leq 0$ be the linear inequality returned by the Separation oracle; this inequality holds true for $u \in C$ and is violated at y_+. The wizard converts this inequality into the linear inequality
>
> $$\ell(x) \equiv e\left(A_0(x) + \sigma \textstyle\sum_{i=1}^d \eta_i^\tau A_i(x)\right) \leq 0.$$
>
> Since $A_i(\cdot)$ are affine, this indeed is a linear inequality in variables x, and since $e(y_+) > 0$, this inequality is violated at x_{t-1}.

---

[19] In standard applications, situation, of course, is not that strange: the problem we are solving is known in advance and is $\min_x \left\{c^T x : x \in X, g(x) \leq 0\right\}$, where g is a convex function. The set $\mathcal{L}$ of linear inequalities is comprised of inequalities of the form $\ell_y(x) \equiv g(y) + (x-y)^T g'(y) \leq 0$, $y \in \mathbb{R}^n$, and the inequality returned by the wizard invoked at point x_{t-1} is $\ell_{x_{t-1}}(x) \leq 0$. In this case, $\widehat{x}$, if defined, is a feasible solution to the problem of interest, and if the feasible set of the latter problem contains a ball of radius r, then $\widehat{x}$ is well-defined and is an ω-optimal solution to the problem of interest, provided that the number of steps M is as in (P).

We have specified the wizard and thus – a (randomized) optimization process; a realization of this process and the corresponding result $\widehat{x}$, if any, are uniquely defined by a realization of MN-element sample with independent elements drawn from the trial distribution. The resulting approximation scheme for (47) is successful if and only if $\widehat{x}$ is well defined, and in this case $\widehat{x}$ is the resulting approximate solution to (47).

Let us investigate the properties of our new approximation scheme. Our first observation is that the scheme is reliable.

Theorem 9. *The reliability of the iterative approximation is at least* $1 - M\delta$*, that is, the probability to generate a sample such that* $\widehat{x}$ *is well defined and is* not *feasible for* (47) *is at most* $M\delta$*.*

Proof. If $\widehat{x}$ is well defined, it is one of the productive points x_{t-1}, $1 \leq t \leq M$. Observe that for a given t the probability of "bad sampling" at step t, that is, probability of the event E_t that x_{t-1} is declared productive and at the same time $\mathbf{F}(\{\eta : G_\sigma(x_{t-1}, \pm\xi) \in C\}) < \theta$, is at most δ. Indeed, by wizard's construction, the conditional, given what happened before step t, probability of this event is at most the probability to get N successes in N independent Bernoulli experiments "check whether $G_\sigma(x_{t-1}, \pm\zeta^p) \in C$" with $\zeta^p \sim \mathbf{F}$, $p = 1, ..., N$, with probability of success in a single experiment is $< \theta$; by (38), this probability is at most δ. Since the conditional, given the past, probability of E_t is $\leq \delta$, so is the unconditional probability of E_t, whence the probability of the event $E = E_1 \cup ... \cup E_M$ is at most $M\delta$. If the event E does not take place and $\widehat{x}$ is well-defined, then $\widehat{x}$ satisfies the requirement $\mathbf{F}(\{\eta : G_\sigma(\widehat{x}, \pm\eta) \in C\}) \geq \theta$, whence, by properties of our analysis test, $\mathbf{P}(\{\xi : G_\sigma(\widehat{x}, \pm\xi) \in C\}) \geq 1 - \varepsilon$. By construction, $\widehat{x}$, if well defined, belongs to X. Thus, $\widehat{x}$ indeed is feasible for (47) "modulo event E of probability $\leq M\delta$". ∎

Our next observation is that *when* $\mathbf{P} = \bar{\mathbf{F}}$*, the iterative scheme is nearly tight up to factor* s. The precise statement is as follows.

Theorem 10. *Let* $\bar{\mathbf{F}} = \mathbf{P}$*, and let there exist an Euclidean ball* $U_r \subset X$ *of radius* nr *such that all points* $x \in U_r$ *are feasible for* (47)*, the uncertainty level being increased by factor* s*:*

$$\mathbf{P}(\{\xi : G_{s\sigma}(x, \pm\xi) \in C\}) \geq 1 - \varepsilon, \quad \text{for all } x \in U_r. \tag{55}$$

Then, with reliability at least $1 - (n+2)MN\varepsilon$*,* $\widehat{x}$ *is well defined and satisfies the relation*

$$c^T \widehat{x} \leq \mathrm{Opt}(s\sigma, \varepsilon) + \omega, \tag{56}$$

where s *is the amplification parameter of the scheme. In other words, the probability to generate a sample* $\eta^1, ..., \eta^{MN}$ *such that* $\widehat{x}$ *is undefined or is well defined but fails to satisfy* (56) *is at most* $(n+2)MN\varepsilon$*.*

Proof. Let $\kappa > 0$, and let $\bar{x}_\kappa \in X$ be such that

$$c^T\bar{x}_\kappa \le \mathrm{Opt}(s\sigma,\varepsilon)+\kappa \text{ and } \mathbf{P}(\{\xi : G_{s\sigma}(\bar{x}_\kappa, \pm\xi)\in C\}) \ge 1-\varepsilon. \qquad (57)$$

Now, let Δ be a perfect simplex with vertices $z_0, ..., z_n$ on the boundary of U_r; since the radius of U_r is nr, Δ contains a ball V of radius r. We now claim that up to probability of bad sampling $p = (n+2)MN\varepsilon$, the $n+2$ points $z_0, ..., z_n, \bar{x}_\kappa$ belong to $X^{\mathcal{I}}$. Indeed, let $z \in X$ be a fixed point satisfying the chance constraint

$$\mathbf{P}(\{\xi : G_{s\sigma}(z, \pm\xi)\in C\}) \ge 1-\varepsilon$$

(as it is the case for $z_0, ..., z_n, \bar{x}_\kappa$). Due to $z \in X$ and the construction of our wizard, the event $z \notin X^{\mathcal{I}}$ takes place if and only if the underlying sample $\eta^1, ..., \eta^{MN}$ of MN independent realizations of random vector $\eta \sim \mathbf{F} = \mathbf{P}^{(s)}$ contains an element η^t such that either $e\left(G_\sigma(z,\eta^t)\right) > 0$ or $e\left(G_\sigma(z,-\eta^t)\right) > 0$, or both, where e is an affine function (depending on the sample) such that $e(y) \le 0$ for all $y \in C$ Thus, at least one of the two points $G_\sigma(z, \pm\eta^t)$ fails to belong to C. It follows that the event $z \notin X^{\mathcal{I}}$ is contained in the union, over $t = 1, ..., MN$, of the *complements* to the events $F_t = \{\eta : G_\sigma(z, \pm\eta^t) \in C\}$. Due to $\mathbf{F} = \mathbf{P}^{(s)}$, the $\mathbf{F}$-probability of F_t is nothing but the $\mathbf{P}$-probability of the event $\{\xi : G_\sigma(z, \pm\eta^t) \in C\}$, that is, $\mathbf{F}(F_t) \ge 1-\varepsilon$. It follows that the probability of the event $z \notin X^{\mathcal{I}}$ is at most $MN\varepsilon$. Applying this result to every one of the points $z_0, ..., z_n, \bar{x}_\kappa$, we conclude that the probability for at least one of these points to be outside of $X^{\mathcal{I}}$ is at most $(n+2)MN\varepsilon$, as claimed.

We are nearly done. Indeed, let E be the event

$$\{\eta^1, ..., \eta^{MN} : z_0, ..., z_n, \bar{x}_\kappa \in X^{\mathcal{I}}\}.$$

As we just have seen, the probability of this event is at least $1-(n+2)MN\varepsilon$. Since $X^{\mathcal{I}}$ is convex, in the case of E the set $X^{\mathcal{I}}$ contains the entire simplex Δ with the vertices $z_0, ..., z_n$ and thus contains the ball V_r of radius r. Invoking (P), we see that in this case $\widehat{x}$ is well defined and

$$c^T\widehat{x} \le \omega + \min_{x\in X^{\mathcal{I}}} c^T x \le \omega + c^T\bar{x}_\kappa \le \omega+\kappa+\mathrm{Opt}(s\sigma,\varepsilon),$$

where the second inequality is given by the fact that in the case of E we have $\bar{x}_\kappa \in X^{\mathcal{I}}$. Thus, the probability of the event "$\widehat{x}$ is well defined and satisfies $c^T\widehat{x} \le \mathrm{Opt}(s\sigma,\varepsilon)+\omega+\kappa$" is at least the one of E, that is, it is $\ge 1-(n+2)MN\varepsilon$. Since $\kappa > 0$ is arbitrary, (56) follows. ■

Discussion.

With the Ellipsoid method as the working horse, the number M of steps in the iterative approximation scheme is about $2n^2\ln\left(\frac{nR^2\|c\|}{r\omega}\right)$. It follows that the *un*reliability level guaranteed by Theorem 9 does not exceed $2n^2\ln\left(\frac{nR^2\|c\|}{r\omega}\right)\delta$; in order to make this unreliability at most a given $\chi << 1$, it suffices to take $\delta = \frac{1}{2}\chi n^{-2}\ln^{-1}\left(\frac{nR^2\|c\|}{r\omega}\right)$. Since relation (38) requires "per step" sample size

$N = \text{ceil}\left[\frac{\ln(\delta)}{\ln(\theta)}\right]$, with our δ the total sample size MN is polynomial in $\frac{n}{1-\theta}$ and in *logarithms* of all remaining parameters (R, r, ω, χ). Thus, our approximation scheme is polynomial, which are good news. Further, with the outlined setup the *un*reliability level $\bar{\chi} = (n+2)MN\varepsilon$ indicated in Theorem 10 is linear in ε and polynomial in $\frac{n}{1-\theta}$ and logarithms of the remaining parameters, which again are good news. A not so good news is that the scheme requires an "ad hoc" choice of r. This, however, seems not that disastrous, since the only element of the construction which is affected by this choice (and affected just logaritmically) is the number of steps M. In reality, we can choose M as large as is allowed by side considerations like restrictions on execution time, thus making r as small as possible under these restrictions (or, equivalently, arriving at approximation as tight as possible, since the less is r, the more likely becomes the premise in Theorem 10).

As far as practicality of the iterative approximation scheme is concerned, the factor of primary importance is the design dimension n, since the reliability characteristics and the computational complexity of the scheme are much more sensitive to n than to parameters like $R, r, \omega, ...$ Let us look at this phenomenon in more details. With $(nR/r) \cdot (R\|c\|/\omega)$ bounded from above by 10^{12} (which seems to be sufficient for real life applications), we have $M = 55n^2$. Bounding the total number of scenarios MN by 10^6 and setting the reliability parameter χ to 10^{-6}, we get $N = 10^6 M^{-1} = 1.82 \cdot 10^4 \cdot n^{-2}$ and $\delta = M^{-1}\chi = 1.82 \cdot 10^{-8} \cdot n^{-2}$. Via (38), the resulting N and δ correspond to

$$\theta = \theta(n) := \exp\{-\ln(1/\delta)/N\} = \exp\left\{-n^2 \frac{17.8 - 2\ln(n)}{1.82 \cdot 10^4}\right\}.$$

Let the pre-trial distribution be normal. Then $\theta(n)$ should be $> \bar{\theta} = 0.5$, which is the case for $n \leq 34$ only. For $n \leq 34$ and $\theta = \theta(n)$, the associated confidence parameter $\varepsilon = \text{Err}(s, \theta(n))$ depends solely on the amplification parameter s; the tradeoff between s and ε is presented in Table 3. As we see, the required amplification level rapidly grows (i.e., tightness rapidly deteriorates) as n grows. This is exactly what should be expected, given that the per step number of scenarios N under our assumptions is inverse proportional to n^2. The influence of n can be moderated by replacing our working horse, the Ellipsoid method, with more advanced convex optimization algorithms; this issue, however, goes beyond the scope of this paper.

3.3 The case of chance semidefinite constraint

In this section, we focus on the case of "relatively simple" geometry of C, specifically, assume that C can be represented as the intersection of the cone $\mathbf{S}^m_+$ of positive semidefinite symmetric $m \times m$ matrices and an affine plane, or, equivalently, that the randomly perturbed constraint in question is Linear Matrix Inequality (LMI)

n	2	6	10	14	18	22	26	30
$\theta(n)$	0.9964	0.9723	0.9301	0.8738	0.8074	0.7432	0.6576	0.5805
N	4450	506	182	93	57	38	27	21
ε				s				
1.0e-3	1.17	1.88	2.54	3.39	4.63	6.68	10.80	23.07
1.0e-4	1.50	2.19	2.93	3.88	5.25	7.51	12.02	25.38
1.0e-5	1.70	2.46	3.27	4.31	5.80	8.26	13.14	27.49
1.0e-6	1.88	2.71	3.58	4.70	6.31	8.95	14.16	29.45
1.0e-7	2.05	2.93	3.86	5.06	6.78	9.58	15.12	31.30
1.0e-8	2.20	3.14	4.13	5.40	7.21	10.18	16.02	33.03
1.0e-9	2.34	3.33	4.38	5.71	7.63	10.75	16.87	34.67
1.0e-10	2.47	3.52	4.61	6.02	8.02	11.28	17.68	36.25
1.0e-11	2.60	3.69	4.84	6.30	8.39	11.79	18.45	37.76
1.0e-12	2.72	3.86	5.05	6.57	8.75	12.28	19.20	39.22
1.0e-13	2.83	4.02	5.26	6.84	9.09	12.75	19.91	40.63
1.0e-14	2.94	4.17	5.45	7.09	9.42	13.21	20.61	41.98

Table 3. Tradeoff between amplification s and confidence parameter ε for iterative approximation scheme (total sample size 10^6, normal trial distribution)

$$x \in X \text{ and } A_\xi(x) := \sum_{i=1}^{d} \xi_i A_i(x) \preceq A_0(x), \tag{58}$$

where $X \subset \mathbb{R}^n$ is the domain of our constraint (we assume the domain to be convex and compact), $A_i(x)$, $i = 0, ..., d$, are symmetric matrices affinely depending on $x \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}$ are random perturbations. Without loss of generality we have set the level of perturbations σ to 1, so that σ is not present in (58) at all. Note that the family of cross-sections of the semidefinite cone is very rich, which allows to reformulate in the form of (58) a wide spectrum of systems of convex constraints, e.g., (finite) systems of linear and conic quadratic inequalities. Besides this, LMI constraints arise naturally in many applications, especially in Control [2].

The question we address is as follows. Let $\mathbf{P}$ be the distribution of the perturbation $\xi = (\xi_1, ..., \xi_d)$, and let X_ε be the solution set of the chance constraint associated with (58):

$$X_\varepsilon = \{x \in X : \mathbf{P}(\{\xi : A_\xi(x) \succeq 0\}) \geq 1 - \varepsilon\}. \tag{59}$$

Now suppose that we choose somehow a symmetric pre-trial distribution $\bar{\mathbf{F}}$, draw an N-element sample $\eta[N] = \{\eta^j\}_{j=1}^N$ from the trial distribution $\mathbf{F} = \bar{\mathbf{F}}^{(s)}$ (s is the amplification level) and thus obtain the "scenario approximation" of X_ε – the set

$$X(\eta[N]) = \left\{x \in X : A_{\eta^j}(x) \succeq 0,\, j = 1, ..., N\right\}. \tag{60}$$

The question we are interested in is: *Under which circumstances the random scenario approximation* $X(\eta[N])$ *is, with reliability at least* $1-\delta$*, a subset of* X_ε*, that is,*

$$\text{Prob}\,\{\eta[N] : X(\eta[N]) \subset X_\varepsilon\} \geq 1-\delta. \tag{61}$$

Note the difference between this question and the one addressed in Section 2. The results of Section 2, when translated into our present situation, explain under which circumstances, *given in advance* a point x and having observed that $x \in X(\eta[N])$, we may be pretty sure that $x \in X_\varepsilon$. Now we require much more: having observed $\eta[N]$ (and thus $X(\eta[N])$), we want to be pretty sure that *all* points from $X(\eta[N])$ belong to X_ε. Note that in the latter case every point of $X(\eta[N])$, e.g., the one which minimizes a given objective $c^T x$ over $X(\eta[N])$, belongs to X_ε. In other words, in the case of (61), an approximation scheme where one minimizes $c^T x$ over $X(\eta[N])$ allows to find, with reliability $1-\delta$, *feasible* suboptimal solution to the problem $\min\limits_{x \in X_\varepsilon} c^T x$ of minimization under the chance constraint.

Preprocessing the situation.

For the moment, let us restrict ourselves to the case where $\mathbf{P} = \mathbf{P}_1 \times ... \times \mathbf{P}_d$, where $\mathbf{P}_i$, $i = 1, ..., d$, is the distribution of ξ_i assumed to be symmetric. Note that if $a_i > 0$ are deterministic scalars, we can replace the perturbations ξ_i with $a_i\xi_i$, and mappings $A_i(x)$ with the mappings $a_i^{-1}A_i$ without affecting the feasible set of the chance constraint. In other words, we lose nothing when assuming that "typical values" of ξ_i are at least of order of 1, specifically, that $\mathbf{P}_i(\{|\xi_i| \geq 1\}) \geq 0.2$, $i = 1, ..., d$. With this normalization, we immediately arrive at a rough *necessary* condition for the inclusion $x \in X_\varepsilon$, namely,

$$\pm A_i(x) \preceq A_0(x),\ i = 1, ..., d. \tag{62}$$

Indeed, let $x \in X_\varepsilon$ with $\varepsilon < 0.45$. Given $p \leq d$ and setting

$$A_\xi(x) := \xi_p A_p(x) + S_\xi^p(x),$$

observe that ξ_p and $S_\xi^p(x)$ are independent and symmetrically distributed, which combines with $x \in X_\varepsilon$ to imply that

$$\mathbf{P}(\{\xi : \xi_p A_p(x) \pm S_\xi^p(x) \preceq A_0(x)\}) \geq 1 - 2\varepsilon.$$

By our normalization and due to the symmetry of $\mathbf{P}_p$, we have that $\mathbf{P}(\{\xi : \xi_p \geq 1\}) \geq 0.1$. It follows that

$$\mathbf{P}(\{\xi : \xi_p \geq 1 \ \&\ \xi_p A_p(x) \pm S_\xi^p(x) \preceq A_0(x)\}) \geq 0.9 - 2\varepsilon > 0,$$

that is, the set $\{\xi : \xi_p \geq 1 \ \&\ \xi_p A_p(x) \pm S_\xi^p(x) \preceq A_0(x)\}$ is nonempty, which is possible only when $t_+ A_p(x) \preceq A_0(x)$ for certain $t_+ \geq 1$. Similar reasoning proves that $-t_- A_p(x) \preceq A_0(x)$ for certain $t_- \geq 1$; due to these observations, $\pm A_i(x) \preceq A_0(x)$.

Note that (62) is a nice deterministic convex constraint, and it makes sense to include it into the definition of X; with this modification of X, we have $A_0(x) \succeq 0$ everywhere on X (since (62) implies $A_0(x) \succeq 0$). In order to simplify our subsequent analysis, let us strengthen the latter inequality to $A_0(x) \succ 0$ (which can be ensured by slight shrinkage of X to a point $\bar{x}$ such that $A_0(\bar{x}) \succ 0$, provided that such a point exists). Thus, from now on we make the following assumption:

> **A.I.** *X is a closed and convex compact set such that relations* (62) *and $A_0(x) \succ 0$ take place everywhere on X.*

Now we formulate our assumptions on the actual and the pre-trial distributions. We discard temporary assumptions on $\mathbf{P}$ made at the beginning of this subsection (their only goal was to motivate **A.I**); what we actually need are similar in spirit assumptions on the pre-trial distribution. Here is what we assume from now on:

> **A.II.** *The actual distribution $\mathbf{P}$ is with zero mean and is majorized by symmetric pre-trial distribution $\bar{\mathbf{F}} \in \mathcal{C}(\bar{\theta}, \psi)$ with known $\bar{\theta}$, $\psi(\cdot, \cdot)$. In addition,*
> 1) *For certain $\widehat{\theta} \in (\bar{\theta}, 1)$ and all $\gamma \geq 1$ one has*
> $$\psi(\widehat{\theta}, \gamma) \geq a + b\gamma^2/2 \tag{63}$$
> *with $b > 0$;*
> 2) *For certain c, random vector $\eta \sim \bar{\mathbf{F}}$ satisfies the bound*
> $$\mathbb{E}\{\|\eta\|^2\} \leq c^2 d. \tag{64}$$

The result we are about to establish (for the case of normal distributions, it was announced in [8]) is as follows.

Theorem 11. *Let* **A.I-II** *hold true. Given confidence and reliability parameters $\varepsilon, \delta \in (0, 1/2)$, let us set, for $s > 1$,*

$$\mathrm{Err}(r) = \inf_{1 \leq \beta < s} \frac{1}{r - \beta} \int_{\beta}^{\infty} \exp\{-\psi(\widehat{\theta}, \gamma)\} d\gamma$$

(cf. (32)*) and specify the amplification parameter s in such a way that*

$$\mathrm{Err}(s) = \varepsilon;$$

note that

$$s \leq 2 + \sqrt{\tfrac{|a| + \ln(1/\varepsilon)}{b}} \tag{65}$$

in view of (33)*.*

Let, further, the sample size N be specified as

$$N = \mathrm{ceil}\left[\tfrac{\kappa}{1 - \widehat{\theta}} \left(\ln(\delta^{-1}) + \kappa m^2 d \ln(Csd)\right)\right], \tag{66}$$

with appropriately chosen absolute constant κ and constant C depending solely on $\widehat{\theta}, a, b, c$. Then, with sample $\eta[N]$ drawn from the trial distribution $\mathbf{F} = \bar{\mathbf{F}}^{(s)}$, one has

$$\text{Prob}\left\{X(\eta[N]) \subset X_\varepsilon\right\} \geq 1 - \delta. \tag{67}$$

For proof, see Appendix.

Note that when treating the parameters $\widehat{\theta}$, a, b, c involved into **A.I-II** as absolute constants (which is possible, e.g., for the pre-trial distributions given by Examples 1 – 3, see Section 2.2), the sample size N as given by (66) is polynomial in the sizes m, d of the problem and in $\ln(1/\delta)$, $\ln(\ln(1/\varepsilon))$.

Tightness of the approximation scheme suggested by Theorem 11 admits the following evident quantification.

Proposition 6. *Let, in addition to Assumptions* **A.I-II**, *the pre-trial distribution $\bar{\mathbf{F}}$ be identical to the actual distribution $\mathbf{P}$, and let x be a fixed in advance point of X which is feasible for the chance constraint with increased by factor s level of perturbations:*

$$\mathbf{P}\left(\{\xi : s\sum_{i=1}^d \xi_i A_i(x) \preceq A_0(x)\}\right) \geq 1 - \varepsilon, \tag{68}$$

where s is the amplification parameter specified in Theorem 11. Then $x \in X(\eta[N])$, the sample being drawn from the trial distribution as defined in Theorem 11, with probability at least $1 - N\varepsilon$, where N is given by (66)*. In particular, optimizing a given objective $c^T x$ over $X(\eta[N])$, we, with reliability at least $1 - \delta - N\varepsilon$, get a point $\widehat{x} \in X_\varepsilon$ with the value of the objective not exceeding*

$$\min_x \left\{c^T x : x \in X \text{ satisfies } (68)\right\}$$

Note that the amplification factor s specified in Theorem 11 is $O(1)\sqrt{\ln(1/\varepsilon)}$, provided that we treat a, b as absolute constants; thus, under the premise of Proposition 6 the tightness of our approximation scheme is nearly independent of ε.

Concluding remarks.

In this paper, our goal was to to get reliable *inner* approximations of the feasible set of optimization problem (47) with chance constraint; we have seen that in good cases (e.g., when the perturbations have normal or uniform distributions, and C is the semidefinite cone), the scenario approach allows to achieve this goal with polynomial in the sizes of the problem and logarithms of the reliability and confidence parameters number of scenarios and level of conservativeness as moderate as $O(1)\sqrt{\ln(1/\varepsilon)}$. A natural question is whether something similar can be done for *outer* approximation of the feasible set in question. The answer, in general, seems to be negative, as can be seen from the following example. Assume that the chance constraint is

$$\text{Prob}\left\{x^T\xi \le 1\right\} \ge 1 - \varepsilon,$$

where $\xi \sim N(0, I_n)$. The true feasible set X_ε of the chance constraint is the centered at the origin Euclidean ball E^ε of the radius $r = r(\varepsilon)$ given by $\frac{1}{\sqrt{2\pi}} \int\limits_r^\infty \exp\{-\gamma^2/2\}d\gamma = \varepsilon$, so that $r = (1 + o(1))\sqrt{2\ln(1/\varepsilon)}$ as $\varepsilon \to +0$. At the same time, the radius of the largest centered at the origin ball U contained in the feasible set $\{x : x^T\xi^j \le 1,\ j = 1, ..., N\}$ of the scenario counterpart, where ξ^j are drawn from $N(0, \sigma^2 I_n)$, is, with probability approaching 1 as $n \to \infty$, as small as $\sigma^{-1}n^{-1/2}$ (since typical values of $\|\xi^j\|$ are as large as $\sigma\sqrt{n}$). Thus, unless σ we use goes to 0 as $O(n^{-1/2})$ as n grows (which would make no much sense), the scenario approximation of X_ε with high probability is much "thinner" along certain *sample-depending* directions than X_ε itself.

Appendix: proof of Theorem 11

Recall that d is the dimension of the perturbation vectors, m is the row size of the matrices $A_i(x)$. From now on, $O(1)'$s stand for appropriate positive *absolute* constants, and C_i are positive quantities depending solely on the quantities $\widehat{\theta}$, a, b, c involved into Assumption **A.II**.

Lemma 3. *Let $\eta \sim \mathbf{F}$. Then for $\rho \ge 0$,*

$$\mathbf{F}\left(\left\{\eta : \|\eta\| > \rho s\sqrt{d}\right\}\right) \le 2\exp\left\{-C_1\rho^2\right\}. \tag{69}$$

Proof. By **A.II**.2) and Tchebyshev Inequality,

$$\mathbf{F}\left(\{\eta : \|\eta\| \le C_{1,1}s\sqrt{d}\}\right) \ge \widehat{\theta}$$

for appropriately chosen $C_{1,1}$. Due to the Concentration property and **A.II.**1), it follows that whenever $\gamma \ge 1$, we have

$$\mathbf{F}\left(\{\eta : \|\eta\| \ge C_{1,1}s\sqrt{d}\gamma\}\right) \ge \exp\{-a - b\gamma^2/2\},$$

and (69) follows. ∎

Our next technical result is as follows.

Lemma 4. *Let $\mathcal{A} = \{(A_1, ..., A_d) : A_i \in \mathbf{S}^m, -I \preceq A_i \preceq I\}$. For $A = (A_1, ..., A_d) \in \mathcal{A}$, let*

$$B(A) = \left\{u \in \mathbb{R}^d : 0.9\sum_{i=1}^d u_i A_i \preceq I\right\}.$$

Further, let $\eta \sim \mathbf{F}$, N be a positive integer, let η^j, $j = 1, ..., N$, be independent realizations of η, and let $\mathbf{F}_N$ be the distribution of $\eta[N] = \{\eta^j\}_{j=1}^N$. Finally, let $\Delta := \frac{1-\widehat{\theta}}{4}$ and

$$\Xi^N := \left\{\eta[N] : \forall \left(A \in \mathcal{A} : \mathbf{F}(B(A)) < \widehat{\theta}\right) \ \exists t \leq N : \textstyle\sum_{i=1}^d \eta_i^j A_i \not\preceq I\right\}. \quad (70)$$

Then

$$\mathbf{F}_N(\Xi^N) \geq 1 - \exp\{O(1)m^2 d \ln(C_2 sd) - O(1)(1-\widehat{\theta})N\} \quad (71)$$

with properly chosen C_2*.*

Proof. Let us equip the space of k-tuples of $m \times m$ symmetric matrices with the norm

$$\|(A_1, ..., A_d)\|_\infty = \max_i \|A_i\|,$$

where $\|A_i\|$ is the standard spectral norm of a symmetric matrix. Given $\omega > 0$, let $\mathcal{A}^\omega$ be a minimal ω-net in $\mathcal{A}$; by the standard reasons, we have

$$\mathrm{Card}(\mathcal{A}^\omega) \leq \exp\{O(1)m^2 d \ln(2 + \omega^{-1})\}. \quad (72)$$

Note that if $A, A' \in \mathcal{A}$, then

$$0.9 \sum_{i=1}^d \eta_i A_i \preceq 0.9 \sum_{i=1}^d \eta_i A_i' + 0.9\|\eta\|_1 \|A' - A\|_\infty I,$$

whence

$$\left\{\eta : 0.9 \sum_{i=1}^d \eta_i A_i \preceq I\right\} \supset \left(\{\eta : \sum_{i=1}^d \eta_i A_i' \preceq 1.1I\} \cap \{\eta : 0.9\|\eta\|_1 \|A' - A\|_\infty \leq 0.01\}\right),$$

so that

$$\begin{aligned} \mathbf{F}(B(A)) &\geq \underbrace{\mathbf{F}(\{\eta : \sum_{i=1}^d \eta_i A_i' \preceq 1.1I\})}_{\phi(A')} - \mathbf{F}(\{\eta : 0.9\|\eta\|_1 \|A' - A\|_\infty > 0.01\}) \\ &\geq \phi(A') - 2\exp\{-C_{2,1}\|A' - A\|_\infty^{-2} (ds)^{-2}\} \end{aligned} \quad (73)$$

for appropriately chosen $C_{2,1}$, where the concluding $\geq$ is given by (69) due to $\|\eta\|_1 \leq \sqrt{d}\|\eta\|$.

Now let

$$\mathcal{B}^\omega = \{A' \in \mathcal{A}^\omega : \mathbf{F}(\{\eta : \sum_{i=1}^d \eta_i A_i' \preceq 1.1I\}) \leq \widehat{\theta} + \Delta\}$$

where Δ is given by (70). According to (69), we can find $C_{2,2}$ such that

$$\mathbf{F}\left(\{\eta : \|\eta\| \geq C_{2,2} s\sqrt{d}\}\right) \leq \Delta,$$

so that $A' \in \mathcal{B}^\omega$ implies

$$\mathbf{F}\left(\{\eta : \sum_{i=1}^{d} \eta_i A'_i \preceq 1.1I \text{ or } \|\eta\|_1 > C_{2,2}sd\}\right) \le \widehat{\theta} + 2\Delta = \frac{1+\widehat{\theta}}{2} < 1.$$

Setting

$$\Xi_\omega^N[A'] = \left\{\eta[N] : \forall (j \le N) : \|\eta^j\| > C_{2,2}s\sqrt{d} \text{ or } \sum_{i=1}^{d} \eta_i^t A'_i \preceq 1.1I\right\},$$

we have by evident reasons

$$A' \in \mathcal{B}^\omega \Rightarrow \mathbf{F}_N(\Xi_\omega^N[A']) \le \exp\{-O(1)(1-\widehat{\theta})N\},$$

whence

$$\begin{aligned}\mathbf{F}_N\left\{\cup_{A' \in \mathcal{B}^\omega} \Xi_\omega^N[A']\right\} &\le \mathrm{Card}(\mathcal{A}^\omega)\exp\{-O(1)N\} \\ &\le \exp\{O(1)m^2 d\ln(2+\omega^{-1}) - O(1)(1-\widehat{\theta})N\}\end{aligned} \qquad (74)$$

(we have used (72)). Now let us set $\omega = C_{2,3}(sd)^{-1}$ with $C_{2,3}$ chosen in such a way that $C_{2,2}\omega sd < 0.1$ and (73) implies that

$$A, A' \in \mathcal{A}, \|A' - A\|_\infty \le \omega \Rightarrow \phi(A') \le \mathbf{F}(B(A)) + \Delta. \qquad (75)$$

Let E be the complement of the set $\cup_{A' \in \mathcal{B}^\omega} \Xi_\omega^N[A']$; due to (74) and to our choice of ω, we have

$$\mathbf{F}_N(E) \ge 1 - \exp\{O(1)m^2 d\ln(C_2 sd) - O(1)(1-\widehat{\theta})N\}. \qquad (76)$$

In view of this relation, in order to prove Lemma it suffices to verify that $E \subset \Xi^N$, that is,

$$\eta[N] \in E \Rightarrow \left[\forall \left(A \in \mathcal{A} : \mathbf{F}(B(A)) < \widehat{\theta}\right) \; \exists j \le N : \sum_{i=1}^{d} \eta_i^j A_i \not\preceq I\right]. \qquad (77)$$

Indeed, given $A \in \mathcal{A}$ such that $\mathbf{F}(B(A)) < \widehat{\theta}$, let A' be the $\|\cdot\|_\infty$-closest to A point from $\mathcal{A}^\omega$, so that $\|A - A'\|_\infty \le \omega$. By (75),

$$\phi(A') := \mathbf{F}\left(\left\{\eta : \sum_{i=1}^{d} \eta_i A_i \preceq 1.1I\right\}\right) \le \mathbf{F}(B(A)) + \Delta \le \widehat{\theta} + \Delta,$$

whence $A' \in \mathcal{B}^\omega$. It follows that whenever $\eta[N] \in \Xi^N$, there exists $j \le N$ such that

$$\|\eta^j\| \le C_{2,2}s\sqrt{d} \quad \text{and} \quad \sum_{i=1}^{d} \eta_i^j A'_i \not\preceq 1.1I.$$

Since

$$\sum_{i=1}^{d} \eta_i^j A_i' \preceq \sum_{i=1}^{d} \eta_i^j A_i + \underbrace{\|\eta^j\|_1 \|A - A'\|_\infty}_{\leq C_{2,2} s d\omega \leq 0.1} I \preceq \sum_{i=1}^{d} \eta_i^j A_i + 0.1 I,$$

it follows that $\sum_{i=1}^{d} \eta_i^j A_i \not\preceq I$, as claimed. ∎

We are ready to complete the proof of Theorem 11. Let Ξ^N be the set from Lemma 4. For $x \in X$, let

$$\begin{aligned} B_x &= \left\{ u : 0.9 \sum_{i=1}^{d} u_i A_i(x) \preceq A_0(x) \right\} = B(A_x), \\ A_x &= \left(A_0^{-1/2}(x) A_1(x) A_0^{-1/2}(x), ..., A_0^{-1/2}(x) A_d(x) A_0^{-1/2}(x) \right) \in \mathcal{A}, \end{aligned}$$

where the concluding inclusion is given by Assumption **A.I**. We claim that

$$\forall \left(\eta[N] \in \Xi^N, x \in X(\eta[N]) \right) : \mathbf{F}(B_x) \geq \widehat{\theta}. \tag{78}$$

Indeed, let $\eta[N] \in \Xi^N$ and $x \in X(\eta[N])$, so that $\eta^j \in B_x = B(A_x)$ for $j = 1, ..., N$. Assuming on contrary to (78), that $\mathbf{F}(B_x) < \widehat{\theta}$, or, which is the same due to $B_x = B(A_x)$, $\mathbf{F}(B(A_x)) < \widehat{\theta}$, we derive from (70) and the inclusion $\eta[N] \in \Xi^N$ that $\sum_{i=1}^{d} \eta_i^j (A_x)_i \not\preceq I$ for certain $t \leq N$; but then $\eta^j \notin B_x$, which is a contradiction.

Now let $\eta[N] \in \Xi^N$ and $x \in X(\eta[N])$. Setting $Q_x = s^{-1} B_x$, by (78) we have

$$\widehat{\theta} \geq \mathbf{F}(B_x) \equiv \bar{\mathbf{F}}^{(s)}(B_x) \equiv \bar{\mathbf{F}}(\{\zeta : s\zeta \in B_x\}) = \bar{\mathbf{F}}(Q_x),$$

whence $\mathbf{P}(\{\xi \notin sQ_x \equiv B_x\}) \leq \mathrm{Err}(s) = \varepsilon$ by Theorem 5. Recalling definition of B_x, we conclude that

$$\eta[N] \in \Xi^N \Rightarrow X(\eta[N]) \subset X_\varepsilon.$$

Invoking (71), we see that with N as given by (66), the probability of generating a sample $\eta[N]$ with $X(\eta[N]) \not\subset X_\varepsilon$ is $\leq \delta$, provided that C is a properly chosen function of a, b, c and κ is a properly chosen absolute constant. ∎

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