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ROBUST OPTIMIZATION AND DYNAMICAL DECISION MAKING

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- Data uncertainty in Optimization
- Robust Counterpart of uncertain optimization program
  <u>Example:</u> NETLIB LP Case Study
- Adding flexibility: Affinely Adjustable RC
  <u>Example:</u> Flexible Supplier-Retailer contracts
- Taking care of global sensitivities: Comprehensive RC
  <u>Example:</u> Stable control of serial inventory

DATA UNCERTAINTY IN OPTIMIZATION

♣ Consider a generic optimization problem of the form

$$\min_x \{f(x,\zeta) : F(x,\zeta) \leq 0\}$$

$x \in \mathbf{R}^n$: decision vector $\zeta \in \mathbf{R}^M$: data

♠ More often than not the data ζ is *uncertain* – not known exactly when problem is to be solved.

Sources of data uncertainty:

- part of the data is measured/estimated $\Rightarrow$ *estimation errors*

- part of the data (e.g., future demands/prices) does not exist when problem is solved $\Rightarrow$ *prediction errors*

- some components of a solution cannot be implemented exactly as computed $\Rightarrow$ *implementation errors* which in many models can be mimicked by appropriate data uncertainty

♠ With traditional modelling methodology,

• "small" data uncertainty is just ignored and the problem is solved for "nominal" values of the data $\Rightarrow$ *nominal* optimal solution.

Fact: *In many situations, small data perturbations can make the nominal optimal solution severely infeasible and/or "highly expensive" in terms of the objective, and thus practically meaningless.*

Example: NETLIB Case Study.

• *We substitute into LP problems from NETLIB Library their optimal solutions as found by CPLEX*

6.0 and then perturb at random "ugly coefficients" of inequality constraints, like -1.353783, by small margin in order to find out how the nominal solution can withstand data perturbations.

- *With 0.01% perturbations, in 19 of totally* ≈ 100 *NETLIB problems the nominal solution violated some of the perturbed constraints by 50% or more.*

♠ With traditional modelling methodology,

• "large" data uncertainty is modelled in a stochastic fashion and then processed via Stochastic Programming techniques

> Fact: *In many cases, it is difficult to specify reliably the distribution of uncertain data and/or to process the resulting Stochastic Programming program.*

♠ The ultimate goal of *Robust Optimization* is to take into account data uncertainty already at the modelling stage in order to "immunize" solutions against uncertainty.

• In contrast to Stochastic Programming, Robust Optimization does not assume stochastic nature of the uncertain data (although can utilize, to some extent, this nature, if any).

“NON-ADJUSTABLE” ROBUST OPTIMIZATION:
Robust Counterpart of Uncertain Problem

$$\boxed{\min_x \{f(x,\zeta) : F(x,\zeta) \leq 0\}} \qquad \text{(U)}$$

♣ The initial (“Non-Adjustable”) Robust Optimization paradigm (Soyster ’73, B-T&N ’97–, El Ghaoui et al. ’97–, Bertsimas et al. ’03–,...) is based on the following tacitly accepted assumptions:

A.1. All decision variables in (U) represent “here and now” decisions which should get specific numerical values as a result of solving the problem and *before* the actual data “reveals itself”.

A.2. The uncertain data are "unknown but bounded": one can specify an appropriate (typically, bounded) *uncertainty set* $\mathcal{U} \subset \mathbf{R}^M$ of possible values of the data. The decision maker is fully responsible for consequences of the decisions to be made when, and only when, the actual data is within this set.

A.3. The constraints in (U) are "hard" – we cannot tolerate violations of constraints, even small ones, when the data is in $\mathcal{U}$.

$$\boxed{\begin{array}{c}\min_x \{f(x,\zeta) : F(x,\zeta) \le 0\} \\ \zeta \in \mathcal{U}\end{array}} \qquad \text{(U)}$$

♠ Conclusions:

• The only meaningful candidate solutions are the *robust* ones – those which remain feasible whatever be a realization of the data from the uncertainty set:

$$x \text{ robust feasible } \Leftrightarrow F(x,\zeta) \le 0 \ \forall \zeta \in \mathcal{U}$$

• "Robust optimal" solution to be used is a robust solution with the smallest possible *guaranteed* value of the objective, that is, the optimal solution of the optimization problem

$$\min_{x,t}\{t : f(x,\zeta) \le t,\ F(x,\zeta) \le 0\ \forall \zeta \in \mathcal{U}\} \qquad \text{(RC)}$$

called the *Robust Counterpart* of (U).

$$\boxed{(\mathrm{U}): \qquad \min_x \{f(x,\zeta) : F(x,\zeta) \le 0\},\ \zeta \in \mathcal{U}}$$

$$\Downarrow$$

$$\boxed{(\mathrm{RC}): \qquad \min_{x,t} \{t : f(x,\zeta) \le t,\, F(x,\zeta) \le 0\ \ \forall \zeta \in \mathcal{U}\}}$$

Note: (RC) is a semi-infinite problem and as such can be difficult even when all instances of (U) are nice convex programs. However:

♣ There are generic cases (most notably, uncertain Linear Programming problems with computationally tractable uncertainty sets) when (RC) is computationally tractable.

♣ What can we gain? – *In our NETLIB Case Study, applying the Robust Counterpart methodology to "immunize" solutions against 0.1% data uncertainty, we always succeeded, and the price of robustness, in terms of the objective, was never greater than 1%.*

ADDING ADJUSTABILITY:
Affinely Adjustable Robust Counterpart

• "A.1. *All decision variables in uncertain problem represent "here and now" decisions"...*

♣ Assumption A.1 is *not* satisfied in many applications:

• In Dynamical Decision Making, *some of x_i can represent "wait and see" decisions* to be made when the uncertain data become (partially) known *and thus can be allowed to depend on (part of) the uncertain data.*

Example: In an inventory affected by uncertain demand, orders of day t can depend on actual demands in days $1, ..., t-1$.

• *Some of x_i can be "analysis variables" not representing decisions at all and thus can be allowed to depend on the uncertain data.*

Example: When converting a convex constraint $\sum_i |a_i^T x - b_i| \leq 1$ with uncertain data a_i, b_i into the Linear Programming form

$$-y_i \leq a_i^T x - b_i \leq y_i, \ \sum_i y_i \leq 1$$

the “certificates” y_i can be allowed to depend on the actual data.

$$\boxed{\min_x \{f(x,\zeta) : F(x,\zeta) \leq 0\}} \qquad \text{(U)}$$

♠ A natural way to relax Assumption A.1 is

— to allow for every decision variable x_j to depend on a prescribed portion of the uncertain data:

$$x_j = X_j(P_j\zeta)$$
$$\left[P_j : \text{ given matrices}\right]$$

— to seek for the *decision rules* $\{X_j(\cdot)\}$ which are *robust feasible* and optimize the *guaranteed* value of the objective. The resulting *Adjustable Robust Counterpart* of the uncertain problem is the *infinite-dimensional* optimization program

$$\min_{\{X_j(\cdot)\}_{j=1}^n, t} \left\{ t : \begin{array}{l} f(X_1(P_1\zeta), ..., X_n(P_n\zeta), \zeta) \le t \\ F(X_1(P_1\zeta), ..., X_n(P_n\zeta), \zeta) \le 0 \end{array} \right\} \forall \zeta \in \mathcal{U} \right\} \tag{ARC}$$

$$\min_x \{f(x,\zeta) : F(x,\zeta) \leq 0\},\ \zeta \in \mathcal{U} \tag{U}$$

$$\min_{x,t} \left\{ t : \begin{array}{l} f(x,\zeta) \leq t \\ F(x,\zeta) \leq 0 \end{array} \right\} \forall \zeta \in \mathcal{U} \right\} \tag{RC}$$

$$\min_{\{X_i(\cdot)\}_{i=1}^n, t} \left\{ t : \begin{array}{l} f(X_1(P_1\zeta), ..., X_n(P_n\zeta), \zeta) \leq t \\ F(X_1(P_1\zeta), ..., X_n(P_n\zeta), \zeta) \leq 0 \end{array} \right\} \forall \zeta \in \mathcal{U} \right\} \tag{ARC}$$

♠ (ARC) becomes (RC) in the trivial case when $P_i = 0$, $i = 1, ..., m$ and in the case of uncertain LP with *constraint-wise uncertainty*:

$$\min_x \left\{ c_{\zeta^0}^T x : a_{i,\zeta^i}^T x \leq b_{i,\zeta^i},\ i = 1, ..., m,\ Ax \leq b \right\}, \zeta^i \in \mathcal{U}_i,\ i = 0, ..., m \tag{U}$$

• all c_{ζ^0}, a_{i,ζ^i}, b_{i,ζ^i} are affine in ζ, • all $\mathcal{U}_i$ are convex compact sets,
• the set $\{x : Ax \leq b\}$ is bounded.

♠ In general, (ARC) is essentially less conservative than (RC)

♣ Major drawback of (ARC): severe computational intractability already in the case of uncertain general-type LP's.

Seemingly the only way to process ARC is Dynamic Programming
$\Rightarrow$ severe limitations on problem's structure and sizes.

$$\min_x \{f(x,\zeta) : F(x,\zeta) \le 0\}, \ \zeta \in \mathcal{U} \tag{U}$$

$$\Downarrow$$

$$\min_{\{X_j(\cdot)\}_{j=1}^n, t} \left\{ t : \begin{array}{l} f(X_1(P_1\zeta), ..., X_n(P_n\zeta), \zeta) \le t \\ F(X_1(P_1\zeta), ..., X_n(P_n\zeta), \zeta) \le 0 \end{array} \right\} \forall \zeta \in \mathcal{U} \Bigg\} \tag{ARC}$$

♣ How to cope with computational intractability of (ARC):

• Let us restrict $X_j(\cdot)$ to be simple – just affine:

$$X_j(P_j\zeta) = \xi_j + \eta_j^T P_j \zeta \tag{Aff}$$

• With decision rules (Aff), the infinite-dimensional problem (ARC) becomes the *Affinely Adjustable Robust Coun-*

terpart of (U) – *the semi-infinite problem*

$$\min_{\{\xi_j,\eta_j\}_{j=1}^n,t}\left\{t:\ \begin{array}{l} f(\xi_1+\eta_1{}^T P_1\zeta,...,\xi_n+\eta_n{}^T P_n\zeta,\zeta)\le t \\ F(\xi_1+\eta_1{}^T P_1\zeta,...,\xi_n+\eta_n{}^T P_n\zeta,\zeta)\le 0 \end{array}\right\}\ \forall\zeta\in\mathcal{U}\right\}$$

(AARC)

[A.B-T, A. Goryashko, E. Gustlizer, A.N '03]

♣ Uncertain problems with convex inclusion constraints are of the form

$$\min_x \left\{ c^T[\zeta]x : A[\zeta]x - b[\zeta] \in \mathcal{Q} \right\} \qquad \text{(U)}$$

where

- $(c[\zeta], A[\zeta], b[\zeta])$ are *affinely* parameterized by the data vector ζ
- $\mathcal{Q}$ is a given closed convex set (common for all instances of the uncertain problem)

Examples: Uncertain Linear/Conic Quadratic/Semidefinite programs.

♣ For uncertain problem with convex inclusion constraints the

Affinely Adjustable Robust Counterpart is the semi-infinite convex program

$$\min_{\chi=(\{\xi_j,\eta_j\}_{j=1}^n,t)} \left\{t : \mathcal{A}[\chi,\zeta] \equiv \left[\begin{array}{c} t - \sum\limits_{j=1}^n c_j[\zeta]\cdot\left[\xi_j + \eta_j{}^T P_j\zeta\right] \\ \sum\limits_{j=1}^n \left[\xi_j + \eta_j{}^T P_j\zeta\right]\cdot A^j[\zeta] - b[\zeta] \end{array}\right] \in \underbrace{\mathbf{R}_+ \times \mathcal{Q}}_{\mathcal{Q}_+} \ \forall \tag{AARC}$$

♠ Definition: (U) has *fixed recourse*, if for every j such that x_j is adjustable (that is, $P_j \neq 0$), both $c_j[\zeta]$ and $A^j[\zeta]$ are certain – independent of ζ.

$\Rightarrow$ The mapping $\mathcal{A}[\chi,\zeta]$ is bi-affine in χ, ζ.

$$\min_x \left\{ c^T[\zeta]x : A[\zeta]x - b[\zeta] \in \mathcal{Q} \right\} \qquad \text{(U)}$$

$$\Downarrow$$

$$\min_{\chi=(\{\xi_j,\eta_j\}_{j=1}^n,t)} \left\{ e^T\chi : \mathcal{A}[\chi,\zeta] \in \mathcal{Q}_+ \ \forall \zeta \in \mathcal{U} \right\} \qquad \text{(AARC)}$$

Proposition. *Assume that*

A. $\mathcal{Q} = \mathbf{R}^N_+$,
and

B. (U) *has fixed recourse.*

Then (AARC) *is computationally tractable whenever* $\mathcal{U}$ *is so. In particular, when* $\mathcal{U}$ *is a polyhedral set given as*

$$\mathcal{U} = \{\zeta : \exists \nu : P\zeta + Q\nu + r \geq 0\},$$

then (AARC) *is equivalent to an explicit LP program which can be obtained in polynomial time from the data specifying* $\mathcal{A}[\cdot,\cdot]$, $\mathcal{Q}$ *and* $\mathcal{U}$.

<u>Remark I.</u> Preserving assumption B and assuming that $\mathcal{U}$ is an ellipsoid, one can relax assumption A by allowing $\mathcal{Q}$ to be a direct product of Second Order cones.

Remark II. Preserving assumption A and removing assumption B, one still has a "tight approximation" result:

Proposition. *Let* $\mathcal{Q} = \mathbf{R}^N_+$, *let* $\mathcal{U}$ *be the intersection of* L *ellipsoids centered at the origin:*

$$\mathcal{U} = \mathcal{U}(\rho) = \left\{\zeta : \zeta^T Q_\ell \zeta \le \rho^2,\, \ell = 1, ..., L\right\} \qquad [Q\ell \succeq 0, \sum_\ell Q_\ell \succ 0]$$

and let

$$\mathrm{Opt}_{AARC}(\rho) = \min_\chi \left\{e^T\chi : \mathcal{A}[\chi, \zeta] \in \mathcal{Q}_+ \ \forall \zeta \in \mathcal{U}(\rho)\right\}.$$

Then for an explicit semidefinite program (SDP[ρ]) *readily given by* $\mathcal{A}[\cdot, \cdot]$ *and* $\{Q_\ell\}_{\ell=1}^L$ *it holds:*

(i) *every feasible solution to* (SDP[ρ]) *is feasible for* (AARC[ρ]) *as well;*

(ii) $\mathrm{Opt}_{AARC}(\rho) \leq \mathrm{Opt}(SDP[\rho]) \leq \mathrm{Opt}_{AARC}(\vartheta\rho)$ *with*

$$\vartheta \leq O(1)\ln(L).$$

♣ Example: Flexible Supplier-Retailer contracts via AARC [A.B-T,
B. Golany, A.N., J.-Ph. Vial '05]

- The story: A single-product inventory affected by uncertain demand should be run over the period of T months. At the very beginning, inventory management commits itself for certain monthly replenishment orders. These orders should not be followed exactly, but there are penalties for deviations of actual orders from the commitments.
- The goal: To specify commitments (non-adjustable variables) and actual replenishment orders (adjustable variables allowed to depend on past demands) in order to

minimize the overall inventory management cost which includes:

- cost of replenishment,
- holding cost,
- penalties for backlogged demands,
- penalties for deviations of actual orders from commitments

♠ With no uncertainty in the demands, the Commitments problem is just an LP program. Assuming the demand uncertain, it becomes an uncertain LP program with fixed recourse
$\Rightarrow$ *the Affinely Adjustable RC is computationally tractable, provided that the uncertainty set is so.*

• The Adjustable RC asks to minimize the worst case, over all demand trajectories from a given uncertainty set, inventory management cost over commitments and decision rules specifying the actual replenishment orders as functions of past demands.

• The Affinely Adjustable RC asks to minimize the same objective over commitments and decision rules specifying the actual replenishment orders as *affine* functions of the past demands.

♠ *In contrast to ARC, which suffers from "curse of dimensionality", AARC is just an explicit LP program with polynomial in T number of variables and constraints, provided that the uncertainty set is polyhedral.*

Computational tractability of AARC is preserved when adding new linear constraints, e.g., when forbidding backlogged demand, adding bounds on instant and cumulative orders, etc.

♠ In the Commitments problem, AARC demonstrates remarkably nice behaviour. In particular,

- among ≈ 300 different data sets with $T = 12$, we found just 4 where the optimal value of ARC (still available when $T = 12$) was better than the one of AARC, and the difference was less than 4%;

- the AARC results in guaranteed management costs which can be by as much as 30% less than those yielded by RC.

Uncertainty %%	Opt(ARC)	Opt(AARC)	Opt(RC)
10	13531.8	13531.8(+0.0%)	15033.4(+11.1%)
20	15063.5	15063.5(+0.0%)	18066.7(+19.9%)
30	16595.3	16595.3(+0.0%)	21100.0(+27.1%)
40	18127.0	18127.0(+0.0%)	24300.0(+34.1%)
50	19658.7	19658.7(+0.0%)	27500.0(+39.9%)
60	21190.5	21190.5(+0.0%)	30700.0(+44.9%)
70	22722.2	22722.2(+0.0%)	33960.0(+49.5%)

CONTROLLING CONSTRAINT VIOLATIONS OUTSIDE OF UNCERTAINTY SET:
Comprehensive Robust Counterpart

• "A.2. ... *The decision maker is fully responsible for consequences of the decisions to be made when, and <u>only</u> when, the actual data is within a given bounded uncertainty set.*"

♣ In some applications, Assumption A.2 is too restrictive.

<u>Example:</u> Consider building a communication network with uncertain information traffic demands. On special rare occasions, these demands may become unusually high.

• including “large deviations” of the demand in the uncertainty set could be too expensive...

• just ignoring “large deviations” could be too irresponsible...

♠ With “large deviations” in the data, it is natural to ensure

• required performance when uncertain data vary in their “normal range” – a not too large uncertainty set $\mathcal{U}$;

• *controlled* deterioration of performance when the uncertain data are outside of the uncertainty set.

♣ A natural way to relax Assumption A.2 is as follows.

♠ Consider an uncertain problem with convex inclusion constraints

$$\min_x \left\{ c^T[\zeta]x : A[\zeta]x - b[\zeta] \in \mathcal{Q} \right\} \tag{U}$$

♠ Assume that the set $\mathcal{Z}$ of *all* "*physically possible*" values of ζ is of the form

$$\mathcal{Z} = \underset{\substack{\uparrow \\ \text{convex compact}}}{\mathcal{U}} + \underset{\substack{\uparrow \\ \text{closed convex cone}}}{\mathcal{L}}$$

where $\mathcal{U}$ is the "normal range" of ζ.

♠ Let us say that affine decision rules

$$x = X(\xi, \eta; \zeta) := (\xi_1 + \eta_1^T P_1 \zeta, ..., \xi_n + \eta_n^T P_n \zeta)^T$$

• form a robust feasible solution to (U) with global sensitivity α, if

$$\forall(\zeta \in \mathcal{Z}) : \mathrm{dist}(A[\zeta]X(\xi,\eta;\zeta) - b[\zeta], \mathcal{Q}) \le \alpha\, \mathrm{dist}(\zeta, \mathcal{U}|\mathcal{L}) \equiv \min_{\substack{u\in\mathcal{U},\ell\in\mathcal{L}\\ u+\ell=\zeta}} \|\ell\|.$$

• has robust objective value $t \in \mathbf{R}$ with global sensitivity α_0, if

$$c^T[\zeta]X(\xi,\eta;\zeta) \le t + \alpha_0\, \mathrm{dist}(\zeta, \mathcal{U}|\mathcal{L})$$

$$\min_x \left\{ c^T[\zeta]x : A[\zeta]x - b[\zeta] \in \mathcal{Q} \right\} \tag{U}$$

♣ The *Comprehensive Robust Counterpart* of (U) [A.B-T,S. Boyd,A.N. '05] is the problem

$$\min_{\{\xi_j,\eta_j\},t} \left\{ t : \begin{array}{l} c^T[\zeta]X(\xi,\eta;\zeta) - t \leq \alpha_0 \,\mathrm{dist}(\zeta, \mathcal{U}|\mathcal{L}) \\ \mathrm{dist}(A[\zeta]X(\xi,\eta;\zeta) - b[\zeta], \mathcal{Q}) \leq \alpha \,\mathrm{dist}(\zeta, \mathcal{U}|\mathcal{L}) \end{array} \right\} \forall \zeta \in \mathcal{Z} = \mathcal{U} +$$

$$\left[X(\xi,\eta;\zeta) = (\xi_1 + \eta_1^T P_1 \zeta, ..., \xi_n + \eta_n^T P_n \zeta)^T \right] \tag{CRC}$$

of minimizing, given the global sensitivities α_0, α, the robust objective value over robust feasible affine decision rules.

♠ Note:

• when $(\{\xi_j, \eta_j\}, t)$ is feasible for (CRC) *and* $\zeta \in \mathcal{U}$, the decisions $x_j = \xi_j + \eta_j^T P_j \zeta$ satisfy the constraints in (U) and make the value of the objective $\leq t$

• With $\mathcal{L} = \{0\}$, (CRC) recovers the Affinely Adjustable Robust Counterpart of (U). If, in addition, $P_j = 0$ for all j, (CRC) recovers the Robust Counterpart of (U)

• Extensions of CRC:

• In may cases, ζ and the constraints in (U) are "structured":

$$D[\zeta]x - b[\zeta] \in \mathcal{Q} \Leftrightarrow D_i[\zeta]x - b_i[\zeta] \in \mathcal{Q}_i,\ i = 1, ..., m$$
$$\mathcal{Z} = \left\{\zeta = (\zeta^1, ..., \zeta^k) : \zeta_s \in \mathcal{U}_s + \mathcal{L}_s,\ s = 1, ..., k\right\}$$

In these cases, it makes sense to use "structured" Comprehensive Robust Counterpart

$$\min_{\{\xi_j,\eta_j\},t}\left\{t:\ \begin{array}{l} c^T[\zeta]X(\xi,\eta;\zeta)-t\leq\sum_{s=1}^k\alpha_{0s}\,\mathrm{dist}(\zeta^s,\mathcal{U}_s|\mathcal{L}_s)\\ \mathrm{dist}(D_i[\zeta]X(\xi,\eta;\zeta)-b_i[\zeta],\mathcal{Q}_i)\leq\sum_s\alpha_{is}\,\mathrm{dist}(\zeta^s,\mathcal{U}_s|\mathcal{L}_s)\\ \quad i=1,...,m\\ \hfill\forall\zeta\in\mathcal{Z}=\mathcal{U}+\mathcal{L}\end{array}\right\}\right\}$$

$$\left[X(\xi,\eta;\zeta)=(\xi_1+\eta_1^TP_1\zeta,...,\xi_n+\eta_n^TP_n\zeta)^T\right]$$

(SCRC)

- We can add more flexibility to (SCRC) by

— specifying different norms in different dist terms;
— treating α_{is} as variables rather than given constants, replacing the objective t with a function of t and α_{is} and adding constraints on α_{is}.

♣ Computational tractability of (CRC)

♠ Assumptions:

- $\mathcal{Q}_i$ are closed convex sets, $\mathcal{U}_s$ are convex compacts, $\mathcal{L}_s$ are closed convex cones;
- (U) has fixed recourse.

Under these assumptions, Comprehensive Robust Counterpart is of the form

$$\min_{\alpha\in\Lambda,\chi} \phi(\chi,\alpha)$$

s.t.

$$\mathrm{dist}_{\|\cdot\|_i}\left(D_{i0}[\chi]+\sum_{s=1}^{k} D_{is}[\chi]\zeta^s,\mathcal{Q}_i\right)\leq \sum_{s=1}^{k}\alpha_{is}\,\mathrm{dist}_{\|\cdot\|_{is}}(\zeta^s,\mathcal{U}_s|\mathcal{L}_s)$$
$$\forall i=0,1,...,m\,\forall\,(\zeta^s\in\mathcal{U}_s+\mathcal{L}_s, s=1,...,k) \tag{CRC}$$

with *affine in* χ vectors/matrices $D_{is}[\cdot]$.

♠ <u>Observation:</u> (CRC) is equivalent to the semi-infinite problem

$$\min_{\alpha\in\Lambda,\chi} \phi(\chi,\alpha)$$

$$\text{s.t.} \quad D_{i0}[\chi] + \sum_{s=1}^{k} D_{is}[\chi]\zeta^s \in \mathcal{Q}_i \;\forall i = 0,1,...,m \,\forall\, (\zeta^s \in \mathcal{U}_s, s = 1,...,k)$$

$$\mathrm{dist}_{\|\cdot\|_i}(D_{is}[\chi]\zeta^s, \mathcal{RQ}_i) \le \alpha_{is}\|\zeta^s\|_{is} \;\forall i = 0,1,...,m \,\forall\, (\zeta^s \in \mathcal{L}_s, s = 1,...,$$

where $\mathcal{RQ}_i$ is the recessive cone of $\mathcal{Q}_i$.

$$\min_{\alpha\in\Lambda,\chi} \phi(\chi,\alpha)$$

$$\text{s.t.}\quad D_{i0}[\chi]+\sum_{s=1}^k D_{is}[\chi]\zeta^s\in\mathcal{Q}_i\ \forall i=0,1,...,m\forall\,(\zeta^s\in\mathcal{U}_s, s=1,...,k)$$

$$\mathrm{dist}_{\|\cdot\|_i}(D_{is}[\chi]\zeta^s,\mathcal{R}\mathcal{Q}_i)\le\alpha_{is}\|\zeta^s\|_{is}\ \forall i=0,1,...,m\forall\,(\zeta^s\in\mathcal{L}_s, s=1,...,$$

(CRC)

Theorem. *Assume that we are in polyhedral case:*

(1) all $\mathcal{Q}_i$ are polyhedral sets given as $\mathcal{Q}_i=\{y: Q_iy\ge q_i\}$,

(2) $\mathcal{Q}_i$ and $\|\cdot\|_i$ are such that $\mathrm{dist}_{\|\cdot\|_i}(y,\mathcal{R}\mathcal{Q}_i)=\max\limits_{1\le\nu\le N_i}\alpha_{i\nu}^Ty$ for given $\alpha_{i\nu}$,

(3) all $\mathcal{L}_s$, $s=1,...,k$, are polyhedral cones given as $\mathcal{L}_s=\{\zeta^s:\exists u^s: L_s\zeta^s\ge R_su^s\}$,

(4) all $\mathcal{U}_s$ are polyhedral sets given as $\mathcal{U}_s=\{\zeta^s:\exists v^s: U_s\zeta^s+V_sv^s\ge w^s\}$,

(5) unit balls of all norms $\|\cdot\|_{is}$ are given as $\{\zeta^s:\exists u^{is}: S_{is}\zeta^s+T_{is}u^{is}\ge r^{is}\}$.

Then the system of semi-infinite constraints (a), (b) in (CRC) is equivalent to an explicit finite system $\mathcal{S}$ of linear inequalities in χ, α and additional variables, and $\mathcal{S}$ can be built in polynomial time from the data of the above representations of $\mathcal{Q}_i$, $\mathcal{L}_i$, $\mathcal{U}_i$, $\mathrm{dist}_{\|\cdot\|_i}(\cdot, \mathcal{R}\mathcal{Q}_i)$, $\|\cdot\|_{is}$.

Conditions (1) – (2) for sure take place when $\mathcal{Q}_i$ are one-dimensional, that is, the original problem (U) is an uncertain Linear Programming program with fixed recourse.

$$\min_{\alpha\in\Lambda,\chi} \phi(\chi,\alpha)$$

$$\text{s.t.} \quad D_{i0}[\chi] + \sum_{s=1}^{k} D_{is}[\chi]\zeta^s \in \mathcal{Q}_i \;\forall i = 0,1,...,m \forall\,(\zeta^s \in \mathcal{U}_s, s = 1,...,k)$$

$$\mathrm{dist}_{\|\cdot\|_i}(D_{is}[\chi]\zeta^s, \mathcal{R}\mathcal{Q}_i) \le \alpha_{is}\|\zeta^s\|_{is} \;\forall i = 0,1,...,m \forall\,(\zeta^s \in \mathcal{L}_s, s = 1,...,$$

Remark: *Under assumptions*

(1) all $\mathcal{Q}_i$ *are polyhedral sets given as* $\mathcal{Q}_i = \{y : Q_i y \ge q_i\}$,

(2) $\mathcal{Q}_i$ *and* $\|\cdot\|_i$ *are such that* $\mathrm{dist}_{\|\cdot\|_i}(y^i, \mathcal{R}\mathcal{Q}_i) = \max\limits_{1\le\nu\le N_i} \alpha_{i\nu}^T y$ *for given* $\alpha_{i\nu}$,

the Comprehensive Robust Counterpart is efficiently solvable whenever $\mathcal{U}_s$, $\mathcal{L}_s$ *and* Λ *are computationally tractable, and the norms* $\|\cdot\|_{is}$ *and the objective* $\phi(\cdot,\cdot)$ *are efficiently computable, and* Λ, $\phi(\cdot)$ *are convex.*

♣ Generic application: *Affine control of uncertainty-affected Linear Dynamical Systems.*

♠ Consider Linear Time-Varying Dynamical system

$$\begin{array}{rcl} x_{t+1} & = & A_t x_t + B_t u_t + R_t d_t \\ y_t & = & C_t x_t \\ x_0 & = & z \end{array} \qquad (S)$$

- x_t: state; • u_t: control • y_t: output;
- d_t: uncertain input; • z: initial state

to be controlled over finite time horizon $t = 0, 1, ..., T$.

♠ Assume that a "desired behaviour" of the system is given by a system of convex inclusions

$$D_i w - b_i \in Q_i,\ i = 1, ..., m$$

on the state-control trajectory

$$w = (x_0, x_1, ..., x_{T+1}, u_0, u_1, ..., u_T),$$

and the goal of the control is to minimize a given linear objective $f(w)$.

$$\begin{array}{rcl} x_{t+1} &=& A_t x_t + B_t u_t + R_t d_t \\ y_t &=& C_t x_t \\ x_0 &=& z \end{array} \qquad \text{(S)}$$

♠ Restricting ourselves with affine output-based control laws

$$u_t = \xi_{t0} + \sum_{\tau=0}^{t} \Xi_{t\tau} y_\tau, \qquad (*)$$

the problem of interest is

(!) *Find an affine control law* $(*)$ *which ensures that the resulting state-control trajectory* w *satisfies the system of convex inclusions*

$$D_i w - b_i \in \mathcal{Q}_i,\ i = 1, ..., m$$

and minimizes, under this restriction, a given linear objective $f(w)$.

Dynamics (S) makes w a known function of inputs $d = (d_0, d_1, ..., d_T)$, the initial state z and the parameters ξ of the control law $(*)$:

$$w = W(\xi; d, z).$$

Consequently, (!) is the optimization problem

$$\min_{\xi} \{f(W(\xi; d, z)) : D_i W(\xi; d, z) - b_i \in \mathcal{Q}_i,\ i = 1, ..., m\} \qquad \text{(U)}$$

open loop dynamics: $\left\{\begin{array}{rcl} x_{t+1} & = & A_t x_t + B_t u_t + R_t d_t \\ y_t & = & C_t x_t \\ x_0 & = & z \end{array}\right.$

control law: $u_t = \xi_{t0} + \sum\limits_{\tau=0}^{t} \Xi_{t\tau} y_\tau$

$$\Downarrow$$

$$w := (u_0, ..., u_T, x_0, ..., x_{T+1}) = W(\xi; d, z)$$

$$\Downarrow$$

$$\min_{\xi} \{ f(W(\xi; d, z)) : D_i W(\xi; d, z) - b_i \in \mathcal{Q}_i, \, i = 1, ..., m \} \qquad \text{(U)}$$

Note: Due to presence of uncertain input trajectory d and possible uncertainty in the initial state, (U) is an uncertain problem.

Difficulty: While linearity of the dynamics and the control law make $W(\xi; d, z)$ linear in (d, z), the dependence of $W(\cdot, \cdot)$ on the parameters $\xi = \{\xi_{t0}, \Xi_{t\tau}\}_{0 \le \tau \le t \le T}$ of the control law is highly nonlinear

$\Rightarrow$ (U) is *not* a problem with convex inclusions, which makes inapplicable the theory we have developed. In fact, (U) seems to be intractable already when there is no uncertainty in d, z!

Remedy: suitable re-parameterization of affine control laws.

♣ Affine control laws revisited. Consider a closed loop system along with its *model*:

closed loop system:	model:
$\begin{array}{rcl} x_{t+1} &=& A_t x_t + B_t u_t + R_t d_t \\ y_t &=& C_t x_t \\ x_0 &=& z \\ u_t &=& U_t(y_0,...,y_t) \end{array}$	$\begin{array}{rcl} \widehat{x}_{t+1} &=& A_t \widehat{x}_t + \widehat{B}_t u_t \\ \widehat{y}_t &=& C_t \widehat{x}_t \\ \widehat{x}_0 &=& 0 \end{array}$

♠ Observation: We can run the model in an on-line fashion, so that at time t, before the decision on u_t should be made, we have in our disposal *purified outputs*

$$v_t = y_t - \widehat{y}_t.$$

♠ Fact I: Every transformation $(d,z) \mapsto w = (u_0,...,u_t,x_0,...,x_{T+1})$ which can be obtained from an affine control law based on outputs:

$$u_t = \xi_{t0} + \sum_{\tau=0}^{t} \Xi_{t\tau} y_\tau \qquad (*)$$

can be obtained from an affine control law based on purified outputs:

$$u_t = \eta_{t0} + \sum_{\tau=0}^{t} H_{t\tau} v_\tau \qquad (**)$$

and vice versa.

system:	model:
$x_{t+1} = A_t x_t + B_t u_t + R_t d_t$ $y_t = C_t x_t$ $x_0 = z$	$\widehat{x}_{t+1} = A_t \widehat{x}_t + \widehat{B}_t u_t$ $\widehat{y}_t = C_t \widehat{x}_t$ $\widehat{x}_0 = 0$

control law:

$$
\begin{array}{rcl}
v_t &=& y_t - \widehat{y}_t \\
u_t &=& \eta_{t0} + \sum_{\tau=0}^{t} H_{t\tau} v_\tau \qquad (**)
\end{array}
$$

(S)

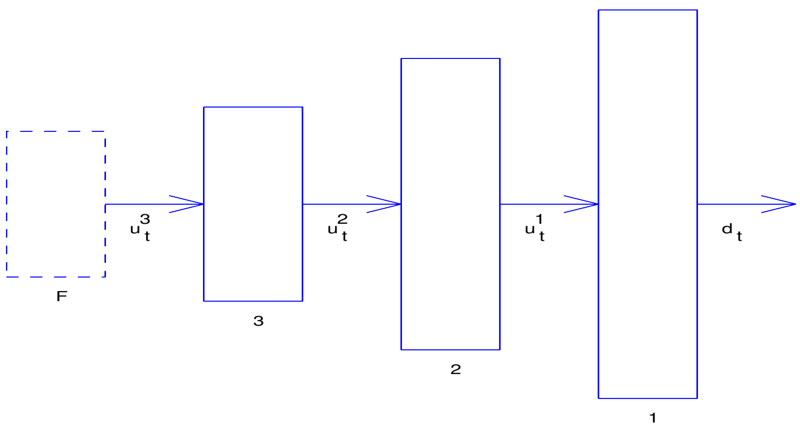
♠ Fact II: The state-control trajectory $w = W(\eta; d, z)$ of (S) is affine in (d, z) when the parameters $\eta = \{\eta_{t0}, H_{t\tau}\}_{0 \le \tau \le t \le T}$ of the control law $(**)$ are fixed, *and is affine in η when (d, z) is fixed.*

♠ Corollary: *With parameterization $(**)$ of affine control laws, problem of interest becomes an uncertain optimization problem with convex inclusions, and as such can be processed via the CRC approach.*

In particular, in the case when $\mathcal{Q}_i$ are one-dimensional, the CRC of the problem of interest is computationally tractable, provided that the normal range $\mathcal{U}$ of (d,z) and the associated cone $\mathcal{L}$ are so. If $\mathcal{U}$, $\mathcal{L}$ and the norms used to measure distances are polyhedral, CRC is just an explicit LP program.

♣ Note: While the outlined approach "as it is" is aimed at building optimal *finite-horizon* affine control, it can be combined with existing Control techniques to get *infinite-horizon* stabilizing control laws with desired transition characteristics.

♠ Illustration: Serial Multi-Level Inventory.



3-Level Inventory. 1 – 3: warehouses; F: factory

- External demand is satisfied by inventory of level 1;
- Inventory of level $i = 1, 2$ is replenished from inventory of level $i + 1 = 2, 3$, inventory of level 3 is replenished from factory;
- There is a delay of 2 time units in executing replenishment orders

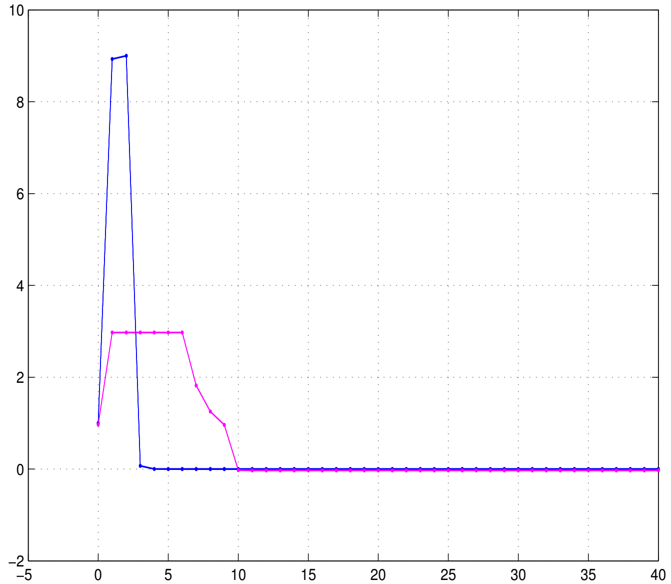
♠ The 3-level inventory with 2-unit delays in executing replenishing orders can be modelled as the Linear Dynamical system

$$x_{t+1} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_t + \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} u_t + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} d_t$$

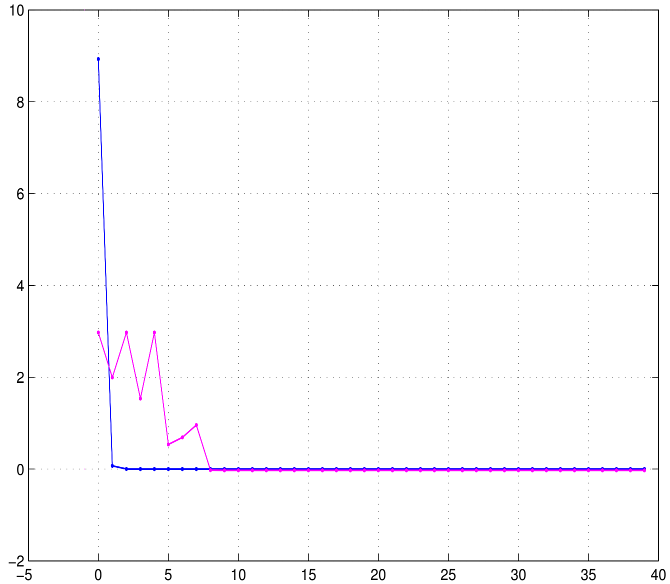
- $x = (x_1, ..., x_9)^T$ – states (x^i, $i = 1, 2, 3$, is the amount of product inventory of level i)
- $u = (u^1, u^2, u^3)^T$ – replenishment orders
- d_t – external demand.

♠ In serial multi-level inventories with delays, variations in external demand usually are "amplified" – they result in much larger variations of replenishment orders and inventory levels.

We have applied the CRC approach in combination with the standard Control techniques in order to moderate this phenomenon. The resulting infinite-horizon affine control law makes the closed loop system essentially more stable than the standard linear feedback control.



$z \mapsto x$ gains



$z \mapsto u$ gains

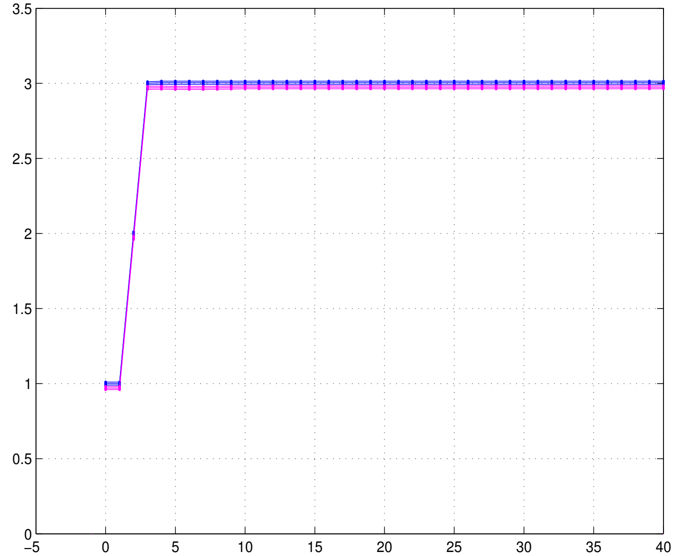
Magenta: CRC-based control

Blue: Feedback control yilded by Lyapunov Stability Synthesis

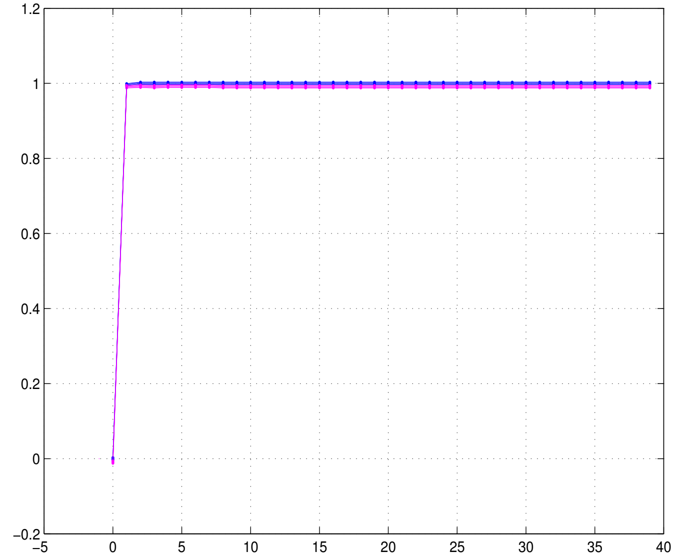
- $z \mapsto x$ gain at time t is the maximal $\|\cdot\|_\infty$-variation of the state at time t which can be caused by a unit $\|\cdot\|_\infty$-

variation in the initial state.

- $z \mapsto u$ gain at time t is the maximal $\| \cdot \|_\infty$-variation of the control at time t which can be caused by a unit $\| \cdot \|_\infty$-variation in the initial state.



$d \mapsto x$ gains



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- $d \mapsto x$ gain at time t is the maximal $\|\cdot\|_\infty$-variation of the state at time t which can be caused by a unit $\|\cdot\|_\infty$-

variation in the sequence $d_0, d_1, ..., d_{t-1}$ of demands.

- $d \mapsto u$ gain at time t is the maximal $\|\cdot\|_\infty$-variation of the control at time t which can be caused by a unit $\|\cdot\|_\infty$-variation in the sequence $d_0, d_1, ..., d_{t-1}$ of demands.

Sample trajectories: