Appendix A

A.1 Notation

Vectors and matrices. $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}$ stand for the spaces of complex, respectively, real $m \times n$ matrices. We write $\mathbb{C}^n$ and $\mathbb{R}^n$ as shorthands for $\mathbb{C}^{n\times 1}$, $\mathbb{R}^{n\times 1}$, respectively.

For $A \in \mathbb{C}^{m\times n}$, A^T stands for the transpose, and A^H for the conjugate transpose of A:

$$(A^H)_{rs} = \overline{A_{sr}},$$

where $\overline{z}$ is the conjugate of $z \in \mathbb{C}$.

Both $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}$ are equipped with the inner product

$$\langle A, B\rangle = \mathrm{Tr}(AB^H) = \sum_{r,s} A_{rs}B^*_{rs}.$$

The norm associated with this inner product is denoted by $\|\cdot\|_2$.

We use the notation I_m, $O_{m\times n}$ for the unit $m \times m$, respectively, the zero $m \times n$ matrices.

• $\mathcal{H}^m$, $\mathbf{S}^m$ are real vector spaces of $m \times m$ Hermitian, respectively, real symmetric matrices. Both are Euclidean spaces w.r.t. the inner product $\langle\cdot,\cdot\rangle$.

We use "MATLAB notation": when $A_1, ..., A_k$ are matrices with the same number of rows, $[A_1, ..., A_k]$ denotes the matrix with the same number of rows obtained by writing, from left to right, first the columns of A_1, then the columns of A_2, and so on. When $A_1, ..., A_k$ are matrices with the same number of columns, $[A_1; A_2; ...; A_k]$ stands for the matrix with the same number of columns obtained by writing, from top to bottom, first the rows of A_1, then the rows of A_2, and so on.

For a Hermitian/real symmetric $m \times m$ matrix A, $\lambda(A)$ is the vector of eigenvalues $\lambda_r(A)$ of A taken with their multiplicities in the non-ascending order:

$$\lambda_1(A) \geq \lambda_2(A) \geq ... \geq \lambda_m(A).$$

For a $m \times n$ matrix A, $\sigma(A) = (\sigma_1(A), ..., \sigma_n(A))^T$ is the vector of singular values of A:

$$\sigma_r(A) = \lambda_r^{1/2}(A^H A),$$

and

$$\|A\| = \sigma_1(A) = \max\{\|Ax\|_2 : x \in \mathbb{C}^n, \|x\|_2 \leq 1\}$$

(by evident reasons, when A is real, one can replace $\mathbb{C}^n$ in the right hand side with $\mathbb{R}^n$).

For Hermitian/real symmetric matrices A, B, we write $A \succeq B$ ($A \succ B$) to express that $A - B$ is positive semidefinite (resp., positive definite). We denote by $\mathcal{H}^n_+$ ($\mathbf{S}^n_+$) the cones of positive semidefinite Hermitian (resp., positive semidefinite real symmetric) $n \times n$ matrices.

A.2 $\mathcal{S}$-Lemma

Theorem A.1 [$\mathcal{S}$-Lemma] (i) [homogeneous version] *Let A, B be symmetric matrices of the same size such that $\bar{x}^T A \bar{x} > 0$ for certain $\bar{x}$. Then the implication*

$$x^T Ax \geq 0 \Rightarrow x^T Bx \geq 0$$

holds true if and only if

$$\exists \lambda \geq 0 : B \succeq \lambda A.$$

(ii) [inhomogeneous version] *Let A, B be symmetric matrices of the same size, and let the quadratic form $x^T Ax + 2a^T x + \alpha$ be strictly positive at certain point $\bar{x}$. Then the implication*

$$x^T Ax + 2a^T x + \alpha \geq 0 \Rightarrow x^T Bx + 2b^T x + \beta \geq 0 \tag{A.1}$$

holds true if and only if

$$\exists \lambda \geq 0 : \left[\begin{array}{c|c} B - \lambda A & b^T - \lambda a^T \\ \hline b - \lambda a & \beta - \lambda \alpha \end{array}\right] \succeq 0.$$

Proof.

(i): In one direction the statement is evident: if $B \succeq \lambda A$ with $\lambda \geq 0$, then $x^T Bx \geq \lambda x^T Ax$ for all x and therefore $x^T Ax \geq 0$ implies $x^T Bx \geq 0$.

Now assume that $x^T Ax \geq 0$ implies $x^T Bx \geq 0$, and let us prove that $B \succeq \lambda A$ for certain $\lambda \geq 0$. Consider the optimization problem

$$\mathrm{Opt} = \min_X \left\{ \mathrm{Tr}(BX) : \mathrm{Tr}(AX) \geq 0, \mathrm{Tr}(X) = 1, X \geq 0 \right\}. \tag{A.2}$$

This problem clearly is strictly feasible. Indeed, by assumption there exists $\bar{X} = \bar{x}\bar{x}^T \succeq 0$ such that $\mathrm{Tr}(A\bar{X}) > 0$; adding to $\bar{X}$ a small positive definite matrix and normalizing the result to have unit trace, we get a strictly feasible solution to (A.2). Moreover, the problem is below bounded, since its feasible set is compact. Applying Semidefinite Duality, we conclude that there exists $\lambda \geq 0$ such that $B - \lambda A \succeq \mathrm{Opt} I$. We see that it suffices for us to prove that $\mathrm{Opt} \geq 0$.

Problem (A.2) is clearly solvable. Let X_* be its optimal solution, and let $\bar{A} = X_*^{1/2} A X_*^{1/2}$, $\bar{B} = X_*^{1/2} B X_*^{1/2}$. Then

$$\mathrm{Tr}(\bar{A}) = \mathrm{Tr}(AX_*) \geq 0,\ \ \mathrm{Tr}(\bar{B}) = \mathrm{Tr}(BX_*) = \mathrm{Opt}, x^T \bar{A} x \geq 0 \Rightarrow x^T \bar{B} x \geq 0.$$

Now let $\bar{A} = U \Lambda U^T$ be the eigenvalue decomposition of $\bar{A}$, so that U is orthogonal and Λ is diagonal, and let ζ be a random vector with independent of each other coordinates taking values ± 1 with probability 1/2, and let $\xi = U\zeta$. For all realizations of ζ, we have

$$\xi^T \bar{A}\xi = \zeta^T U^T (U\Lambda U^T)U\zeta = \zeta^T \Lambda\zeta = \mathrm{Tr}(\Lambda) = \mathrm{Tr}(\bar{A}) \geq 0,$$

whence $\xi^T \bar{B}\xi \geq 0$. Taking expectation, we have

$$\begin{array}{l} 0 \leq \mathbb{E}\{\xi^T \bar{B}\xi\} = \mathbb{E}\{\zeta^T (U^T \bar{B}U)\zeta\} = \mathbb{E}\{\mathrm{Tr}([U^T \bar{B}U][\zeta\zeta^T])\} \\ = \mathrm{Tr}([U^T \bar{B}U] \underbrace{\mathbb{E}\{\zeta\zeta^T\}}_{=I}) = \mathrm{Tr}([U^T \bar{B}U]) = \mathrm{Tr}(\bar{B}) = \mathrm{Opt}. \end{array}$$

Thus, $\mathrm{Opt} \geq 0$, as claimed.

(ii): Let us pass from original inhomogeneous quadratic forms on $\mathbb{R}^n$ to their homogenizations:

$$\begin{array}{l} f_A(x) \equiv x^T Ax + 2a^T x + \alpha \mapsto \widehat{f}_A([x;t]) = [x;t]^T \widehat{A}[x;t] \equiv x^T Ax + 2ta^T x + \alpha t^2 \\ f_B(x) \equiv x^T Bx + 2b^T x + \beta \mapsto \widehat{f}_B([x;t]) = [x;t]^T \widehat{B}[x;t] \equiv x^T Bx + 2tb^T x + \beta t^2 \end{array}$$

Claim: *In the situation of* (ii), $\exists \bar{y} : \bar{y}^T \widehat{A}\bar{y} > 0$, *and implication (A.1) is equivalent to the implication*

$$y^T \widehat{A}y \geq 0 \Rightarrow y^T \widehat{B}y \geq 0 \qquad (*)$$

Claim $\Rightarrow$ Inhomogeneous $\mathcal{S}$-Lemma: Combining Claim and Homogeneous $\mathcal{S}$-Lemma as applied to matrices $\widehat{A}$, $\widehat{B}$, we conclude that (A.1) is equivalent to the existence of a $\lambda \geq 0$ such that $\widehat{B} \succeq \lambda\widehat{A}$, which is exactly what is stated by the Inhomogeneous $\mathcal{S}$-Lemma.

Justifying Claim: We clearly have $[\bar{x};1]^T \widehat{A}[\bar{x};1] = f_A(\bar{x}) > 0$. Further, if $(*)$ is valid, then so is (A.1), since $f_A(x) = \widehat{f}_A([x;1])$, $f_B(x) = \widehat{f}_B([x;1])$. We see that all we need is to show that the validity of implication (A.1) implies the validity of implication $(*)$. Thus, assume that (A.1) is valid, and let us prove that $(*)$ takes place. Let $[x;t]$ be such that $[x;t]^T \widehat{A}[x;t] \geq 0$; we should prove that then $[x;t]^T \widehat{B}[x;t] \geq 0$. The case of $t \neq 0$ is trivial due to

$$\begin{array}{l} [x;t]^T \widehat{A}[x;t] \geq 0 \Rightarrow \underbrace{[t^{-1}x;1]^T \widehat{A}[t^{-1}x;1]}_{f_A(x)} \geq 0 \Rightarrow \underbrace{[t^{-1}x;1]^T \widehat{B}[t^{-1}x;1]}_{f_B(x)} \\ \Rightarrow [x;t]^T \widehat{B}[x;t] \geq 0. \end{array}$$

In order to prove that $[x;0]^T \widehat{A}[x;0] \geq 0$ implies that $[x;0]^T \widehat{B}[x;0] \geq 0$, it suffices to verify that the point $y = [x;0]$ can be represented as the limit of a sequence $y^i = [x^i;t^i]$ with $t^i \neq 0$ and $[y^i]^T \widehat{A}y^i \geq 0$. Indeed, in this situation, due to the already proved part of $(*)$, we would have $[y^i]^T \widehat{B}y^i \geq 0$ for all i, and passing to limit as $i \to \infty$, we would get the required relation $y^T \widehat{B}y \geq 0$.

To prove the aforementioned approximation result, let us pass to the coordinates z_j of a point z in the eigenbasis of $\widehat{A}$, so that

$$y^T \widehat{A}y = \sum_j \lambda_j y_j^2 \geq 0,$$

where $\lambda_1 \geq \lambda_2 \geq ...$ are the eigenvalues of $\widehat{A}$. Observe that $\lambda_1 > 0$, since, as we remember, there exists $\bar{y}$ such that $\bar{y}^T \widehat{A}\bar{y} > 0$. it follows that replacing the first coordinate in y with $(1+1/i)y_1$ and keeping the remaining coordinate intact, we get points $\widehat{y}^i$ such that $\widehat{y}^i \to y$, $i \to \infty$, and $[\widehat{y}^i]^T \widehat{A} \widehat{y}^i > 0$. Since the latter inequalities are strict, we can perturb slightly the points $\widehat{y}^i$ to get a sequence $\{y^i\}$ which still converges to y, still satisfies $[y^i]^T \widehat{A} y^i > 0$ and, in addition, is comprised of points with nonzero t-coordinates. □

A.3 Approximate $\mathcal{S}$-Lemma

Theorem A.2 [6] *Let $\rho > 0$, $A, B, B_1, ..., B_L$ be symmetric $m \times m$ matrices such that $B = bb^T$, $B_\ell \succeq 0$, $\ell = 1, ..., L \geq 1$, and $B + \sum_{\ell=1}^{L} B_\ell \succ 0$.*

Consider the optimization problem

$$Opt(\rho) = \max_x \left\{ x^T A x : x^T B x \leq 1, x^T B_\ell x \leq \rho^2, \ \ell = 1, ..., L \right\} \tag{A.3}$$

along with its semidefinite relaxation

$$\begin{aligned} \mathrm{SDP}(\rho) &= \max_X \Big\{ \mathrm{Tr}(AX) : \mathrm{Tr}(BX) \leq 1, \mathrm{Tr}(B_\ell X) \leq \rho^2, \\ &\qquad \ell = 1, ..., L, X \succeq 0 \Big\} \\ &= \min_{\lambda, \{\lambda_\ell\}} \Big\{ \lambda + \rho^2 \sum_{\ell=1}^{L} \lambda_\ell : \lambda \geq 0, \lambda_\ell \geq 0, \ \ell = 1, ..., L, \\ &\qquad \lambda B + \sum_{\ell=1}^{L} \lambda_\ell B_\ell \succeq A \Big\}. \end{aligned} \tag{A.4}$$

Then there exists $\bar{x}$ such that

$$\begin{array}{llcl} (a) & \bar{x}^T B \bar{x} & \leq & 1 \\ (b) & \bar{x}^T B \bar{x} & \leq & \Omega^2(L)\rho^2, \ell = 1, ..., L \\ (c) & \bar{x}^T A \bar{x} & = & \mathrm{SDP}(\rho), \end{array} \tag{A.5}$$

where $\Omega(L)$ is a universal function of L such that $\Omega(1) = 1$ and

$$\Omega(L) \leq 9.19\sqrt{\ln(L)}, \ L \geq 2. \tag{A.6}$$

In particular,

$$Opt(\rho) \leq \mathrm{SDP}(\rho) \leq Opt(\Omega(L)\rho). \tag{A.7}$$

Proof.
1^0. First of all, let us derive the "in particular" part of the statement from its general part. Indeed, given that $\bar{x}$ satisfying (A.5) does exist, observe that $\bar{x}$ is a feasible solution to the problem defining $\mathrm{Opt}(\Omega\rho)$, whence $\mathrm{Opt}(\Omega\rho) \geq \bar{x}^T A \bar{x} = \mathrm{SDP}(\rho)$. The first inequality in (A.7) is evident.
2^0. The problem

$$\mathrm{SDP}(\rho) = \max_X \left\{ \mathrm{Tr}(AX) : \mathrm{Tr}(BX) \leq 1, \mathrm{Tr}(B_1 X) \leq \rho^2, \ X \succeq 0 \right\} \tag{A.8}$$

clearly is strictly feasible and solvable; by this reason, the semidefinite dual of this problem is solvable with the optimal value $\mathrm{SDP}(\rho)$, which is nothing but the second equality in (A.4).
3^0. Consider the case $L = 1$, where we should prove that here $\Omega(L) = 1$. This can be immediately derived from the following nice fact:

> **Theorem** [31] Let A, B, B_1 be three $m \times m$ symmetric matrices with $m \geq 3$ such that certain linear combination of the matrices if $\succ 0$. Then the joint range $\mathcal{I} = \{(x^TAx, x^TBx, x^TB_1x)^T : x \in \mathbb{R}^m\} \subset \mathbb{R}^3$ of the associated quadratic forms is a closed convex set.

We, however, prefer to present an alternative straightforward proof.
3^0.0. Since $B \succeq 0$, $B_1 \succeq 0$ and $B + B_1 \succ 0$, problem (A.8) clearly is solvable. All we need is to prove that this problem admits an optimal solution X_* of rank ≤ 1. Indeed, such a solution is representable in the form $\bar{x}\bar{x}^T$ for certain vector $\bar{x}$; from the constraints of (A.8) it follows that $\bar{x}$ satisfies (A.5.*a*-*b*) with $\Omega(1) = 1$, and from the optimality of $X_* = \bar{x}\bar{x}^T$ – that $\bar{x}$ satisfies (A.5.*c*) as well. Now, when proving that (A.8) admits an optimal solution with rank ≤ 1, we may assume that $B_1 \succ 0$. Indeed, assuming that in the latter case the statement we are interested in is true, we would conclude that whenever $\epsilon > 0$, the optimization problem

$$\mathrm{Opt}_\epsilon = \max_X \left\{ \mathrm{Tr}(AX) : \mathrm{Tr}(BX) \leq 1, \mathrm{Tr}([B_1 + \epsilon I]X) \leq \rho^2, X \succeq 0 \right\} \qquad (P_\epsilon)$$

has an optimal solution X_*^ϵ of rank ≤ 1. Since $B + B_1 \succ 0$, the matrices X_*^ϵ are bounded, so that, by compactness argument, there exists a matrix X_* of rank ≤ 1 such that

$$\mathrm{Tr}(AX_*) \leq \lim \sup_{\epsilon \to +0} \mathrm{Opt}_\epsilon, \mathrm{Tr}(BX_*) \leq 1, \mathrm{Tr}(B_1X_*) \leq \rho^2.$$

We see that X_* is a feasible solution to (A.8), and all we need is to prove that this solution is optimal, that is, to prove that $\mathrm{SDP}(\rho) \leq \liminf_{\epsilon \to +0} \mathrm{Opt}_\epsilon$. To this end, let Y_* be an optimal solution to (A.8). For every γ, $0 < \gamma < 1$, the matrix γY_* clearly is feasible for problems (P_ϵ) with all small enough ϵ, whence $\gamma\mathrm{SDP}(\rho) \leq \limsup_{\epsilon \to +0} \mathrm{Opt}_\epsilon$. Since $\gamma < 1$ is arbitrary, we get $\mathrm{SDP}(\rho) \leq \liminf_{\epsilon \to +0} \mathrm{Opt}_\epsilon$, as required.

Thus, we may focus on the case when $B_1 \succ 0$, and all we need is to prove that in this case (A.8) has an optimal solution of rank ≤ 1.
3^0.1. Passing in the optimization problem in (A.3) from variables x to variables $B_1^{1/2}x$, we may assume w.l.o.g. that $B_1 = I$; in the latter case, passing to the orthonormal eigenbasis of A, we may further assume that A is diagonal: $A = \mathrm{Diag}\{\lambda_1, ..., \lambda_m\}$ with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$.
3^0.2. Problem (A.8) clearly is solvable; all we need to prove is that this problem has an optimal solution X_* which is a matrix of rank ≤ 1.
3^0.3. Assuming that $\mathrm{SDP}(\rho) \leq 0$, the optimization problem in (A.8) clearly has an optimal rank 0 solution $X_* = 0$, and we are done. Thus, assume that $\mathrm{SDP}(\rho) > 0$, which implies that $\lambda_1 > 0$. Note that since A is diagonal and $B_1 = I$, we have

$$\mathrm{SDP}(\rho) \leq \max_X \left\{ \mathrm{Tr}(AX) : \mathrm{Tr}(X) \leq \rho^2, X \succeq 0 \right\} = \lambda_1\rho^2. \qquad (\mathrm{A}.9)$$

$3^0.4$. It is possible that $\lambda_1 = \lambda_2$. We clearly have $\mathrm{SDP}(\rho) \leq \lambda_1\rho^2$; on the other hand, there exists a vector $\bar{x}$, $\|\bar{x}\|_2 = \rho$, which is in the linear span of the first two basic orths and is orthogonal to b. The rank 1 matrix $X_* = \bar{x}\bar{x}^T$ clearly is feasible for (A.8), and for this matrix $\mathrm{Tr}(AX_*) = \lambda_1\rho^2$, so that X_* is an optimal solution to (A.8) by (A.9). Thus, (A.8) has a rank 1 optimal solution, and we are done.

$3^0.5$. From now on we assume that $\lambda_1 > \lambda_2$. There are to possible cases: one where (A.8) has an optimal solution X_* with $\mathrm{Tr}(BX_*) < 1$ ("Case I") and another one where $\mathrm{Tr}(BX_*) = 1$ for every optimal solution X_* to (A.8). Assume, first, that we are in Case I, and let X_* be an optimal solution to (A.8) with $\mathrm{Tr}(BX_*) < 1$. Let $X_* = V^TV$, let v_i be the columns of V and p_i be the Euclidean norms of the vectors v_i, so that $v_i = p_if_i$, $\|f_i\|_2 = 1$. We clearly have

$$\begin{array}{llcl} (a) & \mathrm{SDP}(\rho) & = & \sum_i \lambda_i (X_*)_{ii} = \sum_i \lambda_i p_i^2, \\ (b) & \rho^2 & \geq & \mathrm{Tr}(B_1X_*) = \mathrm{Tr}(X_*) = \sum_i p_i^2, \\ (c) & 1 & \geq & \mathrm{Tr}(BX_*) = \mathrm{Tr}(bb^TV^TV) = \|\sum_i b_iv_i\|_2^2. \end{array} \tag{A.10}$$

We claim that in fact $p_i = 0$ for $i > 0$, so that X_* is a rank 1 optimal solution, as required. Indeed, assuming that there exists $i_* > 1$ with $p_{i_*} > 0$ and given ϵ, $0 \leq \epsilon < p_{i_*}^2$, let us pass from V to a new matrix V_+ as follows: we replace in V the column $v_{i_*} = p_{i_*}f_{i_*}$, with its multiple $v_{i_*}^+ = \gamma f_{i_*}$, where $\gamma > 0$ is such that $\|v_{i_*}^+\|_2^2 = p_{i_*}^2 - \epsilon$, and replace column $v_1 = p_1f_1$ in V with the column $v_1^+ = \theta f_1$, where $\theta > 0$ is such that $\|v_1^+\|_2^2 = p_1^2 + \epsilon$; all remaining columns in V_+ are exactly the same as in V. Setting $X_+ = [V_+]^TV_+$, we clearly have $X_+ \succeq 0$, $\mathrm{Tr}(B_1X_+) = \mathrm{Tr}(X_+) = \mathrm{Tr}(X_*) \leq \rho^2$ and $\mathrm{Tr}(AX_+) = \mathrm{Tr}(AX_*) + (\lambda_1 - \lambda_{i_*})\epsilon > \mathrm{Tr}(AX_*)$ (recall that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$ and $\lambda_1 > \lambda_2$). On the other hand, for small $\epsilon > 0$, X_+ is close to X_*, so that for small enough $\epsilon > 0$ we have $\mathrm{Tr}(BX_+) < 1$ due to $\mathrm{Tr}(BX_*) < 1$. Thus, for small $\epsilon > 0$ X_+ is a feasible solution to (A.8) which is better than X_* in terms of the objective, which is a contradiction. Thus, $p_i = 0$ for $i > 1$, as claimed.

$3^0.6$. It remains to consider Case II. Let X_* be an optimal solution to (A.8), and let V, v_i, p_i be defined exactly as in $3^0.5$, so that (A.10) takes place. Since we are in Case II, the vector $e = \sum_i b_iv_i$ is of Euclidean norm 1. Let $I = \{i : b_i \neq 0\}$. We claim that all vectors v_i, $i \in I$, are proportional to e. Indeed, assume that $i \in I$ and v_i is not proportional to e, so that the vectors v_i and $w_i = e - b_iv_i$ are nonzero and are not proportional to each other. Let v_i^+ be the vector of exactly the same Euclidean norm as v_i and of the direction opposite to the one of the vector b_iw_i, let V_+ be the matrix obtained from V by replacing the column v_i with the column v_i^+, and let $X[V_+]^TV_+$. By construction, the Euclidean norms of the columns in V_+ are the same as those of columns in V, whence

$$\mathrm{Tr}(AX_+) = \mathrm{Tr}(AX_*) = \mathrm{SDP}(\rho),\ \mathrm{Tr}(B_1X_+) = \mathrm{Tr}(B_1X_*) \leq \rho^2. \tag{A.11}$$

At the same time, by construction

$$\begin{array}{l} \mathrm{Tr}(BX_+) = \|V_+b\|_2^2 = \|b_iv_i^+ + w_i\|_2^2 = b_i^2\|v_i^+\|_2^2 + 2(v_i^+)^T(b_iw_i) + \|w_i\|_2^2 \\ = b_i^2\|v_i\|_2^2 - 2\|b_iv_i\|_2\|w_i\|_2 + \|w_i\|_2^2 \\ < b_i^2\|v_i\|_2^2 + 2b_iv_i^T\|w_i\|_2 + \|w_i\|_2^2 = \|b_iv_i + w_i\|_2^2 \leq 1, \end{array}$$

where the strict inequality is given by the fact that $b_i \neq 0$ and the nonzero vectors v_i and w_i are not proportional to each other. Invoking (A.11), we conclude that X_+ is an optimal solution to (A.8) with $\mathrm{Tr}(BX_+) < 1$, which is impossible, since we are in Case II.

Thus, all v_i, $i \in I$, are proportional to e. Replacing in V columns v_i, $i \notin$, with columns of the same Euclidean norms proportional to e, we get a matrix V_+ such that (a) all columns in V_+ are proportional to e, (b) the columns in V_+ are of the same Euclidean norms as the corresponding columns in V, and (c) $V_+b = Vb$. From (b), (c) it follows that $X_+ = [V_+]^T V_+$ is a feasible solution to (A.8) with the same value of the objective as the one at X_*, i.e., is optimal for the problem, while (a) implies that $V_+ = ef^T$ for certain f, so that X_+ is a rank 1 solution, and we are done.

4^0. Now consider the case of $L > 1$. Let X_* be an optimal solution to the semidefinite program defining $\mathrm{SDP}(\rho)$, and let

$$\widehat{A} = X_*^{1/2} A X_*^{1/2}.$$

Let also

$$\widehat{A} = U \Lambda U^T$$

be the eigenvalue decomposition of $\widehat{A}$, so that U is orthogonal and Λ is diagonal. Consider the random vector

$$\xi = X_*^{1/2} U \zeta,$$

where $\zeta \in \mathbb{R}^m$ is random vector with independent coordinates taking values ± 1 with probabilities 0.5. We have

$$\begin{array}{llcl}
(a) & \xi^T A \xi & = & \zeta^T U^T X_*^{1/2} A X_*^{1/2} U \zeta = \zeta^T U^T \widehat{A} U \zeta = \zeta^T \Lambda \zeta \\
 & & = & \mathrm{Tr}(\Lambda) = \mathrm{Tr}(U \Lambda U^T) = \mathrm{Tr}(\widehat{A}) = \mathrm{Tr}(A X_*) \\
 & & = & \mathrm{SDP}(\rho), \\
(b) & \mathbf{E}\{\xi^T B \xi\} & = & \mathrm{Tr}(B \mathbf{E}\{\xi \xi^T\}) = \mathrm{Tr}(B X_*^{1/2} U \mathbf{E}\{\zeta \zeta^T\} U^T X_*^{1/2}) \\
 & & = & \mathrm{Tr}(B X_*) \leq 1, \\
(c) & \mathbf{E}\{\xi^T B_\ell \xi\} & = & \mathrm{Tr}(B_\ell \mathbf{E}\{\xi \xi^T\}) = \mathrm{Tr}(B_\ell X_*^{1/2} U \mathbf{E}\{\zeta \zeta^T\} U^T X_*^{1/2}) \\
 & & = & \mathrm{Tr}(B_\ell X_*) \leq \rho^2
\end{array} \tag{A.12}$$

(we have used the fact that X_* is an optimal solution to the problem defining $\mathrm{SDP}(\rho)$).

5^0. We need the following

Lemma A.1 *One has*

$$\mathrm{Prob}\{\xi^T B \xi > 1\} \leq 2/3. \tag{A.13}$$

Proof.
Recalling that $B = bb^T$, we have $\xi^T B \xi = \zeta^T U^T X_*^{1/2} b b^T X_*^{1/2} U \zeta = (\beta^T \zeta)^2$, where $\beta = U^T X_*^{1/2} b$. From (A.12.b) it follows that $\mathbf{E}\{(\beta^T \zeta)^2\} = \|\beta\|_2^2 \leq 1$; the fact that in this situation one has $\mathrm{Prob}\{|\beta^T \zeta| > 1\} \leq 2/3$ is proved in Lemma A.1 in [6]. □

6^0. We next need the following fact.

Lemma A.2 *Let $e_1,...,e_m$ be deterministic vectors such that $\sum\limits_{i=1}^m \|e_i\|_2^2 \leq 1$. Then*

$$\forall (t>1): \mathrm{Prob}\left\{\|\sum_{\ell=1}^m \zeta_i e_i\|_2 \geq t\right\} \leq \phi(t) = \inf_{r:1<r<t} \frac{r^2\exp\{-(t-r)^2/16\}}{r^2-1}. \tag{A.14}$$

Proof uses the following fundamental fact:

Talagrand Inequality [see, e.g., [23]] *Let $\eta_1,...,\eta_m$ be independent random vectors taking values in unit balls of the respective finite-dimensional vector spaces $(E_1,\|\cdot\|_{(1)})$,...,$(E_m,\|\cdot\|_{(m)})$, and let $\eta=(\eta_1,...,\eta_m)\in E = E_1\times...\times E_m$. Let us equip E with the norm $\|(z^1,...,z^m)\| = \sqrt{\sum\limits_{i=1}^m \|z^i\|_{(i)}^2}$, and let Q be a closed convex subset of E. Then*

$$\mathbf{E}\left\{\exp\{\frac{\mathrm{dist}^2_{\|\cdot\|}(\eta,Q)}{16}\}\right\} \leq \frac{1}{\mathrm{Prob}\{\eta\in Q\}}.$$

Let us specify the spaces $(E_i,\|\cdot\|_{(i)})$, $i=1,...,m$, as $(\mathbb{R},|\cdot|)$, and let $\eta_i=\zeta_i$, $i=1,...,m$. Let, further,

$$Q_1 = \{u\in\mathbb{R}^m : \|\sum_{i=1}^m u_i e_i\|_2 \leq 1\}.$$

Observe that Q_1 is a closed convex set in $\mathbb{R}^m$ and that this set contains the unit $\|\cdot\|_2$-ball; indeed,

$$\|\sum_{i=1}^m u_i e_i\|_2 \leq \sum_{i=1}^m |u_i|\|e_i\|_2 \leq \|u\|_2\sqrt{\sum_{i=1}^m \|e_i\|_2^2} \leq \|u\|_2.$$

Observe, further, that

$$\mathbf{E}\{\|\sum_i \zeta_i e_i\|_2^2\} = \sum_i \|e_i\|_2^2 \leq 1,$$

whence, by Tschebyshev Inequality,

$$\mathrm{Prob}\left\{\|\sum_i \zeta_i e_i\|_2 > r\right\} \equiv \mathrm{Prob}\{\zeta\notin rQ\} \leq \frac{1}{r^2}\ \forall r>1. \tag{A.15}$$

For $t > r > 1$ we have

$$\begin{array}{ll} & \|\sum_i u_i e_i\|_2 > t \Rightarrow u \not\in \frac{t}{r}(rQ_1) \Rightarrow u \not\in (rQ_1) + (\frac{t}{r}-1)(rQ_1) \\ \Rightarrow & \mathrm{dist}_{\|\cdot\|_2}(z, rQ_1) \geq (\frac{t}{r}-1)r = t - r, \end{array}$$

where the concluding inequality follows from the fact that Q_1 contains the unit Euclidean ball centered at the origin. We now have for $t > r > 1$:

$$\begin{array}{rcl} \mathrm{Prob}\{\|\sum_i \zeta_i e_i\|_2 > t\} & \leq & \mathrm{Prob}\left\{\frac{\mathrm{dist}^2_{\|\cdot\|_2}(\zeta, rQ_1)}{16} \geq \frac{(t-r)^2}{16}\right\} \\ & \leq & \exp\{-\frac{(t-r)^2}{16}\}\mathbf{E}\{\frac{\mathrm{dist}^2_{\|\cdot\|_2}(\zeta, rQ_1)}{16}\} \\ & & \text{[Tschebyshev Inequality]} \\ & \leq & \frac{\exp\{-(t-r)^2/16\}}{\mathrm{Prob}\{\zeta\in rQ_1\}} \\ & & \text{[Talagrand Inequality]} \\ & \leq & \frac{\exp\{-(t-r)^2/16\}}{1-1/r^2}, \end{array}$$

where the concluding $\leq$ is due to

$$\mathrm{Prob}\{\zeta \not\in rQ_1\} = \mathrm{Prob}\{\|\sum_i \zeta_i e_i\|_2 > r\} \leq \frac{1}{r^2}. \qquad \square$$

7^0. Given integer $L > 1$, let $\Omega(L) = \inf\{t \geq 1 : \phi(t) > 1/(3L)\}$. Note that from (A.14) it follows immediately that

$$L > 1 \Rightarrow \Omega(L) \leq C\sqrt{\ln(L)} \tag{A.16}$$

where C is an absolute constant (computer says that it can be set to the value 9.19). Denoting by e_i^ℓ the columns of the matrix $\rho^{-1}B_\ell^{1/2}X_*^{1/2}U$, we have

$$\sum_{i=1}^{L} \|e_i^\ell\|_2^2 = \mathbf{E}\{\|\rho^{-1}B_\ell^{1/2}X_*^{1/2}U\zeta\|_2^2\} \leq 1, \tag{A.17}$$

where the concluding inequality is nothing but (A.12.c). Taking into account that

$$\begin{array}{l} \mathrm{Prob}\{\xi^T B_\ell \xi > a\} = \mathrm{Prob}\{\zeta^T[B_\ell^{1/2}X_*^{1/2}U]^T[B_\ell^{1/2}X_*^{1/2}U]\zeta > a\} \\ = \mathrm{Prob}\{\|B_\ell^{1/2}X_*^{1/2}U\zeta\|_2^2 > a\} = \mathrm{Prob}\{\|\sum_i \zeta_i e_i^\ell\|_2^2 > \rho^{-2}a\} \end{array}$$

and invoking Lemma A.2, we get

$$\mathrm{Prob}\{\xi^T B_\ell \xi > \rho^2 t^2\} = \mathrm{Prob}\{\|\sum_i \zeta_i e_i^\ell\|_2 > t\} \leq \phi(t),$$

whence

$$t > \Omega(L) \Rightarrow \mathrm{Prob}\{\xi^T B_\ell \xi > \rho^2 t^2\} < \frac{1}{3L}. \tag{A.18}$$

Invoking Lemma A.1, it follows that when $t > \Omega(L)$, one has

$$\mathrm{Prob}\{\xi : \xi^T B \xi > 1 \text{ or } \exists \ell : \xi^T B_\ell \xi > \rho^2 t^2\} < \frac{2}{3} + L\frac{1}{3L} = 1,$$

that is, there exists a realization $\widetilde{\xi}$ of ξ such that

$$\widetilde{\xi}^T B\widetilde{\xi} \leq 1,\ \widetilde{\xi}^T B_\ell \widetilde{\xi} \leq t^2\rho^2\ \forall \ell, 1 \leq \ell \leq L.$$

Since $t > \Omega(L)$ is arbitrary, there exists a realization $\bar{\xi}$ of ξ such that

$$\bar{x}^T B\bar{x} \leq 1,\ \bar{x}^T B_\ell \bar{x} \leq \Omega^2(L)\rho^2\ \forall \ell, 1 \leq \ell \leq L.$$

Since $\bar{x}$ is a realization of ξ, we have also

$$\bar{x}^T A\bar{x} = \mathrm{SDP}(\rho)$$

by (A.12.a). Thus, $\bar{x}$ satisfies (A.5). □

A.4 Matrix Cube Theorem

A.4.1 Matrix Cube Theorem, Complex case

The “Complex Matrix Cube” problem is as follows:

CMC: *Let m, $p_1, q_1, ..., p_L, q_L$ be positive integers, and $A \in \mathcal{H}^m_+$, $L_j \in \mathbb{C}^{p_j\times m}$, $R_j \in \mathbb{C}^{q_j\times m}$ be given matrices, $L_j \neq 0$. Let also a partition $\{1, 2, ..., L\} = I^{\mathrm{r}}_{\mathrm{s}} \cup I^{\mathrm{c}}_{\mathrm{s}} \cup I^{\mathrm{c}}_{\mathrm{f}}$ of the index set $\{1, ..., L\}$ into three non-overlapping sets be given, and let $p_j = q_j$ for $j \in I^{\mathrm{r}}_{\mathrm{s}} \cup I^{\mathrm{c}}_{\mathrm{s}}$. With these data, we associate a parametric family of “matrix boxes”*

$$\begin{array}{rcl}\mathcal{U}[\rho] & = & \left\{ A + \rho \sum_{j=1}^{L} [L_j^H \Theta^j R_j + R_j^H [\Theta^j]^H L_j] : \begin{array}{l}\Theta^j \in \mathcal{Z}^j,\\ 1 \leq j \leq L\end{array} \right\} \\ & \subset & \mathcal{H}^m,\end{array} \quad \text{(A.19)}$$

where $\rho \geq 0$ is the parameter and

$$\mathcal{Z}^j = \left\{ \begin{array}{ll} \{\Theta^j = \theta I_{p_j} : \theta \in \mathbb{R}, |\theta| \leq 1\}, & j \in I^{\mathrm{r}}_{\mathrm{s}} \\ \quad [\text{“real scalar perturbation blocks”}] & \\ \{\Theta^j = \theta I_{p_j} : \theta \in \mathbb{C}, |\theta| \leq 1\}, & j \in I^{\mathrm{c}}_{\mathrm{s}} \\ \quad [\text{“complex scalar perturbation blocks”}] & \\ \{\Theta^j \in \mathbb{C}^{p_j\times q_j} : \|\Theta^j\|_{2,2} \leq 1\}, & j \in I^{\mathrm{c}}_{\mathrm{f}} \\ \quad [\text{“full complex perturbation blocks”}]\ . & \end{array}\right. \quad \text{(A.20)}$$

Given $\rho \geq 0$, check whether

$$\mathcal{U}[\rho] \subset \mathcal{H}^m_+ \quad \mathcal{A}(\rho)$$

Remark A.1 *In the sequel, we always assume that $p_j = q_j > 1$ for $j \in I^{\mathrm{c}}_{\mathrm{s}}$. Indeed, one-dimensional complex scalar perturbations can always be regarded as full complex perturbations.*

Consider, along with predicate $\mathcal{A}(\rho)$, the predicate

$$\begin{array}{ll} \exists Y_j \in \mathcal{H}^m,\ j=1,...,L \text{ such that :} & \\ (a) \quad Y_j \succeq L_j^H \Theta^j R_j + R_j^H [\Theta^j]^H L_j \quad \forall (\Theta^j \in \mathcal{Z}^j, 1 \le j \le L) & \\ (b) \quad A - \rho \sum_{j=1}^{L} Y_j \succeq 0. & \end{array} \qquad \mathcal{B}(\rho)$$

Our main result is as follows:

Theorem A.3 [The Complex Matrix Cube Theorem] *One has:*

(i) *Predicate $\mathcal{B}(\rho)$ is stronger than $\mathcal{A}(\rho)$ – the validity of the former predicate implies the validity of the latter one.*

(ii) *$\mathcal{B}(\rho)$ is computationally tractable – the validity of the predicate is equivalent to the solvability of the system of LMIs*

$$\begin{array}{llcl} (s.\mathbb{R}) & Y_j \pm \left[L_j^H R_j + R_j^H L_j\right] & \succeq & 0,\ j \in I_{\rm s}^{\rm r}, \\ (s.\mathbb{C}) & \begin{bmatrix} Y_j - V_j & L_j^H R_j \\ R_j^H L_j & V_j \end{bmatrix} & \succeq & 0,\ j \in I_{\rm s}^{\rm c}, \\ (f.\mathbb{C}) & \begin{bmatrix} Y_j - \lambda_j L_j^H L_j & R_j^H \\ R_j & \lambda_j I_{p_j} \end{bmatrix} & \succeq & 0,\ j \in I_{\rm f}^{\rm c} \\ (*) & A - \rho \sum_{j=1}^{L} Y_j & \succeq & 0. \end{array} \qquad \text{(A.21)}$$

in the matrix variables $Y_j \in \mathcal{H}^m$, $j = 1, ..., k$, $V_j \in \mathcal{H}^m$, $j \in I_{\rm s}^{\rm c}$, and the real variables λ_j, $j \in I_{\rm f}^{\rm c}$.

(iii) *"The gap" between $\mathcal{A}(\rho)$ and $\mathcal{B}(\rho)$ can be bounded solely in terms of the maximal size*

$$p^{\rm s} = \max\{p_j : j \in I_{\rm s}^{\rm r} \cup I_{\rm s}^{\rm c}\} \qquad \text{(A.22)}$$

of the scalar perturbations (here the maximum over an empty set by definition is 0). Specifically, there exists a universal function $\vartheta_{\mathbb{C}}(\cdot)$ such that

$$\vartheta_{\mathbb{C}}(\nu) \le 4\pi\sqrt{\nu},\ \nu \ge 1, \qquad \text{(A.23)}$$

and

$$\text{if } \mathcal{B}(\rho) \text{ is not valid, then } \mathcal{A}(\vartheta_{\mathbb{C}}(p^{\rm s})\rho) \text{ is not valid.} \qquad \text{(A.24)}$$

(iv) *Finally, in the case $L = 1$ of single perturbation block $\mathcal{A}(\rho)$ is equivalent to $\mathcal{B}(\rho)$.*

Remark A.2 *From the proof of Theorem A.3 it follows that $\vartheta_{\mathbb{C}}(0) = \frac{4}{\pi}$, $\vartheta_{\mathbb{C}}(1) = 2$. Thus,*

- *when there are no scalar perturbations: $I_{\rm s}^{\rm r} = I_{\rm s}^{\rm c} = \emptyset$, the factor ϑ in the implication*

$$\neg\mathcal{B}(\rho) \Rightarrow \neg\mathcal{I}(\vartheta\rho) \qquad \text{(A.25)}$$

  can be set to $\frac{4}{\pi} = 1.27...$

- *when there are no complex scalar perturbations (cf. Remark A.1) and all real scalar perturbations are non-repeated ($I_s^c = \emptyset$, $p_j = 1$ for all $j \in I_s^r$), the factor ϑ in* (A.25) *can be set to* 2.

The following simple observation is crucial when applying Theorem A.3.

Remark A.3 *Assume that the data A, $R_1, ..., R_L$ of the Matrix Cube problem are affine in a vector of parameters y, while the data $L_1, ..., L_L$ are independent of y. Then* (A.21) *is a system of LMIs in the variables Y_j, V_j, λ_j and y.*

A.4.2 Proof of Theorem A.3.(i)

Item (i) is evident.

A.4.3 Proof of Theorem A.3.(ii)

The equivalence between the validity of $\mathcal{B}(\rho)$ and the solvability of (A.21) is readily given by the following facts:

Lemma A.3 *Let $B \in \mathbb{C}^{m\times m}$ and $Y \in \mathcal{H}^m$. Then the relation*

$$Y \succeq \theta B + \bar{\theta} B^H \quad \forall(\theta \in \mathbb{C}, |\theta| \leq 1) \tag{A.26}$$

is satisfied if and only if

$$\exists V \in \mathcal{H}^m : \quad \begin{bmatrix} Y - V & B^H \\ B & V \end{bmatrix} \succeq 0. \tag{A.27}$$

Lemma A.4 *Let $L \in \mathbb{C}^{\ell \times m}$ and $R \in \mathbb{C}^{r\times m}$.*

(i) *Assume that L, R are nonzero. A matrix $Y \in \mathcal{H}^m$ satisfies the relation*

$$Y \succeq L^H U R + R^H U^H L \quad \forall(U \in \mathbb{C}^{\ell\times r} : \|U\|_{2,2} \leq 1) \tag{A.28}$$

if and only if there exists a positive real λ such that

$$Y \succeq \lambda L^H L + \lambda^{-1} R^H R. \tag{A.29}$$

(ii) *Assume that L is nonzero. A matrix $Y \in \mathcal{H}^m$ satisfies* (A.28) *if and only if there exists $\lambda \in \mathbb{R}$ such that*

$$\begin{bmatrix} Y - \lambda L^H L & R^H \\ R & \lambda I_r \end{bmatrix} \succeq 0. \tag{A.30}$$

Lemmas A.3, A.4 $\Rightarrow$ Theorem A.3.(ii). All we need to prove is that a collection of matrices Y_j satisfies the constraints in $\mathcal{B}(\rho)$ if and only if it can be extended by properly chosen V_j, $j \in I_f^c$, and λ_j, $j \in I_s^c$, to a feasible solution of

(A.21). This is immediate, since matrices Y_j, $j \in I_{\rm f}^{\rm c}$, satisfy the corresponding constraints $\mathcal{B}(\rho).a$ if and only if these matrices along with some matrices V_j satisfy (A.21.s.$\mathbb{C}$)) (Lemma A.3), while matrices Y_j, $j \in I_{\rm s}^{\rm c}$, satisfy the corresponding constraints $\mathcal{B}(\rho).a$ if and only if these matrices along with some reals λ_j satisfy (A.21.f.$\mathbb{C}$) (Lemma A.4.(ii)). □

Proof of Lemma A.3. "if" part: Assume that V is such that

$$\left[\begin{array}{cc} Y-V & B^H \\ B & V \end{array}\right] \succeq 0.$$

Then, for every $\xi \in \mathbb{C}^n$ and every $\theta \in \mathbb{C}$, $|\theta| = 1$, we have

$$\begin{array}{rcl} 0 \leq \left[\begin{array}{c} \xi \\ -\overline{\theta}\xi \end{array}\right]^H \left[\begin{array}{cc} Y-V & B^H \\ B & V \end{array}\right] \left[\begin{array}{c} \xi \\ -\overline{\theta}\xi \end{array}\right] & = & \xi^H(Y-V)\xi + \xi^H V\xi \\ & & -\xi^H[\theta B + \overline{\theta}B^H]\xi, \end{array}$$

so that $Y \succeq \theta B + \overline{\theta}B^H$ for all $\theta \in \mathbb{C}$, $|\theta| = 1$, which, by evident convexity reasons, implies (A.26).

"only if" part: Let $Y \in \mathcal{H}^m$ satisfy (A.26). Assume, on the contrary to what should be proved, that there does not exist $V \in \mathcal{H}^m$ such that $0 \preceq \left[\begin{array}{cc} Y-V & B^H \\ B & V \end{array}\right]$, and let us lead this assumption to a contradiction. Observe that our assumption means that the optimization program

$$\min_{t,V} \left\{ t : \left[\begin{array}{cc} tI_m + Y - V & B^H \\ B & V \end{array}\right] \succeq 0 \right\} \tag{A.31}$$

has no feasible solutions with $t \leq 0$; since problem (A.31) is clearly solvable, its optimal value is therefore positive. Now, our problem is a conic problem on the (self-dual) cone of positive semidefinite Hermitian matrices; since the problem clearly is strictly feasible, the Conic Duality Theorem says that dual problem

$$\max_{\substack{Z\in\mathcal{H}^m,\\ W\in\mathbb{C}^{m\times m}}} \left\{ -2\Re\left\{\mathrm{Tr}(W^H B)\right\} - \mathrm{Tr}(ZY) : \begin{array}{ll} \left[\begin{array}{cc} Z & W^H \\ W & Z \end{array}\right] \succeq 0, & (a) \\ \mathrm{Tr}(Z) = 1 & (b) \end{array} \right\} \tag{A.32}$$

is solvable with the same – positive – optimal value as the one of (A.31). In (A.32), we can easily eliminate the W-variable; indeed, constraint (A.32.a), as it is well-known, is equivalent to the fact that $Z \succeq 0$ and $W = Z^{1/2}XZ^{1/2}$ with $X \in \mathbb{C}^{m\times m}$, $\|X\|_{2,2} \leq 1$. With this parameterization of W, the W-term in the objective of (A.32) becomes $-2\Re\{\mathrm{Tr}(X^H Z^{1/2}BZ^{1/2})\}$; as it is well-known, the maximum of the latter expression in X, $\|X\|_{2,2} \leq 1$, is $2\|\sigma(Z^{1/2}BZ^{1/2})\|_1$. Since the optimal value in (A.32) is positive, we arrive at the following intermediate conclusion:

(*) *There exists $Z \in \mathcal{H}^m$, $Z \succeq 0$, such that*

$$2\|\sigma(Z^{1/2}BZ^{1/2})\|_1 > \mathrm{Tr}(ZY) = \mathrm{Tr}(Z^{1/2}YZ^{1/2}). \tag{A.33}$$

The desired contradiction is now readily given by the following simple observation:

Lemma A.5 *Let $S \in \mathcal{H}^m$, $C \in \mathbb{C}^{m\times m}$ be such that*

$$S \succeq \theta C + \overline{\theta} C^H \quad \forall(\theta \in \mathbb{C}, |\theta| = 1). \tag{A.34}$$

Then $2\|\sigma(C)\|_1 \leq \mathrm{Tr}(S)$.

To see that Lemma A.5 yields the desired contradiction, note that the matrices $S = Z^{1/2}YZ^{1/2}$, $C = Z^{1/2}BZ^{1/2}$ satisfy the premise of the lemma by (A.26), and for these matrices the conclusion of the lemma contradicts (A.33).

Proof of Lemma A.5: As it was already mentioned,

$$\|\sigma(C)\|_1 = \max_X \left\{ \Re\{\mathrm{Tr}(XC^H)\} : \|X\|_{2,2} \leq 1 \right\}.$$

Since the extreme points of the set $\{X \in \mathbb{C}^{m\times m} : \|X\|_{2,2} \leq 1\}$ are unitary matrices, the maximizer X_* in the right hand side can be chosen to be unitary: $X_*^H = X_*^{-1}$; thus, X_* is a unitary similarity transformation of a diagonal unitary matrix. Applying appropriate unitary rotation $A \mapsto U^H AU$, $U^H = U^{-1}$, to all matrices involved, we may assume that X_* itself is diagonal. Now we are in the situation as follows: we are given matrices C, S satisfying (A.34) and a *diagonal* unitary matrix X_* such that $\|\sigma(C)\|_1 = \Re\{\mathrm{Tr}(X_* C^H)\}$. In other words,

$$\|\sigma(C)\|_1 = \Re\left\{ \sum_{\ell=1}^m (X_*)_{\ell\ell} \overline{C_{\ell\ell}} \right\} \leq \sum_{\ell=1}^m |C_{\ell\ell}| \tag{A.35}$$

(the concluding inequality comes from the fact that X_* is unitary). On the other hand, let e_ℓ be the standard basic orths in $\mathbb{C}^m$. By (A.34), we have

$$\theta C_{\ell\ell} + \overline{\theta C_{\ell\ell}} = e_\ell^H[\theta C + \overline{\theta} C^H]e_\ell \leq e_\ell^H S e_\ell = S_{\ell\ell} \quad \forall(\theta \in \mathbb{C}, |\theta| = 1),$$

whence, maximizing in θ, $2|C_{\ell\ell}| \leq S_{\ell\ell}$, $\ell = 1, ..., m$, which combines with (A.35) to imply that $2\|\sigma(C)\|_1 \leq \mathrm{Tr}(S)$. □

Proof of Lemma A.4 (cf. Section 8.3).

(i), "if" part: Let (A.29) be valid for certain $\lambda > 0$. Then for every $\xi \in \mathbb{C}^m$ one has

$$\begin{array}{rcl} \xi^H Y \xi & \geq & \lambda \xi^H L^H L \xi + \lambda^{-1} \xi^H R^H R \xi \geq 2\sqrt{\xi^H L^H L \xi}\sqrt{\xi^H R^H R \xi} \\ & = & 2\|L\xi\|_2 \|R\xi\|_2 \\ \Rightarrow & & \forall(U, \|U\|_{2,2} \leq 1): \\ \xi^H Y \xi & \geq & 2|[L\xi]^H U[R\xi]| \geq 2\Re\{[L\xi]^H U[R\xi]\} \\ & = & \xi^H[L^H UR + R^H U^H L]\xi, \end{array}$$

as claimed.

(i), "only if" part: Assume that Y satisfies (A.28) and $L \neq 0, R \neq 0$; we prove that then there exists $\lambda > 0$ such that (A.29) holds true. First, observe that w.l.o.g. we may assume that L and R are of the same sizes $r \times n$ (to reduce the general case to this particular one, it suffices to add several zero rows either to L (when $\ell < r$), or to R (when $\ell > r$)). We have the following chain of equivalences:

$$
\begin{array}{ll}
 & (A.28) \\
\Leftrightarrow & \forall \xi \in \mathbb{C}^m : \quad \xi^H Y \xi \geq 2\|L\xi\|_2 \|R\xi\|_2 \\
\Leftrightarrow & \forall (\xi \in \mathbb{C}^n, \eta \in \mathbb{C}^r) : \quad \|\eta\|_2 \leq \|L\xi\|_2 \Rightarrow \xi^H Y \xi - \eta^H R \xi - \xi^H R^H \eta \geq 0 \\
\Leftrightarrow & \forall (\xi \in \mathbb{C}^m, \eta \in \mathbb{C}^r) : \\
 & \xi^H L^H L \xi - \eta^H \eta \geq 0 \Rightarrow \xi^H Y \xi - \eta^H R \xi - \xi^H R^H \eta \geq 0 \\
\Leftrightarrow & \exists (\lambda \geq 0) : \left[\begin{array}{cc} Y & R^H \\ R & \end{array}\right] - \lambda \left[\begin{array}{cc} L^H L & \\ & -I_r \end{array}\right] \succeq 0 \quad [\mathcal{S}\text{-Lemma}] \\
\Leftrightarrow (a) & \left[\begin{array}{cc} Y - \lambda L^H L & R^H \\ R & \lambda I_r \end{array}\right] \succeq 0
\end{array}
\tag{A.36}
$$

(Note that $\mathcal{S}$-Lemma clearly holds true in the Hermitian case, since Hermitian quadratic forms on $\mathbb{C}^m$ can be treated as real quadratic forms on $\mathbb{R}^{2m}$.)

Condition (A.36.a), in view of $R \neq 0$, clearly implies that $\lambda > 0$. Therefore, by the Schur Complement Lemma (SCL), (A.36.a) is equivalent to $Y - \lambda L^H L - \lambda^{-1} R^H R \succeq 0$, as claimed.

<u>(ii)</u>: When $R \neq 0$, (ii) is clearly equivalent to (i) and thus is already proved. When $R = 0$, it is evident that (A.30) can be satisfied by properly chosen $\lambda \in \mathbb{R}$ if and only if $Y \succeq 0$, which is exactly what is stated by (A.28) when $R = 0$. □

A.4.4 Proof of Theorem A.3.(iii)

In order to prove (iii), it suffices to prove the following statement:

> **Lemma A.6** *Assume that $\rho \geq 0$ is such that the predicate $\mathcal{B}(\rho)$ is not valid. Then the predicate $\mathcal{A}(\vartheta_{\mathbb{C}}(p^s)\rho)$, with appropriately defined function $\vartheta_{\mathbb{C}}(\cdot)$ satisfying (A.23), is also not valid.*

We are about to prove Lemma A.6. The case of $\rho = 0$ is trivial, so that from now on we assume that $\rho > 0$ and that all matrices L_j, R_j are nonzero (the latter assumption, of course, does not restrict generality). *From now till the end of Section A.4.5, we assume that we are under the premise of Lemma A.6, i.e., the predicate $\mathcal{B}(\rho)$ is not valid.*

First step: duality

Consider the optimization program

$$\min_{\substack{t,\{Y_j\in\mathcal{H}^m\}_{j\in I_s^r},\\ \{U_j,V_j\in\mathcal{H}^m\}_{j\in I_s^c}\\ \{\lambda_j,\nu_j\in\mathbb{R}\}_{j\in I_f^c}}} \left\{ t: \begin{array}{ll} Y_j \pm \underbrace{[L_j^H R_j + R_j^H L_j]}_{2A_j,A_j=A_j^H} \succeq 0,\ j\in I_s^r, & (a)\\ \left[\begin{array}{cc} U_j & R_j^H L_j\\ L_j^H R_j & V_j\end{array}\right] \succeq 0,\ j\in I_s^c, & (b)\\ \left[\begin{array}{cc} \lambda_j & 1\\ 1 & \nu_j\end{array}\right] \succeq 0,\ j\in I_f^c, & (c)\\ tI + A - \rho\Big[\sum_{j\in I_s^r} Y_j + \sum_{j\in I_s^c}[U_j+V_j] & \\ \quad + \sum_{j\in I_f^c}[\lambda_j L_j^H L_j + \nu_j R_j^H R_j]\Big] \succeq 0 & (d)\end{array}\right\}. \tag{A.37}$$

Introducing "bounds" $Y_j = U_j + V_j$ for $j \in I_s^c$ and $Y_j \succeq \lambda_j L_j^H L_j + \nu_j R_j^H R_j$ for $j \in I_f^c$ and then eliminating the variables U_j, $j \in I_s^c$, ν_j, $j \in I_f^c$, we convert (A.37) into the equivalent problem

$$\min_{\substack{t,\{Y_j\in\mathcal{H}^m\}_{j=1}^L,\\ \{V_j\in\mathcal{H}^m\}_{j\in I_s^c}\\ \{\lambda_j\in\mathbb{R}\}_{j\in I_f^c}}} \left\{ t: \begin{array}{l} Y_j \pm [L_j^H R_j + R_j^H L_j] \succeq 0,\ j\in I_s^r,\\ \left[\begin{array}{cc} Y_j - V_j & R_j^H L_j\\ L_j^H R_j & V_j\end{array}\right] \succeq 0,\ j\in I_s^c,\\ \left[\begin{array}{cc} Y_j - \lambda_j L_j^H L_j & R_j^H\\ R_j & \lambda_j I_{p_j}\end{array}\right] \succeq 0,\ j\in I_f^c,\\ tI + A - \rho\sum_{j=1}^L Y_j \succeq 0\end{array}\right\}.$$

By (already proved) item (ii) of Theorem A.3, predicate $\mathcal{B}(\rho)$ is valid if and only if the latter problem, and thus problem (A.37), admits a feasible solution with $t \leq 0$. We are in the situation when $\mathcal{B}(\rho)$ is *not* valid; consequently, (A.37) does not admit feasible solutions with $t \leq 0$. Since the problem clearly is solvable, it means that the optimal value in the problem is positive. Problem (A.37) is a conic problem on the product of cones of Hermitian and real symmetric positive semidefinite matrices. Since (A.37) is strictly feasible and bounded below, the Conic Duality Theorem implies that the conic dual problem of (A.37) is solvable with the same positive optimal value. Taking into account that the cones associated with (A.37) are self-dual, the dual problem, after straightforward simplifications, becomes the conic problem

$$
\begin{array}{lrcl}
\text{maximize} & \multicolumn{3}{l}{-2\rho\left[\sum\limits_{j\in I_{\rm s}^{\rm r}}\mathrm{Tr}([P_j-Q_j]A_j)+\sum\limits_{j\in I_{\rm s}^{\rm c}}\Re\{\mathrm{Tr}\left(S_jR_j^HL_j\right)\}+\sum\limits_{j\in I_{\rm f}^{\rm c}}w_j\right]} \\
& \multicolumn{3}{l}{-\mathrm{Tr}(ZA)} \\
\text{s.t.} & & & \\
(a.1) & P_j,Q_j & \succeq & 0,\ j\in I_{\rm s}^{\rm r}, \\
(a.2) & P_j+Q_j & = & Z,\ j\in I_{\rm s}^{\rm r}; \\
(b) & \left[\begin{array}{cc} Z & S_j^H \\ S_j & Z\end{array}\right] & \succeq & 0,\ j\in I_{\rm s}^{\rm c}; \\
(c) & \left[\begin{array}{cc}\mathrm{Tr}(L_jZL_j^H) & w_j \\ w_j & \mathrm{Tr}(R_jZR_j^H)\end{array}\right] & \succeq & 0,\ j\in I_{\rm f}^{\rm c}; \\
(d) & Z\succeq 0,\mathrm{Tr}(Z) & = & 1.
\end{array}
\tag{A.38}
$$

in matrix variables $Z\in\mathcal{H}^m_+$, $P_j,Q_j\in\mathcal{H}^m$, $j\in I_{\rm s}^{\rm r}$, $S_j\in\mathbb{C}^{m\times m}$, $j\in I_{\rm s}^{\rm c}$, and real variables w_j, $j\in I_{\rm f}^{\rm c}$. Using (A.38.c), we can eliminate the variables w_j, thus coming the following equivalent reformulation of the dual problem:

$$
\begin{array}{lrcl}
\text{maximize} & \multicolumn{3}{l}{2\rho\left[-\sum\limits_{j\in I_{\rm s}^{\rm r}}\mathrm{Tr}([P_j-Q_j]A_j)-\sum\limits_{j\in I_{\rm s}^{\rm c}}\Re\{\mathrm{Tr}\left(S_jR_j^HL_j\right)\}\right.} \\
& \multicolumn{3}{l}{\left.+\sum\limits_{j\in I_{\rm f}^{\rm c}}\underbrace{\sqrt{\mathrm{Tr}(L_jZL_j^H)}}_{\|L_jZ^{1/2}\|_2}\underbrace{\sqrt{\mathrm{Tr}(R_jZR_j^H)}}_{\|R_jZ^{1/2}\|_2}\right]-\mathrm{Tr}(ZA)} \\
\text{s.t.} & & & \\
(a.1) & P_j,Q_j & \succeq & 0,\ j\in I_{\rm s}^{\rm r}, \\
(a.2) & P_j+Q_j & = & Z,\ j\in I_{\rm s}^{\rm r}; \\
(b) & \left[\begin{array}{cc} Z & S_j^H \\ S_j & Z\end{array}\right] & \succeq & 0,\ j\in I_{\rm s}^{\rm c}; \\
(c) & Z\succeq 0,\mathrm{Tr}(Z) & = & 1.
\end{array}
\tag{A.39}
$$

Next we eliminate the variables S_j, Q_j, R_j. It is clear that

1. (A.39.a) is equivalent to the fact that $P_j=Z^{1/2}\widehat{P}_jZ^{1/2}$, $Q_j=Z^{1/2}\widehat{Q}_jZ^{1/2}$ with $\widehat{P}_j,\widehat{Q}_j\succeq 0$, $\widehat{P}_j+\widehat{Q}_j=I_m$. With this parameterization of P_j,Q_j, the corresponding terms in the objective become $-2\rho\mathrm{Tr}([\widehat{P}_j-\widehat{Q}_j](Z^{1/2}A_jZ^{1/2}))$. Note that the matrices A_j are Hermitian (see (A.37)), and observe that if $A\in\mathcal{H}^m$, then

$$
\max_{P,Q\in\mathcal{H}^m}\{\mathrm{Tr}([P-Q]A):0\preceq P,Q,P+Q=I_m\}=\|\lambda(A)\|_1\equiv\sum_\ell|\lambda_\ell(A)|
$$

(w.l.o.g., we may assume that A is Hermitian and diagonal, in which case the relation becomes evident). In view of this observation, partial optimization in P_j, Q_j in (A.39) allows to replace in the objective of the problem the terms $-2\rho\mathrm{Tr}([P_j-Q_j]A_j)$ with $2\rho\|\lambda(Z^{1/2}A_jZ^{1/2})\|_1$ and to eliminate the constraints (A.39.a).

2. Same as in the proof of Lemma A.3, constraints (A.39.b) are equivalent to the fact that $S_j=-Z^{1/2}U_jZ^{1/2}$ with $\|U_j\|_{2,2}\leq 1$. With this parameterization, the corresponding terms in the objective become $2\rho\Re\{\mathrm{Tr}(U_j(Z^{1/2}R_j^HL_jZ^{1/2}))\}$,

and the maximum of this expression in U_j, $\|U_j\|_{2,2} \le 1$, is $2\rho\|\sigma(Z^{1/2}R_j^H L_j Z^{1/2})\|_1$. With this observation, partial optimization in S_j in (A.39) allows to replace in the objective the terms $-2\rho\Re\{\mathrm{Tr}(S_j R_j^H L_j)\}$ with $2\rho\|\sigma(Z^{1/2}R_j^H L_j Z^{1/2})\|_1$ and to eliminate the constraints (A.39.b).

After the above reductions, problem (A.39) becomes

$$\begin{array}{ll}\text{maximize} & 2\rho\left[\sum\limits_{j\in I_\mathrm{s}^\mathrm{r}}\|\lambda(Z^{1/2}A_jZ^{1/2})\|_1 + \sum\limits_{j\in I_\mathrm{s}^\mathrm{c}}\|\sigma(Z^{1/2}R_j^HL_jZ^{1/2})\|_1 \right.\\ & \left. \quad + \sum\limits_{j\in I_\mathrm{f}^\mathrm{c}}\|L_jZ^{1/2}\|_2\|R_jZ^{1/2}\|_2\right] - \mathrm{Tr}(ZA) \\ \text{s.t.} & Z \succeq 0, \mathrm{Tr}(Z) = 1.\end{array} \tag{A.40}$$

Recall that we are in the situation when the optimal value in problem (A.38), and thus in problem (A.40), is positive. Thus, we arrive at an intermediate conclusion as follows.

Lemma A.7 *Under the premise of Lemma A.6, there exists $Z \in \mathcal{H}^m$, $Z \succeq 0$, such that*

$$\begin{array}{l}2\rho\left[\sum\limits_{j\in I_\mathrm{s}^\mathrm{r}}\|\lambda(Z^{1/2}A_jZ^{1/2})\|_1 + \sum\limits_{j\in I_\mathrm{s}^\mathrm{c}}\|\sigma(Z^{1/2}R_j^HL_jZ^{1/2})\|_1 \right.\\ \left. \quad + \sum\limits_{j\in I_\mathrm{f}^\mathrm{c}}\|L_jZ^{1/2}\|_2\|R_jZ^{1/2}\|_2\right] > \mathrm{Tr}(Z^{1/2}AZ^{1/2}).\end{array} \tag{A.41}$$

Here the Hermitian matrices A_j are given by

$$2A_j = L_j^HR_j + R_j^HL_j,\ j \in I_\mathrm{s}^\mathrm{r}. \tag{A.42}$$

Second step: probabilistic interpretation of (A.41)

The major step in completing the proof of Theorem A.3.(iii) is based on a probabilistic interpretation of (A.41). This step is described next.

Preliminaries. Let us define a standard Gaussian vector ξ in $\mathbb{R}^n$ (notation: $\xi \in \mathcal{N}_\mathbb{R}^n$) as a real Gaussian random n-dimensional vector with zero mean and unit covariance matrix; in other words, ξ_ℓ are independent Gaussian random variables with zero mean and unit variance, $\ell = 1, ..., n$. Similarly, we define a standard Gaussian vector ξ in $\mathbb{C}^n$ (notation: $\xi \in \mathcal{N}_\mathbb{C}^n$) as a complex Gaussian random n-dimensional vector with zero mean and unit (complex) covariance matrix. In other words, $\xi_\ell = \alpha_\ell + i\alpha_{n+\ell}$, where $\alpha_1, ..., \alpha_{2n}$ are independent real Gaussian random variables with zero means and variances $\frac{1}{2}$, and i is the imaginary unit.

We shall use the facts established in the next three propositions.

Proposition A.1 *Let ν be a positive integer, and let $\vartheta_\mathbf{S}(\nu)$, $\vartheta_\mathcal{H}(\nu)$ be given by the relations*

$$\begin{array}{rcll}\vartheta_\mathbf{S}^{-1}(\nu) &=& \min\limits_\alpha\left\{\mathbb{E}_\xi\left\{|\sum\limits_{\ell=1}^\nu \alpha_\ell\xi_\ell^2|\right\} : \alpha \in \mathbb{R}^\nu, \|\alpha\|_1 = 1\right\} & [\xi \in \mathcal{N}_\mathbb{R}^\nu], \\ \vartheta_\mathcal{H}^{-1}(\nu) &=& \min\limits_\alpha\left\{\mathbb{E}_\chi\left\{|\sum\limits_{\ell=1}^\nu \alpha_\ell|\chi_\ell|^2|\right\} : \alpha \in \mathbb{R}^\nu, \|\alpha\|_1 = 1\right\} & [\chi \in \mathcal{N}_\mathbb{C}^\nu].\end{array} \tag{A.43}$$

Then

(i) *Both* $\vartheta_{\mathbf{S}}(\cdot)$, $\vartheta_{\mathcal{H}}(\cdot)$ *are nondecreasing functions such that*

$$\begin{array}{llcl} (a.1) & \vartheta_{\mathbf{S}}(1) & = & 1,\ \vartheta_{\mathbf{S}}(2)=\frac{\pi}{2},\\ (a.2) & \vartheta_{\mathbf{S}}(\nu) & \leq & \frac{\pi}{2}\sqrt{\nu},\ \nu\geq 1;\\ (b.1) & \vartheta_{\mathcal{H}}(1) & = & 1,\ \vartheta_{\mathcal{H}}(2)=2,\\ (b.2) & \vartheta_{\mathcal{H}}(\nu) & \leq & \vartheta_{\mathbf{S}}(2\nu)\leq\pi\sqrt{\nu/2},\ \nu\geq 1. \end{array} \tag{A.44}$$

(ii) *For every* $A\in\mathbf{S}^n$, *one has*

$$\mathbb{E}_\xi\left\{|\xi^TA\xi|\right\}\geq\|\lambda(A)\|_1\vartheta_{\mathbf{S}}^{-1}(\mathrm{Rank}(A))\quad[\xi\in\mathcal{N}_{\mathbb{R}}^n], \tag{A.45}$$

and for every $A\in\mathcal{H}^n$ *one has*

$$\mathbb{E}_\chi\left\{|\Theta^HA\chi|\right\}\geq\|\lambda(A)\|_1\vartheta_{\mathcal{H}}^{-1}(\mathrm{Rank}(A))\quad[\chi\in\mathcal{N}_{\mathbb{C}}^n]. \tag{A.46}$$

Proof.
1^0. Observe that $\vartheta_{\mathbf{S}}(\cdot)$ satisfies (A.45). Indeed, since $\xi\in\mathcal{N}_{\mathbb{R}}^n$ implies that $U\xi\in\mathcal{N}_{\mathbb{R}}^n$ for an orthogonal matrix U, it suffices to verify (A.45) for a diagonal matrix $A=\mathrm{Diag}\{\lambda_1,...,\lambda_\nu,0,...,0\}$, where $\nu=\mathrm{Rank}(A)$, in which case (A.45) is readily given by the definition of $\vartheta_{\mathbf{S}}(\cdot)$. By construction, $\vartheta_{\mathbf{S}}(\cdot)$ is nondecreasing. To check that $\vartheta_{\mathbf{S}}(\cdot)$ satisfies (A.44.a), let $\alpha\in\mathbb{R}^\nu$, $\|\alpha\|_1=1$, let $\beta=[\alpha;-\alpha]\in\mathbb{R}^{2\nu}$, and let $\xi\in\mathcal{N}_{\mathbb{R}}^{2\nu}$. Setting

$$J=\int\left|\sum_{i=1}^{\nu}u_i^2\alpha_i\right|p_\nu(u)du,$$

we have

$$\mathbf{E}\left\{\left|\sum_{i=1}^{2\nu}\xi_i^2\beta_i\right|\right\}\leq\mathbf{E}\left\{\left|\sum_{i=1}^{\nu}\xi_i^2\alpha_i\right|+\left|\sum_{i=\nu+1}^{2\nu}\xi_i^2\alpha_{i-\nu}\right|\right\}=2J. \tag{A.47}$$

On the other hand, setting $\eta_i=(\xi_i-\xi_{i+\nu})/\sqrt{2}$, $\zeta_i=(\xi_i+\xi_{i+\nu})/\sqrt{2}$, we get

$$\left|\sum_{i=1}^{2\nu}\xi_i^2\beta_i\right|=\left|\sum_{i=1}^{\nu}2\alpha_i\eta_i\zeta_i\right|=2\left|\widehat{\eta}^T\zeta\right|,\quad\widehat{\eta}=[\alpha_1\eta_1;...;\alpha_\nu\eta_\nu],\ \zeta=[\zeta_1;...;\zeta_\nu]. \tag{A.48}$$

Note that $\zeta\in\mathcal{N}_{\mathbb{R}}^\nu$ and $\widehat{\eta}$, ζ are independent. Setting $\widetilde{\eta}=[|\alpha_1\eta_1|;...;|\alpha_\nu\eta_\nu|]$, we have

$$\begin{array}{rcl} \mathbf{E}\left\{|\widehat{\eta}^T\zeta|\right\} & = & \mathbf{E}\left\{\|\widehat{\eta}\|_2\right\}\int|t|p_1(t)dt\\ & & \quad[\text{since }\widehat{\eta},\ \zeta\text{ are independent and }\zeta\in\mathcal{N}_{\mathbb{R}}^\nu]\\ & = & \mathbf{E}\left\{\|\widehat{\eta}\|_2\right\}\frac{2}{\sqrt{2\pi}}=\frac{2}{\sqrt{2\pi}}\mathbf{E}\left\{\|\widetilde{\eta}\|_2\right\}\\ & \geq & \frac{2}{\sqrt{2\pi}}\|\mathbf{E}\left\{\widetilde{\eta}\right\}\|_2=\frac{2}{\sqrt{2\pi}}\sqrt{\sum_{i=1}^{\nu}\alpha_i^2\left(\frac{2}{\sqrt{2\pi}}\right)^2}\geq\frac{2}{\pi\sqrt{\nu}}. \end{array} \tag{A.49}$$

Combining (A.47), (A.48) and (A.49), we get $2J \geq \frac{4}{\pi\sqrt{\nu}}$, i.e., $\frac{1}{J} \leq \frac{\pi\sqrt{\nu}}{2}$, which yields (A.44.$a$.2). Relation (A.44.$a$.1) is given by the following computation:

$$\begin{array}{rl} & \frac{1}{\vartheta_{\mathbf{S}}(2)} \\ = & \min\limits_{\substack{\alpha\in\mathbb{R}^2,\\ \|\alpha\|_1=1}} \left\{\int \left|\alpha_1 u_1^2 + \alpha_2 u_2^2\right| p_2(u)du\right\} = \min\limits_{\theta\in[0,1]} \underbrace{\int \left|\theta u_1^2 - (1-\theta)u_2^2\right| p_2(u)du}_{f(\theta)} \\ = & \frac{1}{2}\int \left|u_1^2 - u_2^2\right| p_2(u)du \\ & \qquad\qquad [\text{since } f(\theta) \text{ is convex and symmetric w.r.t. } \theta = 1/2] \\ = & \left[\int |t| p_1(t)dt\right]^2 = \frac{2}{\pi}. \end{array}$$

2^0. From the definition of $\vartheta_{\mathcal{H}}(\cdot)$ it is clear that this function is nondecreasing. To establish (A.46), by the same reasons as in the case of (A.45), it suffices to verify (A.46) when $A = \mathrm{Diag}\{\lambda_1, ..., \lambda_\nu, 0, ..., 0\}$, where $\nu = \mathrm{Rank}(A)$, in which case (A.46) is readily given by the definition of $\vartheta_{\mathcal{H}}(\cdot)$.

It remains to verify (A.44.b). The relation $\vartheta_{\mathcal{H}}(1) = 1$ is evident. Further, we clearly have

$$\vartheta_{\mathcal{H}}^{-1}(2) = \min_{\beta\in[0,1]} \psi(\beta), \quad \psi(\beta) = \mathbb{E}_\chi\left\{\left|\beta|\chi_1|^2 - (1-\beta)|\chi_2|^2\right|\right\}, \ \chi \in \mathcal{N}_{\mathbb{C}}^2.$$

The function $\psi(\beta)$ is convex in $\beta \in [0,1]$ and is symmetric: $\psi(1-\beta) = \psi(\beta)$. It follows that its minimum is achieved at $\beta = \frac{1}{2}$; direct computation demonstrates that $\psi(1/2) = 1/2$, which completes the proof of (A.44.b.1).

It remains to prove the first inequality in (A.44.b.2). Given $\alpha \in \mathbb{R}^\nu$, $\|\alpha\|_1 = 1$, let $\widetilde{\alpha} = [\alpha; \alpha] \in \mathbb{R}^{2\nu}$. Now, if $\chi = \eta + \imath\omega$ is a standard Gaussian vector in $\mathbb{C}^\nu$, then the vector $\xi = 2^{1/2}[\eta; \omega]$ is a standard Gaussian vector in $\mathbb{R}^{2\nu}$. We now have

$$\begin{array}{rcl} \mathbb{E}_\chi\left\{|\sum\limits_{\ell=1}^{\nu} \alpha_\ell |\chi_\ell|^2|\right\} & = & \mathbb{E}_\chi\left\{|\sum\limits_{\ell=1}^{\nu} \alpha_\ell[\eta_\ell^2 + \omega_\ell^2]|\right\} = \frac{1}{2}\mathbb{E}_\xi\left\{|\sum\limits_{\ell=1}^{2\nu} \widetilde{\alpha}_\ell \xi_\ell^2|\right\} \\ & \geq & \frac{1}{2}\|\widetilde{\alpha}\|_1 \vartheta_{\mathbf{S}}^{-1}(2\nu) = \vartheta_{\mathbf{S}}^{-1}(2\nu), \end{array}$$

whence $\vartheta_{\mathcal{H}}^{-1}(\nu) \geq \vartheta_{\mathbf{S}}^{-1}(2\nu)$, and the desired inequality follows. □

Proposition A.2 *For every $A \in \mathbb{C}^{n\times n}$ one has*

$$\mathbb{E}_\chi\left\{|\Theta^H A\chi|\right\} \geq \|\sigma(A)\|_1 \frac{1}{4}\vartheta_{\mathcal{H}}^{-1}(2\mathrm{Rank}(A)) \quad [\chi \in \mathcal{N}_{\mathbb{C}}^n]. \tag{A.50}$$

Proof.
Let $\widehat{A} = \left[\begin{array}{cc} & A \\ A^H & \end{array}\right]$, so that $\widehat{A} \in \mathcal{H}^{2n}$, $\mathrm{Rank}(\widehat{A}) = 2\mathrm{Rank}(A)$ and the eigenvalues of $\widehat{A}$ are $\pm\sigma_\ell(A)$, $\ell = 1, ..., n$. Let also $\chi = [\eta; \omega]$ be a standard Gaussian vector in $\mathbb{C}^{2n}$ partitioned into two n-dimensional blocks, so that η, ω are independent

standard Gaussian vectors in $\mathbb{C}^n$. We have

$$\begin{array}{rcl}
\Theta^H \widehat{A}\chi & = & 2\Re\{\eta^H A\omega\} \\
& = & \Re\Big\{ \Big[(\eta+\omega)^H A(\eta+\omega) - \eta^H A\eta - \omega^H A\omega\Big] \\
& & \quad +\imath\Big[(\eta-\imath\omega)^H A(\eta-\imath\omega) - \eta^H A\eta - \omega^H A\omega\Big]\Big\} \\
& & \text{[polarization identity]} \\
\Rightarrow \mathbb{E}_\chi\left\{|\Theta^H \widehat{A}\chi|\right\} & \leq & \mathbb{E}_{\eta,\omega}\left\{|(\eta+\omega)^H A(\eta+\omega)|\right\} \\
& & +\mathbb{E}_{\eta,\omega}\left\{|(\eta-\imath\omega)^H A(\eta-\imath\omega)|\right\} \\
& & +2\mathbb{E}_\eta\left\{|\eta^H A\eta|\right\} + 2\mathbb{E}_\omega\{|\Theta^H A\chi|\}.
\end{array} \tag{A.51}$$

Since η, ω are independent standard Gaussian vectors in $\mathbb{C}^n$, the vectors $2^{-1/2}(\eta+\omega)$ and $2^{-1/2}(\eta - \imath\omega)$ also are standard Gaussian. Therefore (A.51) implies that

$$\mathbb{E}_\chi\left\{|\Theta^H \widehat{A}\chi|\right\} \leq 8\mathbb{E}_\eta\left\{|\eta^H A\eta|\right\}. \tag{A.52}$$

Since $\widehat{A}$ is a Hermitian matrix of rank 2Rank(A) and $\|\lambda(\widehat{A})\|_1 = 2\|\sigma(A)\|_1$, the left hand side in (A.52) is at least $2\|\sigma(A)\|_1\vartheta_{\mathcal{H}}^{-1}(2\mathrm{Rank}(A))$, and (A.52) implies (A.50). □

Proposition A.3 (i) *Let $L \in \mathbb{C}^{p\times n}, R \in \mathbb{C}^{q\times n}$, and let χ be a standard Gaussian vector in $\mathbb{C}^n$. Then*

$$\mathbf{E}_\chi\left\{\|L\chi\|_2\|R\chi\|_2\right\} \geq \frac{\pi}{4}\|L\|_2\|R\|_2. \tag{A.53}$$

(ii) *Let $L \in \mathbb{R}^{p\times n}, R \in \mathbb{R}^{q\times n}$, and let ξ be a standard Gaussian vector in $\mathbb{R}^n$. Then*

$$\mathbf{E}_\xi\left\{\|L\xi\|_2\|R\xi\|_2\right\} \geq \frac{2}{\pi}\|L\|_2\|R\|_2. \tag{A.54}$$

Proof.
(i): There is nothing to prove when L or R are zero matrices; thus, assume that both L and R are nonzero.

Let us demonstrate first that it suffices to verify (A.53) in the case when both L and R are rank 1 matrices. Let $L^H L = U^H \mathrm{Diag}\{\lambda\}U$ be the eigenvalue decomposition of $L^H L$, so that U is a unitary matrix and $\lambda \geq 0$. We have

$$\begin{array}{rcl}
\mathbf{E}\left\{\|L\xi\|_2\|R\xi\|_2\right\} & = & \mathbf{E}\left\{\sqrt{\xi^H L^H L\xi}\|R\xi\|_2\right\} \\
& = & \mathbf{E}\left\{\Big((U\xi)^H \mathrm{Diag}\{\lambda\}\underbrace{(U\xi)}_{\chi}\Big)^{1/2}\underbrace{\|RU^H\chi\|_2}_{\phi(\chi)\geq 0}\right\} \\
& = & \mathbf{E}\left\{\phi(\chi)\sqrt{\sum\limits_{\ell=1}^n \lambda_\ell|\chi_\ell|^2}\right\} = \Phi(\lambda), \\
\Phi(x) & = & \mathbf{E}\left\{\phi(\chi)\sqrt{\sum\limits_{\ell=1}^n x_\ell|\chi_\ell|^2}\right\}.
\end{array} \tag{A.55}$$

The function $\Phi(x)$ of $x \in \mathbb{R}^n_+$ is concave; therefore its minimum on the simplex

$$S = \{x \in \mathbb{R}^n_+ : \sum_\ell x_\ell = \sum_\ell \lambda_\ell\}$$

is achieved at a vertex, let it be e. Now let $\widehat{L} \in \mathbb{C}^{d\times n}$ be such that $\widehat{L}^H\widehat{L} = U^H\mathrm{Diag}\{e\}U$. Note that $\widehat{L}$ is a rank 1 matrix (since e is a vertex of S) and that

$$[\|\widehat{L}\|_2^2 =] \quad \mathrm{Tr}(\widehat{L}^H\widehat{L}) = \sum_\ell e_\ell = \sum_\ell \lambda_\ell = \mathrm{Tr}(L^HL) \quad [= \|L\|_2^2].$$

Since the unitary factor in the eigenvalue decomposition of $\widehat{L}^H\widehat{L}$ is U, (A.55) holds true when L is replaced with $\widehat{L}$ and λ with e, so that

$$\mathbf{E}\left\{\|\widehat{L}\chi\|_2\|R\chi\|_2\right\} = \Phi(e) \le \Phi(\lambda) = \mathbf{E}\left\{\|L\chi\|_2\|R\chi\|_2\right\}.$$

Applying the same reasoning to the quantity

$$\mathbf{E}\left\{\|\widehat{L}\chi\|_2\|R\chi\|_2\right\}$$

with R playing the role of L, we conclude that there exists a rank 1 matrix $\widehat{R}$ such that

$$\|\widehat{R}\|_2 = \|R\|_2$$

and

$$\mathbf{E}\left\{\|\widehat{L}\chi\|_2\|\widehat{R}\chi\|_2\right\} \le \mathbf{E}\left\{\|\widehat{L}\chi\|_2\|R\chi\|_2\right\}.$$

Thus, replacing L and R with the rank 1 matrices $\widehat{L}$, $\widehat{R}$, we do not increase the left hand side in (A.53) and do not vary the right hand side, so that it indeed suffices to establish (A.53) in the case when L, R are rank 1 matrices. Note that so far our reasoning did not use the fact that χ is standard Gaussian.

Now let us look what inequality (A.53) says in the case of rank 1 matrices L, R. By homogeneity, we can further assume that $\|L\|_2 = \|R\|_2 = 1$. With this normalization, for rank 1 matrices L, R we clearly have $L\chi = z\ell$ and $R\chi = wr$ for unit deterministic vectors ℓ, r and a Gaussian random vector $[z; w] \in \mathbb{C}^2 = \mathbb{R}^4$ such that $\mathbf{E}\left\{|z|^2\right\} = \mathbf{E}\left\{|w|^2\right\} = 1$ (both z and w are just linear combinations, with appropriate deterministic coefficients, of the entries in χ). Since $\mathbf{E}\left\{|z|^2\right\} = \mathbf{E}\left\{|w|^2\right\} = 1$, we can express (z, w) in terms of a *standard* Gaussian vector $[\eta; \xi] \in \mathbb{C}^2$ as $z = \eta$, $w = \cos(\theta)\eta + \sin(\theta)\xi$, where $\theta \in [0, \frac{\pi}{2}]$ is such that $\cos(\theta)$ is the absolute value of the correlation $\mathbf{E}\{z\overline{w}\}$ between z and w. With this representation, inequality (A.53) becomes

$$\phi(\theta) \equiv \int\limits_{\mathbb{C}\times\mathbb{C}} |\eta||\cos(\theta)\eta + \sin(\theta)\xi| dG(\eta, \xi) \ge \frac{\pi}{4} \equiv \phi(\frac{\pi}{2}), \qquad \text{(A.56)}$$

where $G(\eta, \xi)$ is the distribution of $[\eta; \xi]$. We should prove (A.56) in the range $[0, \frac{\pi}{2}]$ of values of θ; in fact we shall prove this inequality in the larger range $\theta \in [0, \pi]$. Given $\theta \in [0, \pi]$, we set

$$u = \cos(\theta/2)\eta + \sin(\theta/2)\xi, \quad v = -\sin(\theta/2)\eta + \cos(\theta/2)\xi;$$

it is immediately seen that the distribution of (u,v) is exactly G. At the same time,

$$\eta = \cos(\theta/2)u - \sin(\theta/2)v, \quad \cos(\theta)\eta + \sin(\theta)\xi = \cos(\theta/2)u + \sin(\theta/2)v,$$

whence

$$\begin{aligned}\phi(\theta) &= \int\limits_{\mathbb{C}\times\mathbb{C}} |\cos(\theta/2)u - \sin(\theta/2)v||\cos(\theta/2)u + \sin(\theta/2)v|dG(u,v)\\ &= \int\limits_{\mathbb{C}\times\mathbb{C}} |\cos^2(\theta/2)u^2 - \sin^2(\theta/2)v^2|dG(u,v).\end{aligned}$$

We see that

$$\min_{\theta\in[0,\pi]}\phi(\theta) = \min_{0\leq\alpha\leq 1}\psi(\alpha), \quad \psi(\alpha) = \int\limits_{\mathbb{C}\times\mathbb{C}} |\alpha u^2 - (1-\alpha)v^2|dG(u,v).$$

The function $\psi(\alpha)$ clearly is convex and $\psi(1-\alpha) = \psi(\alpha)$ (since the distribution of $[u;v]$ is symmetric in u,v). Consequently, ψ attains its minimum when $\alpha = 1/2$, and ϕ attains its minimum when $\cos^2(\theta/2) = 1/2$, i.e., when $\theta = \pi/2$, which is exactly what is stated in (A.56).

(ii): Applying exactly the same reasoning as in the proof of (i), we conclude that it suffices to verify (A.54) in the case when L, R are real rank 1 matrices. In this case, the same argument as above demonstrates that (A.54) is equivalent to the fact that if ξ,η are independent real standard Gaussian variables and $G(\xi,\eta)$ is the distribution of $[\xi;\eta]$, then the function

$$\phi(\theta) = \int\limits_{\mathbb{R}\times\mathbb{R}} |\xi||\cos(\theta)\xi + \sin(\theta)\eta|dG(\xi,\eta) \tag{A.57}$$

of $\theta\in[0,\pi]$ achieves its minimum when $\theta = \frac{\pi}{2}$. To prove this statement, one can repeat word by word, with evident modifications, the reasoning we have used in the complex case. □

A.4.5 Completing the proof of Theorem A.3.(iii)

We are now in a position to complete the proof of Theorem A.3.(iii). Let us set

$$\begin{array}{rcl} p^{\rm s}_{\mathbb{R}} &=& 2\max\left\{p_j : j\in I^{\rm r}_{\rm s}\right\},\\ p^{\rm s}_{\mathbb{C}} &=& 2\max\left\{p_j : j\in I^{\rm c}_{\rm s}\right\},\\ \vartheta_{\mathbf{S}} &=& \max\left[\vartheta_{\mathcal{H}}(p^{\rm s}_{\mathbb{R}}), 4\vartheta_{\mathcal{H}}(p^{\rm s}_{\mathbb{C}}), \frac{4}{\pi}\right];\end{array} \tag{A.58}$$

here by definition the maximum over an empty set is 0, and $\vartheta_{\mathcal{H}}(0) = 0$. Note that by (A.44) one has

$$\vartheta_{\mathbf{S}} \leq 4\pi\sqrt{p^{\rm s}}$$

(cf. (A.22), (A.23)).

Let χ be a standard Gaussian vector in $\mathbb{C}^n$. Invoking Propositions A.1 – A.3, we have (for notation, see Lemma A.7):

$$\begin{array}{ll}
& \|\lambda(Z^{1/2}A_jZ^{1/2})\|_1 \\
\le & \vartheta_{\mathcal{H}}(\mathrm{Rank}(Z^{1/2}A_jZ^{1/2}))\mathbb{E}_\chi\left\{|\Theta^H Z^{1/2}A_jZ^{1/2}\chi|\right\} \\
\le & \vartheta_{\mathbf{S}}\mathbb{E}_\chi\left\{|\Theta^H Z^{1/2}A_jZ^{1/2}\chi|\right\},\ j\in I_{\mathrm{s}}^{\mathrm{r}} \\
& \qquad\left[\begin{array}{l}\text{by Proposition A.1 since } A_j=A_j^H \text{ and}\\ \mathrm{Rank}(A_j)=\mathrm{Rank}([L_j^HR_j+R_j^HL_j])\le 2p_j\end{array}\right] \\
& \|\sigma(Z^{1/2}R_j^HL_jZ^{1/2})\|_1 \\
\le & 4\vartheta_{\mathcal{H}}(2\mathrm{Rank}(Z^{1/2}R_j^HL_jZ^{1/2}))\mathbb{E}_\chi\left\{|\Theta^H Z^{1/2}R_j^HL_jZ^{1/2}\chi|\right\} \\
\le & \vartheta_{\mathbf{S}}\mathbb{E}_\chi\left\{|\Theta^H Z^{1/2}R_j^HL_jZ^{1/2}\chi|\right\},\ j\in I_{\mathrm{s}}^{\mathrm{c}} \\
& \qquad[\text{by Proposition A.2 since } \mathrm{Rank}(R_j^HL_j)\le p_j] \\
& \|L_jZ^{1/2}\|_2\|R_jZ^{1/2}\|_2 \\
\le & \frac{4}{\pi}\mathbb{E}_\chi\left\{\|L_jZ^{1/2}\chi\|_2\|R_jZ^{1/2}\chi\|_2\right\} \\
\le & \vartheta_{\mathbf{S}}\mathbb{E}_\chi\left\{\|L_jZ^{1/2}\chi\|_2\|R_jZ^{1/2}\chi\|_2\right\} \\
& \qquad[\text{by Proposition A.3.(i)}]
\end{array}$$

and, of course,

$$\mathbb{E}_\chi\left\{\Theta^H Z^{1/2}AZ^{1/2}\chi\right\}=\mathrm{Tr}(Z^{1/2}AZ^{1/2}).$$

In view of these observations, (A.41) implies that

$$\begin{array}{l}
\rho\vartheta_{\mathbf{S}}\left[\sum\limits_{j\in I_{\mathrm{s}}^{\mathrm{r}}}\mathbb{E}_\chi\left\{|\Theta^H Z^{1/2}[L_j^HR_j+R_j^HL_j]Z^{1/2}\chi|\right\}\right. \\
+\sum\limits_{j\in I_{\mathrm{s}}^{\mathrm{c}}}\mathbb{E}_\chi\left\{2|\Theta^H Z^{1/2}R_j^HL_jZ^{1/2}\chi|\right\} \\
\qquad\left.+\sum\limits_{j\in I_{\mathrm{f}}^{\mathrm{c}}}\mathbb{E}_\chi\left\{2\|L_jZ^{1/2}\chi\|_2\|R_jZ^{1/2}\chi\|_2\right\}\right]>\mathbb{E}_\chi\left\{\Theta^H Z^{1/2}AZ^{1/2}\chi\right\}
\end{array}$$

(we have substituted the expressions for A_j, see (A.42)). It follows that there exists a realization $\widehat{\chi}$ of χ such that with $\xi=Z^{1/2}\widehat{\chi}$ one has

$$\begin{array}{l}
\rho\vartheta_{\mathbf{S}}\left[\sum\limits_{j\in I_{\mathrm{s}}^{\mathrm{r}}}|\xi^H[L_j^HR_j+R_j^HL_j]\xi|+\sum\limits_{j\in I_{\mathrm{s}}^{\mathrm{c}}}2|\xi^HR_j^HL_j\xi|+\sum\limits_{j\in I_{\mathrm{f}}^{\mathrm{c}}}2\|L_j\xi\|_2\|R_j\xi\|_2\right] \\
>\xi^HA\xi.
\end{array}\tag{A.59}$$

Observe that

- The quantities $\xi^H[L_j^HR_j+R_j^HL_j]\xi$ are real; we therefore can choose $\theta_j=\pm1$, $j\in I_{\mathrm{s}}^{\mathrm{r}}$, in such a way that with $\Theta^j=\theta_jI_{p_j}$ one has

$$\xi^H[L_j^H\Theta^jR_j+R_j^H[\Theta^j]^HL_j]\xi=|\xi^H[L_j^HR_j+R_j^HL_j]\xi|,\ j\in I_{\mathrm{s}}^{\mathrm{r}};$$

- For $j\in I_{\mathrm{s}}^{\mathrm{c}}$, we can choose $\theta_j\in\mathbb{C}$, $|\theta_j|=1$, in such a way that with $\Theta^j=\theta_jI_{p_j}$ one has

$$\xi^H[L_j^H\Theta^jR_j+R_j^H[\Theta^j]^HL_j]\xi=2|\xi^HR_j^HL_j\xi|,\ j\in I_{\mathrm{s}}^{\mathrm{c}};$$

- For $j \in I_{\rm f}^{\rm c}$, we can choose $\Theta^j \in \mathbb{C}^{p_j \times q_j}$, $\|\Theta^j\|_{2,2} \le 1$, in such a way that
$$\xi^H[L_j^H \Theta^j R_j + R_j^H [\Theta^j]^H L_j]\xi = 2\|L_j\xi\|_2 \|R_j\xi\|_2,\ j \in I_{\rm f}^{\rm c}.$$

With Θ^j's we have defined, (A.59) reads
$$\xi^H \Big[\underbrace{A - \rho\vartheta_{\mathbf{S}} \sum_{j=1}^{L} [L_j^H \Theta^j R_j + R_j^H [\Theta^j]^H L_j]}_{C}\Big] \xi < 0,$$
so that C is not positive semidefinite; on the other hand, by construction $C \in \mathcal{U}[\vartheta_{\mathbf{S}}\rho]$. Thus, the predicate $\mathcal{A}(\vartheta_{\mathbf{S}}\rho)$ is not valid; recalling the definition of $\vartheta_{\mathbf{S}}$, this completes the proof of Lemma A.6. and thus the proof of Theorem A.3.(iii). □

A.4.6 Proof of Theorem A.3.(iv)

The fact that $\mathcal{A}(\rho)$ is equivalent to $\mathcal{B}(\rho)$ in the case of $L = 1$ is evident when the only perturbation block in question is a real scalar one, is readily given by Lemma A.3 when the block is a complex scalar one, and is readily given by Lemma A.4 when the block is full. □

A.4.7 Matrix Cube Theorem, Real case

The Real Matrix Cube problem is as follows:

RMC: *Let m, $p_1, q_1, ..., p_L, q_L$ be positive integers, and $A \in \mathbf{S}^m$, $L_j \in \mathbb{R}^{p_j \times m}$, $R_j \in \mathbb{R}^{q_j \times m}$ be given matrices, $L_j \neq 0$. Let also a partition $\{1, 2, ..., L\} = I_{\rm s}^{\rm r} \cup I_{\rm f}^{\rm r}$ of the index set $\{1, ..., L\}$ into two non-overlapping sets be given. With these data, we associate a parametric family of "matrix boxes"*
$$\mathcal{U}[\rho] = \left\{ A + \rho \sum_{j=1}^{L} [L_j^T \Theta^j R_j + R_j^T [\Theta^j]^T L_j] : \Theta^j \in \mathcal{Z}^j, 1 \le j \le L \right\} \subset \mathbf{S}^m, \tag{A.60}$$
where $\rho \ge 0$ is the parameter and
$$\mathcal{Z}^j = \left\{ \begin{array}{l} \{\theta I_{p_j} : \theta \in \mathbb{R}, |\theta| \le 1\}, j \in I_{\rm s}^{\rm r} \\ \qquad [\text{“scalar perturbation blocks”}] \\ \{\Theta^j \in \mathbb{R}^{p_j \times q_j} : \|\Theta^j\|_{2,2} \le 1\}, j \in I_{\rm f}^{\rm r} \\ \qquad [\text{“full perturbation blocks”}] \end{array} \right. . \tag{A.61}$$
Given $\rho \ge 0$, check whether
$$\mathcal{U}[\rho] \subset \mathbf{S}^m_+ \qquad \mathcal{A}(\rho)$$

Remark A.4 *In the sequel, we always assume that $p_j > 1$ for $j \in I_{\rm s}^{\rm r}$. Indeed, non-repeated ($p_j = 1$) scalar perturbations always can be regarded as full perturbations.*

Consider, along with predicate $\mathcal{A}(\rho)$, the predicate

$$\begin{array}{ll}\exists Y_j \in \mathbf{S}^m,\ j=1,...,L: & \\ (a)\quad Y_j \succeq L_j^T \Theta^j R_j + R_j^T [\Theta^j]^T L_j \ \forall \left(\Theta^j \in \mathcal{Z}^j, 1 \le j \le L\right) & \\ (b)\quad A - \rho \sum_{j=1}^{L} Y_j \succeq 0. & \end{array} \qquad \mathcal{B}(\rho)$$

The Real case version of Theorem A.3 is as follows:

Theorem A.4 [The Real Matrix Cube Theorem] *One has:*

(i) *Predicate $\mathcal{B}(\rho)$ is stronger than $\mathcal{A}(\rho)$ – the validity of the former predicate implies the validity of the latter one.*

(ii) *$\mathcal{B}(\rho)$ is computationally tractable – the validity of the predicate is equivalent to the solvability of the system of LMIs*

$$\begin{array}{lccl} (s) & Y_j \pm \left[L_j^T R_j + R_j^T L_j\right] & \succeq & 0,\ j \in I_{\rm s}^{\rm r}, \\ (f) & \left[\begin{array}{cc} Y_j - \lambda_j L_j^T L_j & R_j^T \\ R_j & \lambda_j I_{p_j}\end{array}\right] & \succeq & 0,\ j \in I_{\rm f}^{\rm r} \\ (*) & A - \rho \sum_{j=1}^{L} Y_j & \succeq & 0. \end{array} \qquad (A.62)$$

in matrix variables $Y_j \in \mathbf{S}^m$, $j = 1, ..., L$, and real variables λ_j, $j \in I_{\rm f}^{\rm r}$.

(iii) *"The gap" between $\mathcal{A}(\rho)$ and $\mathcal{B}(\rho)$ can be bounded solely in terms of the maximal rank*

$$p^{\rm s} = \max_{j \in I_{\rm s}^{\rm r}} \mathrm{Rank}(L_j^T R_j + R_j^T L_j)$$

of the scalar perturbations. Specifically, there exists a universal function $\vartheta_{\mathbb{R}}(\cdot)$ satisfying the relations

$$\vartheta_{\mathbb{R}}(2) = \frac{\pi}{2};\ \vartheta_{\mathbb{R}}(4) = 2;\ \vartheta_{\mathbb{R}}(\mu) \le \pi\sqrt{\mu}/2\ \forall \mu \ge 1$$

such that with $\mu = \max[2, p^{\rm s}]$ one has

$$\text{if } \mathcal{B}(\rho) \text{ is not valid, then } \mathcal{A}(\vartheta_{\mathbb{R}}(\mu)\rho) \text{ is not valid.} \qquad (A.63)$$

(iv) *Finally, in the case $L = 1$ of single perturbation block $\mathcal{A}(\rho)$ is equivalent to $\mathcal{B}(\rho)$.*

The proof of the Real Matrix Cube Theorem repeats word by word, with evident simplifications, the proof of its complex case counterpart and is therefore omitted. Note that Remark A.3 remains valid in the real case.

A.5 Theorem 13.2

Let

$$\mathrm{Erf}(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty \exp\{-s^2/2\}ds, \quad \mathrm{ErfInv}(r) : \frac{1}{\sqrt{2\pi}} \int_{\mathrm{ErfInv}(r)}^\infty \exp\{-s^2/2\}ds = r.$$

Theorem A.5 *Let $\zeta \sim \mathcal{N}(0, I_m)$, and let Q be a closed convex set in $\mathbf{R}^m$ such that*

$$\mathrm{Prob}\{\zeta \in Q\} \geq \chi > \frac{1}{2}. \tag{A.64}$$

Then

$$\alpha \geq 1 \Rightarrow \mathrm{Prob}\{\zeta \notin \alpha Q\} \leq \exp\left\{-\frac{\mathrm{ErfInv}^2(1-\chi)\alpha^2}{2}\right\}. \tag{A.65}$$

Equivalently: for a closed and convex set Q and $\zeta \sim \mathcal{N}(0, \Sigma)$ one has

$$\mathrm{Prob}\left\{\zeta \notin Q\right\} \leq \delta < \frac{1}{2} \Rightarrow \mathrm{Prob}\left\{\zeta \notin \alpha Q\right\} \leq \exp\{-\frac{\mathrm{ErfInv}^2(\delta)\alpha^2}{2}\} \ \forall \alpha > 1. \tag{A.66}$$

Proof.
We need the following fact [17]:

> (!) For every $\alpha > 0$, $\epsilon \geq 0$ and every closed set $X \subset \mathbf{R}^k$ such that $\mathrm{Prob}\{\zeta \in X\} \geq \alpha$ one has
>
> $$\mathrm{Prob}\left\{\mathrm{dist}(\zeta, X) > \epsilon\right\} \leq \mathrm{Erf}(\mathrm{ErfInv}(1-\alpha) + \epsilon)$$
>
> where $\mathrm{dist}(a, X) = \min_{x \in X} \|a - x\|_2$.

Now let ζ, η be independent $\mathcal{N}(0, I_m)$ random vectors, let $\alpha \geq 1$, and let

$$p(\alpha) = \mathrm{Prob}\{\zeta \notin \alpha Q\}.$$

Setting $\gamma = 1/\alpha$, observe that the vector $\gamma\zeta + \sqrt{1-\gamma^2}\eta$ is $\mathcal{N}(0, I_m)$, so that whenever $t \geq 0$, we have

$$\mathrm{Prob}\{\mathrm{dist}(\gamma\zeta + \sqrt{1-\gamma^2}\eta, Q) > t\} \leq \mathrm{Erf}(\mathrm{ErfInv}(1-\chi) + t) \tag{A.67}$$

by (!). On the other hand, let $\zeta \notin \alpha Q$ (i.e., let $\gamma\zeta \notin Q$), and let $e = e(\zeta)$ be a unit vector such that $e^T[\gamma\zeta] > \max_{x \in Q} e^T x$. If η is such that $\sqrt{1-\gamma^2}e^T\zeta > t$, then $\mathrm{dist}(\gamma\zeta + \sqrt{1-\gamma^2}\zeta, Q) > t$, whence

$$\zeta \notin \alpha Q \Rightarrow \mathrm{Prob}\left\{\zeta : \mathrm{dist}(\gamma\zeta + \sqrt{1-\gamma^2}\eta, Q) > t\right\} \geq \mathrm{Erf}(t/\sqrt{1-\gamma^2}),$$

whence for all $t \geq 0$ such that $\delta(t) \equiv \mathrm{ErfInv}(1-\chi) + t - t/\sqrt{1-\gamma^2} \geq 0$ one has

$$\begin{array}{rcl} p(\alpha)\mathrm{Erf}(t/\sqrt{1-\gamma^2}) & \leq & \mathrm{Prob}\{\mathrm{dist}(\gamma\zeta + \sqrt{1-\gamma^2}\zeta, Q) > t\} \\ & \leq & \mathrm{Erf}(\mathrm{ErfInv}(1-\chi)+t) \\ \Rightarrow p(\alpha) & \leq & \frac{\mathrm{Erf}(\mathrm{ErfInv}(1-\chi)+t)}{\mathrm{Erf}(t/\sqrt{1-\gamma^2})} = \frac{\int_{t/\sqrt{1-\gamma^2}}^{\infty} \exp\{-(s+\delta(t))^2/2\}ds}{\int_{t/\sqrt{1-\gamma^2}}^{\infty} \exp\{-s^2/2\}ds} \\ & = & \frac{\int_{t/\sqrt{1-\gamma^2}}^{\infty} \exp\{-s^2/2 - s\delta(t) - \delta^2(t)/2\}ds}{\int_{t/\sqrt{1-\gamma^2}}^{\infty} \exp\{-s^2/2\}ds} \\ & \leq & \exp\{-t\delta(t)/\sqrt{1-\gamma^2} - \delta^2(t)/2\}. \end{array}$$

Setting in the resulting inequality $t = \frac{\mathrm{ErfInv}(1-\chi)(1-\gamma^2)}{\gamma^2}$, we get

$$p(\alpha) \leq \exp\{-\frac{\mathrm{ErfInv}^2(1-\chi)\alpha^2}{2}\}. \qquad \square$$

A.6 Proof of Theorem 7.1

Theorem 7.1 is an immediate corollary of the following statement:

Theorem A.6 *Let $\Psi(z) : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $\psi(u) : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function with bounded level sets such that*

$$\Psi(z) = \sup_u \left\{ z^T(Bu+b) - \psi(u) \right\}. \tag{A.68}$$

Let, further, real ρ and a direction $e \in \mathbb{R}^L$ be such that

$$\rho < \Psi(0) \tag{A.69}$$

and

$$\lim_{t\to\infty} \Psi(z+te) < \rho \ \forall z \in \mathbb{R}^n. \tag{A.70}$$

Let us set

$$Z_o^\rho = \{z : \exists \alpha > 0 : \Psi(\alpha z) \leq \rho\}, \ Z^\rho = \mathrm{cl}\, Z_o^\rho.$$

Then the set

$$\mathcal{U}^\rho = \{u : \psi(u) \leq -\rho\}$$

is a nonempty convex compact set and

$$z \in Z^\rho \Leftrightarrow z^T(Bu+b) \leq 0 \, \forall u \in \mathcal{U}^\rho. \tag{A.71}$$

Theorem A.6 ⇒ Theorem 7.1. Let us set $\Psi(z_0, z1, ..., z_L) = z_0 + \Phi([z_1; ...; z_L])$, $\psi(\cdot) \equiv \phi(\cdot)$, $Bu + b = [1; Au + a]$, $\rho = \ln(\epsilon)$, $e = [-1; 0; ...; 0]$. These data clearly satisfy the premise in Theorem A.6. It remains to note that $Z_o^\rho = Z_\epsilon^0$ (so that $Z^\rho = Z_\epsilon$), $\mathcal{U}^\rho = \mathcal{U}_\epsilon$ and $z^T(Bu+b) \equiv z_0 + [z_1; ...; z_L]^T(Au+a)$.

Proof of Theorem A.6. 1^0. First, let us verify that

$$\inf_u \psi(u) = -\Psi(0) \tag{A.72}$$

and extract from this relation that $\mathcal{U}^\rho$ is a nonempty compact convex set.

Indeed, by (A.68) we have

$$\Psi(0) = \sup_u \left\{ 0^T(Bu+b) - \phi(u) \right\} = -\inf_u \psi(u),$$

and (A.72) follows. Now, since $\rho < \Phi(0)$, (A.72) says that $-\rho > \inf_u \psi(u)$, so that the set $\mathcal{U}^\rho$ is nonempty. This set is convex, closed and bounded due to the fact that ψ is a convex lower semicontinuous function with bounded level sets.

2^0. The result of 1^0 states that $\mathcal{U}^\rho$ is a nonempty convex compact set, which is the first statement of Theorem A.6. To complete the proof, we need to justify the equivalence in (A.71), which is the goal of items 3^0 and 4^0 to follow.

3^0. We claim that whenever $z \in Z^\rho$, one has

$$z^T(Bu+b) \le 0 \;\forall u \in \mathcal{U}^\rho. \tag{A.73}$$

Indeed, assuming, on the contrary, that $z^T(B\bar{u}+b) > 0$ for certain $\bar{u}$ with $\psi(\bar{u}) \le -\rho$, observe that there exists a neighborhood U_z of z and $\delta > 0$ such that $z^T(B\bar{u}+b) > \delta$ whenever $z' \in U_z$. Consequently, for every $\alpha > 0$ and every $z' \in U_z$ we have $\Psi(\alpha z') \ge \alpha[z']^T(B\bar{u}+b) - \psi(\bar{u}) \ge \alpha\delta + \rho > \rho$, so that U_z does not intersect Zo^ρ and therefore $z \notin Z^\rho$, which is a desired contradiction.

4^0. To complete the proof of Theorem A.6, it suffices to justify the following statement:

(!) *If z satisfies* (A.73), *then* $z \in Z^\rho$.

To this end, let us fix z satisfying (A.73).

$4^0.1$. We claim that $e^T(Bu+b) \le 0$ for all $u \in \operatorname{Dom}\psi$.

Indeed, assuming the opposite, there exists $\bar{u} \in \operatorname{Dom}\psi$ such that $e^T(B\bar{u}+b) > 0$, whence $\Psi(te) \ge te^T(B\bar{u}+b) - \psi(\bar{u}) \to \infty$ as $t \to \infty$, which is impossible due to (A.70).

$4^0.2$. Consider the case when z is such that $z^T(Bu+b) \le 0$ for all $u \in \operatorname{Dom}\psi$. We claim that in this case $z + \delta e \in Z^\rho_o$ for all $\delta > 0$, whence, of course, $z \in Z^\rho$. Let us fix $\delta > 0$.

$4^0.2.1$) Let us first prove that $(z+\delta e)^T(Bu+b) < 0$ for every $u \in \mathcal{U}^\rho$. Indeed, assuming the opposite, there exists $\bar{u} \in \mathcal{U}^\rho$ with $(z+\delta e)^T(B\bar{u}+b) \ge 0$. Since $z^T(Bu+b) \le 0$ and $e^T(Bu+b) \le 0$ for all $u \in \operatorname{Dom}\psi$ (by assumption in 2) and by 1), respectively), we conclude that $z^T(B\bar{u}+b) = e^T(B\bar{u}+b) = 0$, whence $(z+te)^T(B\bar{u}+b) \ge 0$ for all $t \ge 0$, and therefore

$$\forall t > 0 : \Psi(z+te) \ge (z+te)^T(B\bar{u}+b) - \psi(\bar{u}) \ge 0 - (-\rho) = \rho,$$

which contradicts (A.70).

$4^0.2.2$) Since $\mathcal{U}^\rho$ is a compact set, $4^0.2.1$) implies that there exists $\gamma > 0$ such that

$$(z+\delta e)^T(Bu+b) \le -\gamma \;\forall u \in \mathcal{U}^\rho.$$

Now let $\alpha > 0$. We have

$$\begin{array}{rcl}\Psi(\alpha(z+\delta e)) & = & \sup\limits_{u\in \mathrm{Dom}\,\psi} \left\{\alpha(z+\delta e)^T(Bu+b)-\psi(u)\right\} \\ & = & \max\Big[\sup\limits_{u\in\mathcal{U}^\rho\cap \mathrm{Dom}\,\psi}\{\alpha(z+\delta e)^T(Bu+b)-\psi(u)\}, \\ & & \sup\limits_{u\in(\mathrm{Dom}\,\psi)\backslash\mathcal{U}^\rho}\{\alpha(z+\delta e)^T(Bu+b)-\psi(u)\}\Big].\end{array}$$

When $u\in\mathcal{U}^\rho$, we have $\alpha(z+\delta e)^T(Bu+b)-\psi(u)\leq -\alpha\gamma+\Psi(0)$; this quantity is $\leq\rho$ for all large enough $\alpha\geq 0$. When $u\in(\mathrm{Dom}\,\psi)\backslash\mathcal{U}^\rho$, we have $\alpha(z+\delta e)^T(Bu+b)-\psi(u)\leq 0+\rho=\rho$ due to $\psi(u)>-\rho$ and $(z+\delta e)^T(Bu+b)\leq 0$ for all $u\in\mathrm{Dom}\,\psi$. We see that $\Psi(\alpha(z+\delta e))\leq\rho$ for all large enough values of α, whence $z+\delta e\in Z^\rho_o$, as claimed in $4^0.2$.
$4^0.3$. We have seen that (!) takes place in the case of $4^0.2$. It remains to verify that (!) takes place when $z^T(Bu+b)>0$ for certain $u\in\mathrm{Dom}\,\psi$. Assume that the latter is the case, and let us set

$$\begin{array}{rcl} S &=& \{[p;q]\in\mathbb{R}^2 : \exists u : p\geq\psi(u), q\geq c(u)\equiv -z^T(Bu+b)\} \\ T &=& \{[p;q]\in\mathbb{R}^2 : p\leq -\rho, q<0\}.\end{array}$$

The sets S, T clearly are convex and nonempty; let us prove that S, T do not intersect. Indeed, assuming that $[\bar{p};\bar{q}]\in S\cap T$, we would have $\bar{p}\leq-\rho)$, $\bar{q}<0$ and for certain $\bar{u}$ it holds $\bar{p}\geq\psi(\bar{u})$, $\bar{q}\geq c(\bar{u})$, that is, $z^T(B\bar{u}+b)>0$, while $\psi(\bar{u})\leq-\rho$; this is the desired contradiction, since z satisfies fact that z satisfies (A.73).

Since S,T are nonempty non-intersecting convex sets, they can be separated: there exists $[\mu;\nu]\neq 0$ such that

$$\inf_{[p;q]\in S}[\mu p+\nu s]\geq\sup_{[p;q]\in T}[\mu p+\nu q].$$

Due to the structure of S,T, this relation implies that $\mu,\nu\geq 0$ and that

$$\inf_{u\in\mathrm{Dom}\,\psi}[\mu\psi(u)+\nu c(u)]\geq-\mu\rho. \tag{A.74}$$

We claim that $\nu>0$. Indeed, otherwise $\mu>0$, and (A.74) would imply that $\psi(u)\geq-\rho$ for all $u\in\mathrm{Dom}\,\psi$, which is not the case (indeed, $\inf_u\psi(u)=-\Psi(0)<-\rho$ according (A.72)). Thus, $\nu>0$. We claim that $\mu>0$ as well. Indeed, otherwise (A.74) would imply that $\inf\limits_{u\in\mathrm{Dom}\,\psi}c(u)\geq 0$, that is, $z^T(Bu+b)\leq 0$ for all $u\in\mathrm{Dom}\,(\psi)$, which contradicts the premise in $4^0.3$. Thus, $\mu>0,\nu>0$ and therefore (A.74) implies that with $\alpha=\nu/\mu>0$ one has $\inf\limits_{u\in\mathrm{Dom}\,\psi}[\psi(u)+\alpha c(u)]\geq -\rho$, that is,

$$\Psi(\alpha z)=\sup_u\left\{\alpha z^T(Bu+b)-\psi(u)\right\}\leq\rho,$$

whence $z\in Z^\rho_o$. The proof of (!) is completed. □

A.7 Proofs of Propositions 7.4, 7.10 and Theorems 7.2, 7.3

Proof of Proposition 7.4

1^0. The set Γ^o_ϵ is of the form

$$\{z : \exists\alpha > 0 : G(\alpha z) \le 0\}$$

with convex and everywhere finite function $G(z) = \Psi(z) - \epsilon$. The set of this form always is convex and satisfies $\alpha\Gamma^o_\epsilon = \Gamma^o_\epsilon$ for every $\alpha > 0$; as a result, the closure Γ_ϵ of Γ^o_ϵ is a closed convex cone, provided that $\Gamma^o_\epsilon \ne \emptyset$. The latter indeed is the case due to $G(te) \to -\epsilon < 0$ as $t \to \infty$ (see (7.18)); as a byproduct, we get that $e \in \mathrm{int}\Gamma^o_\epsilon \in \mathrm{int}\Gamma_\epsilon$.

2^0. Now let us prove that the inequality (7.20) is convex and that Γ_ϵ is exactly the solution set, let it be called Z, of this inequality.
Since the function $G(z) = \Psi(z) - \epsilon$ is convex, the function $F(\beta, z) = \beta G(\beta^{-1}z)$ is convex in (β, z) in the domain $\beta > 0$. As $\beta \to \infty$, this function goes to $+\infty$ due to $\Psi(0) \ge 1$, and as $\beta \to +0$, this function clearly remains below bounded. Thus, $F(\beta, z)$ is below bounded in $\beta > 0$ for every z, whence the function $\inf\limits_{\beta>0} F(\beta, z)$ is convex, so that (7.20) indeed is a convex inequality. In order to prove that $\Gamma_\epsilon = Z$, observe that

$$\Gamma^o_\epsilon = \{z : \exists\beta > 0 : G(\beta^{-1}z) \le 0\}, \quad Z = \{z : \inf_{\beta>0} \beta G(\beta^{-1}z) \le 0\}.$$

It follows that $\Gamma^o_\epsilon \subset Z$. Next let us prove that $\Gamma_\epsilon \subset Z$. Thus, let $z^i \in \Gamma^o_\epsilon$ be such that $z^i \to \bar z$ as $i \to \infty$; we should prove that $\bar z \in Z$. Since $z^i \in \Gamma^o_\epsilon$, there exist $\beta_i > 0$ such that $G(\beta_i^{-1}z^i) \le 0$. Passing to a subsequence of $\{z^i\}$, we may restrict ourselves with the following three possibilities:

1) $\beta_i \to \bar\beta \in (0, \infty)$ as $i \to \infty$; 2) $\beta_i \to +\infty$ as $i \to \infty$; 3) $\beta_i \to +0$ as $i \to \infty$.
In the case of 1), passing to the limit in the relation $G(\beta_i^{-1}z^i) \le 0$, we get $G(\bar\beta^{-1}\bar z) \le 0$, whence $\bar x \in \Gamma^o_\epsilon \subset Z$. In the case of 2) we have $0 \ge G(\beta_i^{-1}z^i) \to G(0) = \Psi(0) - \epsilon \ge \Psi_*(0) - \epsilon = 1 - \epsilon > 0$ as $i \to \infty$, so that 2) is impossible. In the case of 3), let $\mathcal{G} = \{z : G(z) \le 0\}$, so that $\mathcal{G}$ is a closed and nonempty convex set, and $\beta_i^{-1}z^i \in \mathcal{G}$ for all i. From the latter relation, in view of $z^i \to \bar z$ and $\beta_i \to +0$ as $i \to \infty$, it follows that $\bar z$ is a recessive direction of $\mathcal{G}$. Now let $\hat z$ be a point from $\mathcal{G}$ and let $\mathcal{G}^+$ be the convex hull of the union of the ray $\hat z + \mathbb{R}_+\bar z$ and $-\hat z$; observe that since the ray in question belongs to $\mathcal{G}$, G is above bounded on $\mathcal{G}^+$. At the same time, the ray $\mathbb{R}_+\bar z$ clearly belongs to $\mathcal{G}^+$, so that $G(\beta^{-1}\bar z)$ is above bounded on the ray $\beta > 0$, whence $\inf\limits_{\beta>0} \beta G(\beta^{-1}\bar z) \le 0$ and thus $\bar z \in Z$.

To complete the proof of 2^0, it remains to show that Z is contained in $\mathrm{cl}\,\Gamma^o_\epsilon$. Let $\bar z \in Z$; we should prove that $\bar z \in \mathrm{cl}\,\Gamma^o_\epsilon$. Since $\bar z \in Z$, we have $\inf\limits_{\beta>0} \beta G(\beta^{-1}\bar z) \le 0$, which, by convexity of $f(\beta) = \beta G(\beta^{-1}z)$ on the ray $\beta > 0$, leaves us with 3 possibilities:

a) $\exists\beta > 0 : f(\beta) \le 0$;
b) $f(\beta) > 0$ for $\beta > 0$ & $\lim_{\beta\to\infty} f(\beta) = 0$;
c) $f(\beta) > 0$ for $\beta > 0$ & $\lim_{\beta\to+0} f(\beta) = 0$.

In the case of a) we have $\bar{z} \in \Gamma_\epsilon^o$, as required. The case of b) is impossible, since clearly $\lim_{\beta\to+\infty} f(\beta) = \lim_{\beta\to+\infty} \beta[\Psi(\beta^{-1}\bar{z}) - \epsilon] = +\infty$ due to $\Psi(0) \geq 1$. Now consider the case of c). We have $\alpha^{-1}G(\alpha\bar{z}) \to 0$ as $\alpha \to \infty$, so that the convex function $f(\alpha) = G(\alpha\bar{z}) = \Psi(\alpha\bar{z}) - \epsilon$ is nonincreasing on the ray $\alpha \geq 0$. Since this function is positive on the ray (we are in the case of c)), there exists $C > 1$ such that $f(\alpha) \leq C$ for all $\alpha > 0$. Besides this, there exists t such that $\Psi(te) \leq \epsilon/2$, $e = [-1; 0; ...; 0]$. Now let $p = \frac{\epsilon}{4C}$, and let

$$z_\alpha = \bar{z} + \frac{1-p}{p\alpha} te,$$

so that $z_\alpha \to \bar{z}$ as $\alpha \to \infty$. We have

$$\begin{aligned} G(p\alpha z_\alpha) &= G(p\alpha\bar{z} + (1-p)te) \leq pG(\alpha\bar{z}) + (1-p)G(te) \\ &\leq pC + (1-p)(\Psi(te) - \epsilon) \leq pC - (1-p)\tfrac{\epsilon}{2} = \tfrac{\epsilon}{4} - \tfrac{\epsilon}{2}\left(1 - \tfrac{\epsilon}{4C}\right) \\ &\leq \epsilon\left(\tfrac{1}{4} - \tfrac{3}{8}\right) < 0, \end{aligned}$$

meaning that $z_\alpha \in \Gamma_\epsilon^o$. Since $z_\alpha \to \bar{z}$ as $\alpha \to +\infty$, we see that $\bar{z} \in \mathrm{cl}\,\Gamma_\epsilon^o$.

3^0. We are done. Indeed, we have proved that the solution set Γ_ϵ of the convex inequality (7.20) is a closed convex cone with $e \in \mathrm{int}\Gamma_\epsilon$. Besides this, this cone is contained in the feasible set Z_* of the chance constraint (7.1). Indeed the set Z_* contains the set Γ_ϵ^o by (7.19), and since Z_* is closed, it contains Γ_ϵ as well. Invoking Remark 7.1, we conclude that (7.20) is a safe normal approximation of (7.1). □

Proof of Theorem 7.2

We split the proof of Theorem 7.2.(ii) (item (i) is evident) in two steps.

Step 1. Let $\mathcal{M}$ be the family of all even and continuously differentiable functions on the axis which are nondecreasing on $\mathbb{R}_+$, and $\mathcal{M}_*$ be a subset of $\mathcal{M}$ comprised of all bounded functions $f(s)$ from $\mathcal{M}$. We claim that for $p, q \in \mathcal{P}$, one has $q \succeq_{\mathrm{m}} p$ if and only if

$$\forall f \in \mathcal{M}_* : \int f(s)p(s)ds \leq \int f(s)q(s)ds, \tag{A.75}$$

and if this indeed is the case, then

$$\int f(s)p(s)ds \leq \int f(s)q(s)ds \tag{A.76}$$

for every function $f \in \mathcal{M}$ for which both integrals exist.

Indeed, setting $P(a) = \int\limits_a^\infty p(s)ds$, $Q(a) = \int\limits_a^\infty q(s)ds$, observe that when $f \in \mathcal{M}_*$, then

$$\begin{aligned} &\int f(s)p(s)ds = 2\int_0^\infty f(s)(-P'(s))ds = 2\int\limits_0^\infty f'(s)P(s)ds + 2P(0)f(0) \\ &= 2\int\limits_0^\infty f'(s)P(s)ds + f(0) \end{aligned}$$

and similarly $\int f(s)q(s)ds = 2\int\limits_0^\infty f'(s)Q(s)ds + f(0)$. Thus, (A.75) says exactly that

$$\int\limits_0^\infty \phi(s)(Q(s)-P(s))ds \geq 0$$

whenever ϕ is a nonnegative continuous and summable function on $\mathbb{R}_+$, which, due to the continuity of P, Q, is exactly the same as to say that $Q \geq P$ everywhere on $\mathbb{R}_+$; but the latter is nothing but the relation $q \succeq_{\rm m} p$.

The concluding claim can be justified as follows: let $q \succeq_{\rm m} p$. Then, as we have just proved, (A.75) holds true, whence $\int f(s)p(s)ds \leq \int f(s)q(s)ds$ for all bounded $f \in \mathcal{M}$. Since every function from $\mathcal{M}$ is a pointwise limit of a nonincreasing sequence of bounded functions from $\mathcal{M}$, we conclude that $\int f(s)p(s)ds \leq \int f(s)q(s)ds$ whenever $f \in \mathcal{M}$ and both integrals exist.

Step 2. We claim that

(a) If $\xi, \xi' \in \Pi$ are independent, then $\xi + \xi' \in \Pi$,

and

(b) If $p \in \mathcal{P}$ and $f \in \mathcal{M}_*$, then $f_+ := f * p$ belongs to $\mathcal{M}_*$.

Indeed, denoting by p, r the densities of ξ, ξ', in order top prove (a) we should prove that the density $(p*r)(s) = \int p(s-t)r(t)dt$ is even (which is evident) and nondecreasing when $s < 0$. By standard approximation arguments, it suffices to establish the latter fact when p, r are smooth. We have

$$(p*r)'(s) = \int p'(s-t)r(t)dt = \int p(s-t)r'(t)ds = \int\limits_{-\infty}^{0} (p(s-t)-p(s+t))r'(t)dt. \tag{A.77}$$

Since $s < 0$, with $t < 0$ we have $|s-t| \leq |s+t| = |s| + |t|$; and since p is even and nonincreasing on $\mathbb{R}_+$, we conclude that $p(s-t) = p(|s-t|) \geq p(|s+t|) = p(s+t)$, so that $p(s-t) - p(s+t) \geq 0$ when $s, t \leq 0$. Since, in addition, $r'(t) \geq 0$ when $t \leq 0$, the concluding quantity in (A.77) is nonnegative. (a) is proved.

To prove (b), observe that the facts that f_+ is even, continuously differentiable and bounded are evident. All we should prove is that f_+ is nondecreasing on $\mathbb{R}_+$; by standard approximation argument, it suffices to prove this relation when $p \in \mathcal{P}$ is smooth. In the latter case we have $f_+'(s) = \int f(s-t)p'(t)ds = \int_{-\infty}^0 (f(s-t) - f(s+t))p'(t)dt$. Assuming s positive and taking into account that f is even and is nondecreasing on $\mathbb{R}_+$, we have $f(s-t) = f(|s-t|) = f(|s|+|t|) \geq f(|s+t|) = f(s+t)$; since $p'(t) \geq 0$ when $t \leq 0$, we conclude that $\int_{-\infty}^0 (f(s-t) - f(s+t))p'(t)dt \geq 0$ when $s \geq 0$. (b) is proved.

Now we can complete the proof of (ii). Let the premise of (ii) hold true, and let $p, \bar{p}, q, \bar{q}$ be the densities of $\xi, \bar{\xi}, \eta, \bar{\eta}$, respectively. To prove (ii), we should prove that $\xi + \bar{\xi} \in \Pi$, $\eta + \bar{\eta} \in \Pi$ (which is readily given by (a)) and that $\eta + \bar{\eta} \succeq_{\rm m} \xi + \bar{\xi}$. We have

$$\begin{array}{l} \int f(s)(p*\bar{p})(s)ds = \int f(s)\left\{\int p(s-t)\bar{p}(t)dt\right\}ds \\ = \int \underbrace{\left\{\int f(s)p(s-t)ds\right\}}_{F(t)} \bar{p}(t)dt \end{array}$$

and similarly $\int f(s)(p*\bar{q})(s)ds = \int F(t)\bar{q}(t)dt$. By (b), $F \in \mathcal{M}_*$, and since $\bar{q} \succeq_{\rm m} \bar{p}$, we get from Step 1 that

$$\int F(t)\bar{p}(t)dt \leq \int F(t)\bar{q}(t)dt.$$

Thus, $\int f(s)(p*\bar{p})(s)ds \leq \int f(s)(p*\bar{q})(s)ds$ whenever $f \in \mathcal{M}_*$, which, by the same Step 1, implies that

$$p*\bar{q} \succeq_{\rm m} p*\bar{p}.$$

By completely similar reasoning, we have $q*\bar{q} \succeq_{\rm m} p*\bar{q}$; since the relation $\succeq_{\rm m}$, due to its origin, is transitive, we conclude that $q*\bar{q} \succeq_{\rm m} p*\bar{p}$. □

Proof of Theorem 7.3

The case of $L = 1$ is evident, so that in the sequel we assume that $L > 1$.

1^0. Observe that if $p(\cdot)$ is a unimodal and symmetric w.r.t. the origin probability density on the axis, then there exists a sequence $\{p^t(\cdot)\}_{t=1}^{\infty}$ of probability densities on the axis such that

— every $p^t(\cdot)$ is a convex combination of densities of uniform symmetric w.r.t. 0 distributions;

— the sequence $\{p^t\}$ converges to p in the sense that

$$\int f(s)p^t(s)ds \to \int f(s)p(s)ds \text{ as } t \to \infty$$

for every bounded piecewise continuous function f on the axis.

2^0. To proceed, we need the following fundamental fact:

Symmeterization Principle [Brunn-Minkowski] *Let $S \subset \mathbb{R}^n$, $n > 1$, be a nonempty convex compact set, $e \in \mathbb{R}^n$ be a unit vector, and Δ be the projection of S onto the axis $\mathbb{R}e$: $\Delta = [\min_{x\in S} e^T x, \max_{x\in S} e^T x]$. Then the function*

$$f(s) = \left(\mathrm{mes}_{n-1}\{x \in S : e^T x = s\}\right)^{\frac{1}{n-1}}$$

is concave and continuous on Δ.

Now let S be a convex compact set in $\mathbb{R}^L$ symmetric w.r.t. the origin, $p_1(\cdot), ..., p_L(\cdot), q(\cdot)$ be unimodal symmetric w.r.t. 0 probability densities on the axis such that

(a) $p_1(\cdot)$,...,$p_{L-1}(\cdot)$ are densities of uniform distributions;

(b) $p_L \preceq_{\rm m} q$.

We claim that then

$$\int\limits_S p_1(x_1)p_2(x_2)...p_{L-1}(x_{L-1})p_L(x_L)dx \geq \int\limits_S p_1(x_1)p_2(x_2)...p_{L-1}(x_{L-1})q(x_L)dx. \tag{A.78}$$

Indeed, let Σ_ℓ, $1 \leq \ell < L$, be the support of the density p_ℓ, so that Σ_ℓ is a segment on the axis symmetric w.r.t. 0. Let us set $\Sigma = \Sigma_1 \times ... \times \Sigma_{L-1} \times \mathbb{R}$, $\widehat{S} = S \cap \Sigma$, so that $\widehat{S}$ is a convex compact set symmetric w.r.t. the origin, and let

$$f(s) = \mathrm{mes}_{n-1}\{x \in \widehat{S} : x_L = s\}.$$

The function $f(s)$ is even; denoting by Δ the projection of $\widehat{S}$ onto the x_L-axis and applying the Symmeterization Principle, we conclude that $f^{1/(L-1)}(s)$ is concave, even and continuous on Δ, whence, of course, $f^{1/(L-1)}(s)$ is nonincreasing on $\Delta \cap \mathbb{R}_+$. We see that the function $f(s)$ is even and nonnegative and is nonincreasing on $\mathbb{R}_+$, whence

$$\int f(s)p_L(s)ds \geq \int f(s)q(s)ds \tag{A.79}$$

due to $p_L \preceq_{\mathrm{m}} q$. It remains to note that the left and the right hand sides in (A.78) are proportional, with a positive coefficient, to the respective sides in (A.79).

3^0. We are ready to prove Theorem 7.3. By continuity reasons, all we need is to prove the following statement:

> (!) *If* $p_1, ..., p_L$, q *are unimodal symmetric w.r.t. 0 probability densities on the axis such that* $p_L \preceq_{\mathrm{m}} q$ *and* S *is a symmetric w.r.t. the origin convex compact set, then*

$$\int\limits_S p_1(x_1)p_2(x_2)...p_{L-1}(x_{L-1})p_L(x_L)dx \geq \int\limits_S p_1(x_1)p_2(x_2)...p_{L-1}(x_{L-1})q(x_L)dx.$$

By 1^0, it suffices to verify (!) in the particular case where $p_1(\cdot)$,..., $p_{L-1}(\cdot)$ are convex combinations of the densities of uniform distributions supported on symmetric w.r.t. 0 segments, and this case, due to linearity of the both sides of the target inequality in p_1,...,p_{L-1}, reduces to the case where $p_1, ..., p_{L-1}$ are densities of uniform symmetric w.r.t. 0 distributions. In the latter case, the validity of the target inequality is stated in 2^0. □

Proof of Proposition 7.10

As it is explained immediately after formulating Proposition 7.10, the situation can be reduced to the case when $z_0 = 0$ and $z_1 = ... = z_L = 1$, which we assume from now on. Besides this, we assume that $\zeta_1, ..., \zeta_L$ are the random variables (7.66).

1^0. We start with the following conditional statement:

> (!) *If* (7.65) *holds true for all affine functions* f *and all functions* f *of the form*
> $$f(s) = \max[0, a + s],$$
> *then* (7.65) *holds true for all piecewise linear convex functions* f.

Indeed, every piecewise linear convex function $f(\cdot)$ is a linear combination, with nonnegative coefficients λ_i, of an affine function f_0 and functions f_i, $i = 1, ..., I = I(f)$, of the form $\max[0, a + s]$. Under the premise of (!), we have

$$\Phi[f_i, z] = \mathbf{E}\{f_i(\sum_\ell \zeta_\ell)\},\ i = 0, 1, ..., I(f), \tag{A.80}$$

whence by the results of Proposition 7.9

$$\Phi[f,z] = \Phi[\sum_{i=0}^{I} \lambda_i f_i, z] \le \sum_{i=0}^{I} \lambda_i \Phi[f_i, z] = \mathbf{E}\{f(\sum_\ell \zeta_\ell)\},$$

where the concluding equality is given by (A.80) combined with the fact that $\sum_{i=0}^{I} \lambda_i f_i = f$. Thus, $\Phi[f,z] \le \mathbf{E}\{f(\sum\limits_\ell \zeta_\ell)\}$; since the opposite inequality always is true, we get $\Phi[f,z] = \mathbf{E}\{f(\sum\limits_\ell \zeta_\ell)\}$, as claimed in (!).

2^0. In view of (!), all we need in order to prove Proposition 7.10 is to verify that the relation (7.65) indeed takes place when f is linear or $f(s) = \max[0, a+s]$.

When f is linear, relation (7.65) holds true independently of whether $\zeta_1, ..., \zeta_L$ are linked to each other by (7.66) or are arbitrary random variables with given distributions. Indeed, when $f(s) = a + bs$, then, setting

$$\gamma_\ell(u_\ell) = \frac{1}{L} a + b u_\ell,$$

we clearly ensure that

$$\sum_\ell \gamma_\ell(u_\ell) = f(\sum_\ell u_\ell) \ \forall u \in \mathbb{R}^L,$$

which, by (7.61), implies that $\Phi[f,z] \le \sum\limits_\ell \mathbf{E}\{\gamma_\ell(\zeta_\ell)\}$; but the latter quantity is exactly $\mathbf{E}\{f(\sum\limits_\ell \zeta_\ell)\} \le \Phi[f,z]$, so that $\Phi[f,z] = \mathbf{E}\{f(\sum\limits_\ell \zeta_\ell)\}$.

Now let us prove that (7.65) takes place when $f(s) = \max[0, a+s]$. To save notation, let $a = 0$ (the case of an arbitrary a is completely similar). As above, all we need is to verify that

$$\Phi[f,z] \le \mathbf{E}\{\max[0, \sum_\ell \zeta_\ell]\}. \tag{A.81}$$

Let $\phi(t) = \sum\limits_\ell \phi_\ell(t)$, $t \in (0,1)$. There are three possibilities:

a) $\phi(t) \le 0$ for all $t \in (0,1)$;

b) $\phi(t) \ge 0$ for all $t \in (0,1)$;

c) $\phi(t_-) < 0$ and $\phi(t_+) > 0$ for appropriately chosen $t_\pm$, $0 < t_- < t_+ < 1$.

In the case of a), the nondecreasing functions $\phi_\ell(t)$ are above bounded, and $0 \ge \lim_{t\to 1-0} \phi(t) = \sum\limits_\ell d_\ell$, $d_\ell = \lim_{t \to 1-0} \phi_\ell(t)$. Setting $\gamma_\ell(u_\ell) = \max[0, u_\ell - d_\ell]$, we clearly get

$$\sum_\ell \gamma_\ell(u_\ell) \ge \max[0, \sum_\ell (u_\ell - d_\ell)] \ge \max[0, \sum_\ell u_\ell],$$

where the concluding inequality is given by $\sum\limits_\ell d_\ell \le 0$. Invoking (7.61), we conclude that

$$\begin{array}{l} \Phi[f,z] \le \sum_\ell \mathbf{E}\{\max[0, \zeta_\ell - d_\ell]\} = \sum_\ell \int_0^1 \max[0, \phi_\ell(t) - d_\ell] dt \\ = 0 = \int_0^1 \max[0, \sum_\ell \phi_\ell(t)] dt = \mathbf{E}\{f(\sum_\ell \zeta_\ell)\}, \end{array}$$

where the second and the third equalities follow from the fact that $\phi_\ell(t) \le d_\ell$ and $\phi(t) \le 0$ when $t \in (0,1)$. The resulting inequality is exactly the relation (A.81) we need.

In the case of b), the nondecreasing functions ϕ_ℓ are below bounded on $(0,1)$, and $0 \leq \lim_{t\to+0}\phi(t) = \sum_\ell d_\ell$, $d_\ell = \lim_{t\to+0}\phi_\ell(t)$. Setting $\gamma_\ell(u_\ell) = \max[d_\ell, u_\ell]$, we ensure that

$$\sum_\ell \gamma_\ell(u_\ell) \geq \max[0, \sum_\ell u_\ell]\,\forall u$$

due to $\sum_\ell d_\ell \geq 0$, whence, invoking (7.61),

$$\begin{array}{l}\Phi[f,z] \leq \sum_\ell \mathbf{E}\{\max[d_\ell,\zeta_\ell]\} = \sum_\ell \int_0^1 \max[d_\ell,\phi_\ell(t)]dt \\ = \int_0^1[\sum_\ell \phi_\ell(t)]dt = \int_0^1 \max[0,\sum_\ell \phi_\ell(t)]dt = \mathbf{E}\{f(\sum_\ell \zeta_\ell)\},\end{array}$$

where the second and the third equalities follow from the fact that $\phi_\ell(t) \geq d_\ell$ and $\phi(t) \geq 0$ when $t \in (0,1)$. The resulting inequality is exactly (A.81).

In the case of c), the quantity $t_* = \sup\{t \in (0,1) : \phi(t) \leq 0\}$ is well defined and belongs to $(0,1)$; since ϕ_ℓ are continuous from the left at t_*, we have

$$0 \geq \phi(t_*) = \sum_\ell d_\ell^-,\ d_\ell^- = \phi_\ell(t_*),$$

and since $\phi(t) > 0$ when $t > t_*$, we have

$$0 \leq \lim_{t\to t_*+0}\phi(t) = \sum_\ell d_\ell^+,\ d_\ell^+ = \lim_{t\to t_*+0}\phi_\ell(t).$$

Since ϕ_ℓ are nondecreasing, we have $d_\ell^+ \geq d_\ell^-$; since $\sum_\ell d_\ell^- \leq 0 \leq \sum_\ell d_\ell^+$, we can find reals $d_\ell \in [d_\ell^-, d_\ell^+]$ in such a way that $\sum_\ell d_\ell = 0$. Setting $\gamma_\ell(u_\ell) = \max[0, u_\ell - d_\ell]$, we clearly get

$$\sum_\ell \gamma_\ell(u_\ell) \geq \max[0, \sum_\ell u_\ell - \sum_\ell d_\ell] = \max[0, \sum_\ell u_\ell],$$

whence, invoking (7.61),

$$\begin{array}{l}\Phi[f,z] \leq \sum_\ell \mathbf{E}\{\max[0,\zeta_\ell - d_\ell]\} = \sum_\ell \int_0^1 \max[0,\phi_\ell(t) - d_\ell]dt \\ = \sum_\ell \int_{t_*}^1 [\phi_\ell(t) - d_\ell]dt = \int_{t_*}^1 \phi(t)dt \leq \int_{t_*}^1 \max[0,\phi(t)]dt = \mathbf{E}\{f(\sum_\ell \zeta_\ell)\},\end{array}$$

where the second equality follows from the fact that $\phi_\ell(t) - d_\ell \leq d_\ell^- - d_\ell \leq 0$ when $t \leq t_*$ and $\phi_\ell(t) - d_\ell \geq d_\ell^+ - d_\ell \geq 0$ when $t > t_*$, and the third equality is given by $\sum_\ell d_\ell = 0$. The resulting inequality is exactly (A.81). □

A.8 Proofs of Proposition 13.1 and Theorem 13.3

Proof of Proposition ??. Items 1 – 4 are evident.

Item 5: Let $f(x,y) \in \mathcal{CF}_{r+s}$ ($x \in \mathbb{R}^r$, $y \in \mathbb{R}^s$), The functions $p(x) = \int f(x,y)dP_2(y)$, $q(x) = \int f(x,y)dQ_2(y)$ clearly belong to $\mathcal{CF}_r$ and $p \leq q$ due to $P_2 \preceq_c Q_2$. We have $\int f(x,y)d(P_1 \times P_2)(x,y) = \int p(x)dP_1(x) \leq \int p(x)dQ_1(x) \leq \int q(x)dQ_1(x) = \int f(x,y)d(Q_1 \times Q_2)(x,y)$, where the first $\leq$ follows from $P_1 \preceq_c Q_1$, and the second $\leq$ is given by $p \leq q$. The resulting inequality shows that $P_1 \times P_2 \preceq_c Q_1 \times Q_2$. □

Item 6: Let $f \in \mathcal{CF}_1$; we should prove that

$$\int f(s)dP_\xi(s) \leq \int f(s)dP_\eta(s). \qquad (*)$$

When we add to f an affine function, both quantities we are comparing change by the same amount (recall that $\mathcal{R}_1$ is comprised of probability distributions with zero mean). It follows that w.l.o.g. we can assume that $f(-1) = 0$ and $f'(-1+0) = 0$, so that f is nonnegative to the left of the point -1. Replacing f in this domain by 0, we preserve convexity, keep the quantity $\int f(s)dP_\xi(s)$ intact and do not increase $\int f(s)dP_\eta(s)$; it follows that it suffices to prove our inequality when f, in addition to $f'(-1+0) = f(-1) = 0$, is identically zero to the left of -1. Now, either f is identically zero on $[-1,1]$, or $f(1)$ is positive. In the first case, the left hand side in $(*)$ is 0, while the right hand side is nonnegative (since f is nonnegative due to $f(-1) = f'(-1+0) = 0$), and $(*)$ holds true. When $f(1) > 0$, we can, by scaling f, reduce the situation to the one where $f(1) = 1$. In this case, $f(s) \leq (s+1)/2$ on $[-1,1]$ by convexity, whence, recalling that ξ is supported on $[-1,1]$ and has zero mean, we have

$$\int f(s)dP_\xi(s) \leq \int \frac{1}{2}(1+s)dP_\xi(s) = \frac{1}{2}$$

On the other hand, let $\alpha = f'(1+0)$, so that $\alpha > 0$. Besides this, f is nonnegative and $f(s) \geq 1 + \alpha(s-1)$ for all s, whence

$$f(s) \geq \max[0, 1-\alpha+\alpha s] \ \forall s.$$

Consequently, setting $\sigma = \sqrt{2/\pi}$, we have

$$\begin{array}{l}\int f(s)dP_\eta(s) \geq \int \max[0,1-\alpha+\alpha s]\frac{1}{\sqrt{2\pi}\sigma}\exp\{-s^2/(2\sigma^2)\}ds \\ \min\limits_{\alpha\geq 1/2}\int [\max[0,1-\alpha+\alpha s] \underbrace{\frac{1}{\sqrt{2\pi}\sigma}\exp\{-s^2/(2\sigma^2)\}}_{p(s)} ds.\end{array}$$

The function $g(\alpha) = \int \max[1-\alpha+\alpha s]p(s)ds$ clearly is convex in $\alpha \geq 0$, and $g'(\alpha) = \int\limits_{\frac{\alpha-1}{\alpha}}^{\infty} (s-1)p(s)ds$. Taking into account that $\int\limits_0^\infty p(s)ds = \int\limits_0^\infty sp(s)ds = 1/2$, we conclude that $g'(1) = 0$, that is, $\alpha = 1$ is a minimizer of g so that $g(\alpha) \geq g(1) = 1/2$ whenever $\alpha > 0$. Thus, the right hand side in $(*)$ is $\geq 1/2$ and $(*)$ indeed is true. □

Item 7: Due to absolute symmetry, the distribution of ξ is the limit, in the sense of weak convergence, of a sequence of convex combinations of uniform distributions on the vertices of cubes $\{u : \|u\|_\infty \leq r\}$, $r \leq 1$. By item 6, all these distributions are dominated by $\mathcal{N}(0, (\pi/2)I_n)$. It remains to apply item 2. □

Item 8: Since $0 \preceq \Sigma \preceq \Theta$, there exists a nonsingular transformation $x \mapsto Ax : \mathbb{R}^d \to \mathbb{R}^d$ such that the random vectors $\widetilde{\xi} = A\xi$ and $\widetilde{\eta} = A\eta$ are, respectively, $\mathcal{N}(0, \mathrm{Diag}\{\lambda\})$ and $\mathcal{N}(0, \mathrm{Diag}\{\mu\})$; since $\Sigma \preceq \Theta$, we have $\lambda \leq \mu$, whence, by item 4, $\widetilde{\xi} \preceq_c \widetilde{\eta}$, which, of course, is equivalent to $\xi \preceq_c \eta$. □

Proof of Theorem 13.3. Let $\xi \in \mathcal{P}_n$ and $\eta \sim \mathcal{N}(0,\Sigma)$ be a Gaussian n-dimensional random vector with nonsingular covariance matrix Σ such that $\xi \preceq_{\rm c} \eta$. Let, further, $Q \subset \mathbb{R}^n$ be a closed convex set such that $\chi \equiv \mathrm{Prob}\{\eta \in Q\} \in (1/2, 1)$. All we need is to prove that whenever $\gamma > 1$, one has

$$\mathrm{Prob}\{\xi \notin \gamma Q\} \leq \inf_{1\leq\beta<\gamma} \frac{1}{\gamma-\beta} \int\limits_{\beta}^{\infty} \exp\{-r^2 \mathrm{ErfInv}^2(1-\chi)/2\} dr. \tag{A.82}$$

Indeed, since Q is convex and $\mathrm{Prob}\{\eta \in Q\} > 1/2$, the origin is in the interior of Q. Let $\beta \in [1,\gamma)$, let

$$\theta(x) = \inf\{t : t^{-1}x \in Q\}$$

be the Minkowski function of Q, and let $\delta(x) = \max[\theta(x) - \beta, 0]$. We clearly have $\delta(\cdot) \in \mathcal{CF}_n$, so that

$$\int \delta(x) dP_\xi(x) \leq \int \delta(x) dP_\eta(x). \qquad (a)$$

For $r \geq \beta$ let $p(r) = \mathrm{Prob}\{\eta \notin rQ\} = \mathrm{Prob}\{\delta(\eta) > r - \beta\}$. By Theorem A.5, for $r \geq \beta$ we have

$$p(r) \leq \exp\{-r^2 \mathrm{ErfInv}^2(1-\chi)/2\}. \qquad (b)$$

We have

$$\begin{array}{l} \int \delta(x) dP_\eta(x) = -\int\limits_\beta^\infty (r-\beta) dp(r) = \int\limits_\beta^\infty p(r) dr \\ \leq \int\limits_\beta^\infty \exp\{-r^2 \mathrm{ErfInv}^2(1-\chi)/2\} dr, \end{array}$$

whence

$$\int \delta(x) dP_\xi(x) \leq \int\limits_\beta^\infty \exp\{-r^2 \mathrm{ErfInv}^2(1-\chi)/2\} dr$$

by (a). Now, when $\xi \notin \gamma Q$, we have $\delta(\xi) \geq \gamma - \beta$. Invoking Tschebyshev Inequality, we arrive at

$$\mathrm{Prob}\{\xi \notin \gamma Q\} \leq \frac{\mathbb{E}\{\delta(\xi)\}}{\gamma - \beta} \leq \frac{1}{\gamma-\beta} \int\limits_\beta^\infty \exp\{-r^2 \mathrm{ErfInv}^2(1-\chi)/2\} dr.$$

The resulting inequality holds true for all $\beta \in [1, \gamma)$, and (A.82) follows. □

References

[1] Ben-Tal, A., Nemirovski, A. Robust Convex Optimization. *Mathematics of Operations Research* **23:4** (1998).

[2] Ben-Tal, A., Nemirovski, A. Robust solutions of uncertain linear programs. *OR Letters* **25** (1999), 1-13.

[3] Ben-Tal, A., and Nemirovski, A. Robust solutions of Linear Programming problems contaminated with uncertain data. *Math. Progr.* **88** (2000), 411-424.

[4] Ben-Tal, A., and Nemirovski, A., Robust Optimization – Methodology and Applications. *Math. Progr. Series B*, **92** (2002), 453-480.

[5] Ben-Tal, A., Nemirovski, A. On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty. *SIAM J. Optimization* **12** (2002), 811-833.

[6] Ben-Tal, A., Nemirovski, A., Roos, C. Robust solutions of uncertain quadratic and conic-quadratic problems, *SIAM J. Optimization* **13** (2002), 535-560.

[7] Ben-Tal, A., Nemirovski, A., Roos, C. Extended Matrix Cube Theorems with applications to μ-Theory in Control. *Mathematics of Operations Research* **28** (2003), 497-523.

[8] Ben-Tal, A., Goryashko, A., Guslitzer, E. Nemirovski, A. Adjustable Robust Solutions of Uncertain Linear Programs. *Math. Progr.* **99** (2004), 351-376.

[9] Ben-Tal, A., Golany, B., Nemirovski, A., Vial, J.-Ph. Supplier-Retailer Flexible Commitments Contracts: A Robust Optimization Approach. *Manufacturing & Service Operations Management* **7:3** (2005), 248-273.

[10] Ben-Tal, A., Boyd, S., Nemirovski, A. Extending Scope of Robust Optimization: Comprehensive Robust Counterparts of Uncertain Problems. *Math. Progr.* **107:1-2** (2006), 63-89.

[11] Ben-Tal, A., Nemirovski, A. Selected Topics in Robust Convex Optimization. To appear in *Math. Progr. Series B*, Special Issue on International Math. Progr. Symposium 2006.

[12] Ben-Tal, A., Boyd, S., Nemirovski, A. (2005). Control of Uncertainty-Affected Discrete Time Linear Systems via Convex Programming. Submitted to *SIAM J. Control and Optimization*
E-print: http://www.optimization-online.org/DB_HTML/2005/10/1232.html

[13] Ben-Tal, A., Margalit, T., Nemirovski, A. Robust Modeling of Multi-Stage Portfolio Problems. In: H. Frenk, K. Roos, T. Terlaky, S. Zhang, Eds. *High Performance Optimization*, Kluwer Academic Publishers, 2000, 303-328.

[14] Ben-Tal, A., El Ghaoui, L., Nemirovski, A. Robust Semidefinite Programming. In: R. Saigal, H. Wolkowitcz, L. Vandenberghe, Eds. *Handbook on Semidefinite Programming*, Kluwer Academic Publishers, 2000, 139-162.

[15] Ben-Tal, A., Nemirovski, A., Roos, C. Robust versions of convex quadratic and conic-quadratic problems. In: D. Li, Ed. *Proceedings of the 5th International Conference on Optimization Techniques and Applications (ICOTA 2001)*, **4** (2001), 1818-1825.

[16] Boyd, S., El Ghaoui, L., Feron, E., Balakrishnan, V. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, 1994.

[17] Borell, C. The Brunn-Minkowski inequality in Gauss space. *Inventiones Mathematicae* **30:2** (1975), 207-216.

[18] Charnes, A., Cooper, W.W., Symonds, G.H. Cost horizons and certainty equivalents: an approach to stochastic programming of heating oil. *Management Science* **4** (1958), 235-263.

[19] Dan Barb, F., Ben-Tal, A., Nemirovski, A. Robust dissipativity of interval uncertain system. *SIAM J. Control and Optimization* **41** (2003), 1661 - 1695.

[20] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., Vyncke, D. The consept of comonotonicity in actuarial science and finance: theory. *Insurance: Mathematics and Economics* **31** (2002), 3 - 33.

[21] Eldar, Y., Ben-Tal, A., Nemirovski, A. Robust mean-squared error estimation in the presence of model uncertainties. *IEEE Trans. on Signal Processing* **53** (2005), 168-181.

[22] Håstad, J. Some optimal inapproximability results. *J. of ACM* **48** (2001), 798-859.

[23] Johnson, W.B., Schechtman, G. Remarks on Talagrand's deviation inequality for Rademacher functions. Banach Archive 2/16/90, *Springer Lecture Notes* 1470 (1991), 72-77.

[24] Khachiyan, L.G., The problem of calculating the volume of a polyhedron is enumerably hard, *Russian Mathemcatical Surveys*, 44 (1989), 199-200.

[25] Nemirovski, A., Roos, C., Terlaky, T. On maximization of quadratic form over intersection of ellipsoids with common center. *Math. Progr.* **86** (1999), 463-473.

[26] Nemirovski, A. On tractable approximations of randomly perturbed convex constraints. *Proceedings of the 42nd IEEE Conference on Decision and Control Maui, Hawaii USA, December 2003*, 2419-2422.

[27] Nemirovski, A., Shapiro, A. (2005). Convex Approximations of Chance Constrained Programs. Submitted to *SIAM J. on Optimization.* E-print: http://www.optimization-online.org/DB_HTML/2004/12/1033.html

[28] Nemirovski, A., Shapiro, A. Scenario Approximations of Chance Constraints. In: Calafiore, G., Dabbene, F., Eds. *Probabilistic and Randomized Methods for Design under Uncertainty*, Springer, 2006.

[29] Nemirovski, A. Sums of Random Symmetric Matrices and Quadratic Optimization under Orthogonality Constraints. To appear in *Math. Progr. Series B*, Special Issue on Large-Scale and Semidefinite Optimization E-print: http://www.optimization-online.org/DB_HTML/2004/12/1016.html

[30] Nesterov, Yu. Semidefinite relaxation and nonconvex quadratic optimization. *Optim. Methods and Software* **9** (1998), 141-160.

[31] Polyak, B.T. Convexity of quadratic transformations and its use in control and optimization, *J. of Optimization Theory and Applications* **99** (1998), 553-583.

[32] H. Wolkowicz, R. Saigal, L. Vandenberghe, Eds. *Handbook of Semidefinite Programming.* Kluwer Academic Publishers, 2000.