Robust Optimization for Empty Repositioning Problems

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Abstract

This paper develops a robust optimization framework for dynamic empty repositioning problems modeled using time-expanded networks. In such problems, uncertainty arises primarily from forecasts of future supplies and demands for assets at different time epochs. The approach developed in this paper models such uncertainty using intervals about nominal forecast values and a single control parameter k limiting the system-wide absolute deviation from the nominal forecast values (and thus determining the conservatism of a reposition plan). A robust repositioning plan is defined as one in which the typical flow balance constraints and flow bounds are satisfied for the nominal forecast values and the plan is *recoverable* under a limited set of recovery actions, where a plan is recoverable when feasibility can be reestablished for any uncertain outcome. We develop necessary and sufficient conditions for flows to be robust under this definition for three types of allowable recovery actions. When recovery actions allow only flow changes on inventory arcs, we show that the resulting problem is polynomially-solvable. When recovery actions allow limited reactive repositioning flows, we develop feasibility conditions that are independent of the size of the uncertain outcome space.

1 Introduction

Consider a problem faced by most large freight transportation service providers: the management of empty resources over time. Almost all freight transporters serve a set of load requests that is imbalanced in both time and space. Thus, when a resource such as a container or truck driver arrives at the destination location of a loaded move, there may not be an opportunity to match that resource in a timely way with a new loaded move outbound from that location. To correct imbalance, transporters move resources empty between locations and minimizing the costs of such empty moves is a primary challenge.

Currently, most transporters address this problem using simple deterministic flow optimization models over time-expanded networks. Network nodes are defined at relevant decision points, and connect forward in time with other nodes via arcs that represent management decisions (and their costs) such as holding inventory of empty resources, or repositioning such resources between locations. Next, point forecasts are developed for the net expected supply for resources at some or all of the time-space network nodes, and initial and final resource states are specified. A feasible flow on such a network represents a set of feasible empty management decisions, and network optimization algorithms can be used to find an optimal flow. For problems that can be decomposed by resource, the resulting problems are often single-commodity minimum cost network flow problems, which can be solved very efficiently. In practice, these models are used in a rolling horizon implementation where a solution is obtained for a long planning horizon, but only the decisions in an initial set of time periods are implemented.

One major deficiency of this traditional approach is that there may be significant uncertainty in the forecasts of resource net supply at each time-space node, especially when the planning horizon grows large. When realized demands differ from forecasts, the implemented empty allocation plan may be far from optimal. Stochastic models for these problems typically replace point forecasts of expected net supply with distribution forecasts, and attempt to find solutions that minimize total expected cost over the planning horizon; in this case, difficult-to-solve stochastic dynamic programming or stochastic integer programming problems result.

In contrast to existing stochastic approaches that focus on expected cost minimization, this research develops a robust optimization approach for repositioning problems. To do so, we extend the standard single-commodity minimum cost network flow problem on time-expanded networks. To avoid requiring estimates of probability distributions, we model forecast uncertainty using intervals about a nominal expected net supply at each time-space node. The robust approach seeks to find minimum cost solutions that satisfy special feasibility conditions for any demand realization in which (1) each time-space point demand lies within its forecast interval, and (2) the total absolute deviation of the realized demand from the nominal expected demand across all time-space points is bounded by a control parameter k. Parameter k determines the conservatism of the solution: when $k = 0$, the problem reduces to the deterministic problem, and when $k = \infty$, solutions must satisfy the special feasibility conditions for all realizations.

The special feasibility conditions for the developed approach require that a robust repositioning plan (1) satisfies flow balance equalities and flow bounds with respect to the nominal expected net supply values, and (2) is *recoverable*, that is, it can be converted into a plan that satisfies flow balance equalities and flow bounds for every demand realization using some predefined subset of the original allowed decisions, known as the recovery actions. (Note that any solution feasible for a specific realization of net supplies is necessarily infeasible for every other realization.) The recovery actions are similar to recourse actions in two-stage stochastic programming models. The simplest recovery action considered allows flow changes only on inventory arcs between the same space point in consecutive time periods; this scenario, therefore, models the case where each spatial location must hedge independently against uncertain future outcomes. We also consider recovery actions that allow limited reactive repositioning between locations.

The robust repositioning problem that allows only inventory recovery actions is shown to be polynomially-solvable. For the robust repositioning problems that allow reactive repositioning, we develop sets of feasibility conditions whose sizes, while not polynomial, do not grow with the size of the uncertain outcome space. We illustrate the robust repositioning ideas developed with a numerical example. The example provides some insight into how different levels of conservatism, measured by k, and different degrees of flexibility for performing recovery actions affect the cost of a robust repositioning plan, and therefore the price of robustness.

The main contributions of this paper can be summarized as follows.

- This paper introduces a robust optimization framework in which uncertain parameters are assumed to fall within an interval around a nominal value and where the conservatism of a solution is controlled by limiting the total absolute deviation from the nominal values. In addition, the framework explicitly incorporates the notion of recovery actions to dynamically handle realizations of the uncertain parameters.
- This paper shows the implementation and value of the proposed robust optimization framework in the context of empty repositioning problems faced by many freight transportation service providers for three sets of recovery actions.
- This paper shows that for each of the three sets of recovery actions considered for the empty repositing problem, the size of the resulting optimization problem does not depend on the size of the uncertain outcome space, and that for the simplest set of recovery actions the resulting optimization problem can be solved in polynomial time.

The remainder of the paper is organized as follows. Section 2 discusses related literature. Section 3 introduces the robust optimization framework. Section 4 applies the robust optimization framework in the context of empty repositioning problems. Finally, Section 5 presents a numerical example.

2 Related Literature

Empty repositioning problems have received much attention by the research community, and the existing literature for problems of this type is extensive and varied. For operational decision-making, dynamic models are most frequently developed and deployed (see Powell (2003) for an excellent review of dynamic models for transportation operations management). Early important studies of problems in this class assume that decision-makers have complete information for future periods, and deterministic models are developed. Leddon and Wrathall (1967) and Misra (1972) develop models for railcar distribution, and White (1972) for container management. Recent research in this area focuses on solving large-scale problems with time windows, and integrating repositioning decisions with load assignment decisions (see, *e.g.*, Abrache et al. (1999) and Erera et al. (2005)). Deterministic models of dynamic repositioning decisions typically result in linear programs or easy-to-solve integer programs. As input, they require only point forecasts of future resource supplies and demands. For these reasons, they remain popular with many freight transportation companies that still rely on these simple approaches.

Work with stochastic models has focused primarily on expectation minimization approaches, beginning with work in Powell (1986) and Powell (1987). Modelling approaches for stochastic problems, focusing specifically on empty container management, are provided in Crainic et al. (1993). Most computational work in this area has focused on suboptimal solution procedures, developing results for improving value function approximations in iterative dynamic programming solution methods (see, for example, Frantzeskakis and Powell (1990), Cheung and Powell (1996), and Powell and Carvalho (1998)). More recently, adaptive approaches to approximating nonlinear value functions have been successfully applied to both single commodity and multicommodity problems (see, for example, Godfrey et al. (2002a), Godfrey et al. (2002b), and Topaloglu and Powell (2004)). Topaloglu and Powell (2003) provides a theoretical underpinning for much of this work, proving the optimality of a particular variant of the general sampling approach developed first in Godfrey et al. (2002a) when points along the value function are sampled infinitely often.

Robust optimization is emerging as a leading alternative approach for modeling and solving decision optimization problems given uncertainty. Soyster (1973) is the first work to consider coefficient uncertainty in linear programming formulations, and shows that such uncertainty can be handled by an equivalent linear programming model. The approach, however, is very conservative since it protects feasibility against the worst-case outcome. Ben-Tal and Nemirovski (2000) readdresses coefficient uncertainty in linear programs, and develops a solution approach based on a convex optimization problem over a second-order cone. To allow less conservative modelling of coefficient robustness without resorting to a nonlinear model, Bertsimas and Sim (2004) models coefficient variability using real intervals about a nominal value, and uses a control parameter for each constraint that limits the number of coefficient values that can simultaneously take their worst-case value. For this characterization of uncertainty, the resulting robust optimization is still a linear optimization problem. Bertsimas et al. (2004) develops robust versions of linear programming problems with coefficient uncertainty sets described by an arbitrary norm. The approach developed in this paper is motivated by the work in Bertsimas and Sim (2004).

3 Robust Optimization for Equality-constrained Problems with Right-hand-side Uncertainty

Before focusing specifically on empty repositioning flow problems, we first develop a general framework for robust optimization problems with linear equality constraints and right-hand-side uncertainty. The framework is essentially a two-stage approach where we search for first-stage solutions that can be made feasible for any uncertain realization via some restricted set of second-stage solution transformation actions.

Consider first the nominal problem:

$$\mathbf{NP} \quad \min_{x} \left\{ c^T x : Ax = b,\ x \in X \cap \mathbb{R}^n \right\} \tag{1}$$

where c and x are an n-vectors, A is an m by n matrix, b is a deterministic m-vector referred to as the *vector of nominal right-hand-side values*, and set X represents constraints that potentially further restrict the feasibility space. Suppose now that the right-hand side vector of the system $Ax = b$ may be uncertain, and that each realization of the right-hand side is a realization of the random vector $\widetilde{b}$ for which it is assumed that $\widetilde{b} \in [b - \underline{b}, b + \overline{b}]$, where parameter vectors $\underline{b}, \overline{b} \geq 0$ are known. We do not assume knowledge of a distribution for $\widetilde{b}$ on this interval.

In typical robust optimization problems, one searches for a solution x that remains feasible given any realization of the uncertain parameters. Any realization of $\widetilde{b}$ can be characterized by the perturbation set

$$\Gamma = \left\{ \delta \in \mathbb{R}^m : -\underline{b} \leq \delta \leq \overline{b} \right\}, \tag{2}$$

where $\widetilde{b} = b + \delta$ for some $\delta \in \Gamma$. However, it is clear that any feasible solution x to **NP** is infeasible given any non-zero perturbation $\delta \in \Gamma$ to b. Thus, any feasible solution of the nominal problem is not robust by the typical definition.

To develop an alternative concept of solution robustness in problems with equality constraints, we define a special type of feasibility by permitting certain *feasibility recovery transformations* to x given a realization of $\widetilde{b}$. Such solution transformations are similar in spirit to the recourse actions in two-stage stochastic programming approaches, except here we focus only on the question of whether feasibility can be recovered and ignore any costs.

Not all solution transformations should be permitted; in fact, placing no restrictions on allowable solution transformations leads to trivial and impractical problems. We assume instead that feasibility recovery transformations to x are limited by a set W defined by *feasibility recovery constraints*, which should be developed in the context of the application problem. Given W, an *allowable transformation* of solution x is $x + w$, where $w \in W \cap \mathbb{R}^n$. Then, we can define a set $H(W, \delta)$ of nominal problem solutions x that are *recoverable* since they can be made feasible for any uncertain realization $b + \delta$ by some allowable transformation:

$$H(W, \delta) = \{x \mid \exists\, w \;:\; Aw = \delta,\; x + w \in X,\; w \in W \cap \mathbb{R}^n\}$$

The last step in defining a robust optimization problem with equality constraints restricts the size of the potential uncertain outcome space. As noted in Bertsimas and Sim (2004), it may be overly conservative to find robust solutions for every uncertain outcome in Γ. Therefore, let $\varphi \subseteq \Gamma$ represent a limited uncertain outcome space. We then seek a minimum cost feasible solution for the nominal problem for which feasibility can be recovered using allowable transformations allowed by W for uncertain outcomes given by φ. Formally, we define the following robust optimization problem:

$$\mathbf{ROP}(W, \varphi) \quad \min_{x} \left\{ cx : Ax = b,\; x \in X \cap \mathbb{R}^n,\; x \in \bigcap_{\delta \in \varphi} H(W, \delta) \right\} \tag{3}$$

Alternatively, (3) can be rewritten in an extended form as

$$\begin{array}{lll}
\mathbf{ROP}(W, \varphi) & \text{minimize} & c\,x \\
& \text{s.t.} & Ax = b \\
& & x \in X \cap \mathbb{R}^n \\
& & Aw_\delta = \delta \quad \forall\, \delta \in \varphi \\
& & x + w_\delta \in X \cap \mathbb{R}^n \quad \forall\, \delta \in \varphi \\
& & w_\delta \in W \cap \mathbb{R}^n \quad \forall\, \delta \in \varphi
\end{array}$$

where w_δ is a feasibility recovery transformation vector applied to x given uncertain outcome δ. Observe that the number of decision variables and the number of constraints in (3) may dramatically increase from (1) when the size of the outcome space φ is large.

4 Robust Empty Repositioning

We now consider the $ROP(W, \varphi)$ problem framework in the context of an empty repositioning problem. Consider a transport operator managing a homogeneous fleet of reusable resources using a centralized control. Such resources may represent for instance containers, tank-containers, railroad cars or trucks. Further suppose that to manage these resources, the decision-maker need only track two state attributes: location and empty/loaded status. Other potential state attributes (*e.g.*, those required for maintenance) are ignored.

To manage this system, the operator maintains a network of storage depots from which all empty resources are sourced and to which all empty resources are returned. The empty repositioning problem of the operator, then, is to determine a plan for repositioning empty resources between these depots to satisfy loaded move requirements.

In practice this problem arises naturally for tank container fleet operators. Such operators move loads globally using ocean transportation, and face significant loaded flow imbalance. One large operator in this industry manages its empty repositioning flows using a deterministic time-space network flow formulation, considering a six-month planning horizon discretized into weeks (Stolt-Nielsen Transportation Group, 2004). The operator develops point forecasts of the expected net supply of containers at each depot during each week of the horizon; negative net supply corresponds to demand for containers. Using costs for repositioning containers between depots, the operator determines an empty repositioning plan for the horizon and then implements the decisions for the current week. The process is repeated weekly.

4.1 The Nominal Repositioning Problem

Suppose that b, the vector of nominal right-hand side values, is the vector of time-space net supply forecasts in the container repositioning problem described above. The nominal empty repositioning problem can then be modeled using a time-expanded network $G = (N, \mathcal{A})$. Assume for simplicity that a planning horizon has been discretized into $\rho + 1$ periods, $\{0, 1, 2, ..., \rho\}$. Let D be the set of depots in the system, and let $V^d = \{v_0^d, v_1^d, ..., v_\rho^d\}$ for each $d \in D$ be the ordered set of nodes v_t^d representing depot d at each time period t. Let $V = \cup_{d \in D} V^d$. Let $b(v)$ represent the component of b corresponding to $v \in V$.

Containers can be held in inventory at a depot from one time period to the next. Therefore, an *inventory arc* (v_t^d, v_{t+1}^d) exists for each $d \in D$ and $0 \leq t < \rho$. Let I be the set of all inventory arcs. A *repositioning arc* (v_t^i, v_{t+h}^j) is defined between depots i and j at time t when available, where $h \geq 1$ is the travel time in periods along this arc. Let R be the set of all repositioning arcs. To complete the network specification, we add to G a sink node s with net supply $b(s) = -\sum_{v_t^d \in V} b(v_t^d)$ and an arc connecting v_ρ^d to s for all $d \in D$. For consistency we add these arcs to set I; for simplicity, node s can be labeled as $v_{\rho+1}^d$ for any $d \in D$. Let $N = V \cup \{s\}$ and $\mathcal{A} = I \cup R$.

For each $a \in \mathcal{A}$, let $c(a)$ be the unit cost of flow on arc a. For $a \in I$, $c(a)$ represents per period inventory holding costs per unit, and for $a \in R$ repositioning costs per unit. The nominal repositioning problem may now be written as:

$$\textbf{NP} \quad \min_x \left\{ cx : Ax = b, x \in \mathbb{Z}_+^{|\mathcal{A}|} \right\} \tag{4}$$

where the decision vector x corresponds to the empty container flow on each arc and A is the node-arc incidence matrix implied by G defining the typical network flow-balance constraints $Ax = b$. Let $x(a)$ represent the flow on arc $a \in A$. Note that this formulation is equivalent to (1) where $X = \mathbb{Z}_+^{|\mathcal{A}|}$.

It is well-known that problem (4) can be solved to optimality in polynomial time with standard minimum cost network flow algorithms, or via linear programming. We note that a feasible solution may not exist for this problem as posed; well-known techniques can address this problem, but for clarity and simplicity we assume that a feasible solution to (4) exists.

4.2 Three Robust Repositioning Problems

Since the nominal estimate of the net supply $b(v)$ at each node $v \in V$ may be uncertain, we now apply the robust framework $ROP(W, \varphi)$ to this repositioning problem. Let $\widetilde{b}(v) \in \mathbb{Z}$ be the random variable representing net supply at $v \in V$. Suppose that the decision-maker is able to estimate $\underline{b}(v)$ and $\overline{b}(v)$ for each $v \in V$, such that $\widetilde{b}(v) \in [b(v) - \underline{b}(v), b(v) + \overline{b}(v)]$. We assume that the decision-maker knows with certainty the net supplies in the initial period, and therefore $\underline{b}(v_0^i) = \overline{b}(v_0^i) = 0$ for each $i \in D$. Given these parameters, the perturbation set Γ given in (2) is completely specified by additionally letting $\underline{b}(s) = \overline{b}(s) = \infty$.

To control the conservatism of our robust repositioning models, we will restrict the set of allowable perturbations from the nominal vector b. To do so, let k represent the maximum total absolute deviation between $\widetilde{b}$ and b that we might expect. Using k, we define a limited perturbation set φ_k as

$$\varphi_k = \Gamma \cap \left\{ \delta \in \mathbb{Z}^{|N|} \mid \sum_{v \in V} |\delta(v)| \leq k \;,\; \delta(s) = -\sum_{v \in V} \delta(v) \right\}.$$

where $\delta(n)$ represents the component of δ corresponding to each node $n \in N$, *i.e.*, $\widetilde{b}(n) = b(n) + \delta(n)$. Note that the constraint on $\delta(s)$ is a technical condition only to preserve balance (for this reason, we will ignore $\delta(s)$ for the majority of the discussion to follow). Given this restricted set, parameter k gives complete control over the conservatism of the robust optimization model. When $k = 0$, the decision-maker is most aggressive assuming that every realization will conform to nominal. When $k = \infty$, the decision-maker protects against all potential outcomes.

We now specify several different robust repositioning problems, each defined by a different feasibility recovery set W. For each problem, we develop necessary and sufficient conditions for feasible solutions.

4.2.1 The Inventory Robust Repositioning Problem

Suppose that a decision-maker would like each depot to hedge independently against uncertainty using its own inventory. To model this case, let W_1 be the set of feasibility recovery transformation vectors w that allow integer flow changes only on inventory arcs:

$$W_1 = \{w \in \mathbb{Z}^{|\mathcal{A}|} \mid w(a) = 0 \quad \forall \;\; a \in R\}.$$

Flow changes on inventory arcs can be interpreted as using containers in inventory to satisfy a larger-than-expected demand (a negative flow change), or adding extra containers to inventory in the event of a larger-than-expected net supply (a positive flow change).

Setting $X = \mathbb{Z}_+^{|\mathcal{A}|}$, the recoverable set $H(W_1, \delta)$ for a given perturbation δ from nominal is given by:

$$H(W_1, \delta) = \left\{ x \mid \; \exists \; w \;:\; Aw = \delta, \; x + w \in \mathbb{Z}_+^{|\mathcal{A}|} \; w \in W_1 \cap \mathbb{R}^{|\mathcal{A}|} \right\}.$$

We can now define the inventory robust optimization problem using our earlier notation:

$$\mathbf{ROP1} = \mathbf{ROP}(W_1, \varphi_k).$$

Any repositioning plan x satisfying the feasibility conditions of **ROP1** is called *k-robust inventory feasible*. **ROP1** seeks the minimum cost k-robust inventory feasible solution, and can be written in extended form as

$$
\begin{aligned}
\mathbf{ROP1}(W_1, \varphi_k) \quad & \text{minimize} \quad && c\,x \\
& \text{s.t.} && Ax = b && (5)\\
& && x \in \mathbb{Z}_+^{|\mathcal{A}|} && (6)\\
& && Aw_\delta = \delta \quad \forall\, \delta \in \varphi_k && (7)\\
& && x + w_\delta \in \mathbb{Z}_+^{|\mathcal{A}|} \quad \forall\, \delta \in \varphi_k && (8)\\
& && w_\delta(a) = 0 \quad \forall\, a \in R,\ \ \delta \in \varphi_k && (9)\\
& && w_\delta \in \mathbb{Z}^{|\mathcal{A}|} \quad \forall\, \delta \in \varphi_k && (10)
\end{aligned}
$$

While correct, the above formulation requires for each potential uncertain outcome δ a vector w_δ of decision variables representing the feasibility recovery transformation and an associated set of flow balance constraints. Clearly, such a formulation becomes intractable as the size of the outcome space grows. We now show that **ROP1** alternatively can be solved as a minimum cost network flow problem with flow lower bound constraints using only the original flow variables x.

To do so, define the *vulnerability* $\vartheta(a)$ of time-space arc $a = (v_t^d, v_{t+1}^d) \in I$ as

$$\vartheta(a) = \sum_{s=1}^{t} \underline{b}(v_s^d).$$

By this definition, the vulnerability of an inventory arc is the maximum unconstrained cumulative negative deviation from the nominal net supply estimate up to its tail in any realization. Similarly, given a specific uncertain outcome $\delta \in \varphi_k$, let $\sigma(a)$ be the cumulative deviation from nominal net supply at depot d by time t:

$$\sigma(a) = \sum_{s=1}^{t} \delta(v_s^d).$$

Clearly, $-\sigma(a) \leq \vartheta(a)$ for all $a \in I$ and $\delta \in \varphi_k$.

The relationship between the flow on an inventory arc $x(a)$ and its corresponding vulnerability will determine whether or not a solution is k-robust inventory feasible. This motivates the following definition.

Definition 1 (Weak Arc) *For a given repositioning plan x, an inventory arc $a \in I$ is a weak arc if*

$$x(a) < \min\{k, \vartheta(a)\}.$$

Observe that if $a = (v_t^d, v_{t+1}^d)$ is a weak arc, then the inventory at time t at depot d cannot be used to protect against every potential uncertain realization in φ_k.

The following theorem now characterizes the set of feasible solutions for **ROP1**:

Theorem 1 *A feasible solution x for the nominal problem* (4) *is also a feasible solution for* ***ROP1*** *if and only if*

$$x(a) \geq \min\{k, \vartheta(a)\} \quad \forall \; a \in I. \tag{11}$$

Proof: Given the definition of W_1, for any $\delta \in \varphi_k$ the only transformation vector w that can feasibly satisfy constraints (7), (9), and (10) in **ROP1** is given by

$$w(a) = \sigma(a) \quad \forall \; a \in I. \tag{12}$$

Thus, we focus attention on constraints (8).

To show necessity by contradiction, suppose that there exists a feasible solution x for **ROP1** such that $x(a) < \min\{k, \vartheta(a)\}$ for some arc $a = (v_t^d, v_{t+1}^d)$. Defining $a_s^d = (v_s^d, v_{s+1}^d)$, consider perturbation vector $\delta \in \varphi_k$ such that $\delta(v_1^d) = -\min\{k, \vartheta(a_1^d)\}$, $\delta(v_s^d) = -\min\{k, \vartheta(a_s^d)\} - \sigma(a_{s-1}^d)$ for all $1 < s \leq t$, and $\delta(v) = 0$ for all other $v \in V$. Thus, from (12), note that $w(a) = -\min\{k, \vartheta(a)\}$ and therefore that the transformed flow on arc a is $x(a) - \min\{k, \vartheta(a)\} < 0$ and thus violates constraint (8). Therefore, x cannot be a feasible solution for **ROP1**.

Sufficiency can also be shown by contradiction. Let x be a feasible solution of (4) satisfying (11). Now, consider perturbation vector $\delta \in \varphi_k$ such that after applying transformation (12) to x there exists an arc $a = (v_t^d, v_{t+1}^d) \in I$ such that $x(a) + w(a) < 0$. This implies $\min\{k, \vartheta(a)\} \leq x(a) < -\sigma(a)$, which then implies $\delta \notin \varphi_k$. □

Theorem 1 shows that **ROP1** can be solved for any given φ_k by adding a flow lower bound constraint for each inventory arc $a \in I$ to (4). Thus, **ROP1** is polynomially-solvable using standard minimum cost network flow algorithms. Also, observe that the lower bound constraints for a specific depot j are independent of the vulnerability of the arcs for any other depot in the system. A depot is therefore called a *k-robust depot* if and only if all of its inventory arcs satisfy conditions (11) (*i.e.*, all inventory arcs are not weak arcs); hence, a solution is k-robust inventory feasible if and only if all depots in the system are k-robust depots.

It is also important to note that in order to develop the necessary and sufficient conditions in Theorem 1, we need only consider perturbations $\delta \leq 0$. Any positive component in δ implies an unexpected addition of containers into some depot of the system, and such an event does not act against the interests of a decision-maker attempting to determine a feasible container allocation. In the remainder of this paper, we only consider perturbation vectors $\delta \leq 0$.

4.2.2 The Inventory-Pooling Robust Repositioning Problem

Suppose that container depots can hedge against uncertainty not only using their own inventory but also using inventory at other depots in the system. We use the term *reactive repositioning* to refer to depot-to-depot container repositioning conducted in response to a perturbation from expected net supplies. For a given value of k, depots may be able to jointly hedge against uncertainty with fewer total container resources.

This idea is illustrated for a simple two-depot system in Figure 1. The number inside each node corresponds to the nominal net supply value, the number above each arc corresponds to its flow, and the interval above a node determines the range in which δ can take values. Observe that depot A is a 1-robust depot, but depot B is not. However, if we could reactively reposition a unit of inventory at time 1 from depot A to depot B, then we could recover feasibility for this problem given any realization with at most $k = 1$ total deviation from the nominal values across all depots. Since depots A and B are sharing the single container resource in inventory to collectively hedge against uncertainty, such a solution is called an *inventory pooling 1-robust* solution.

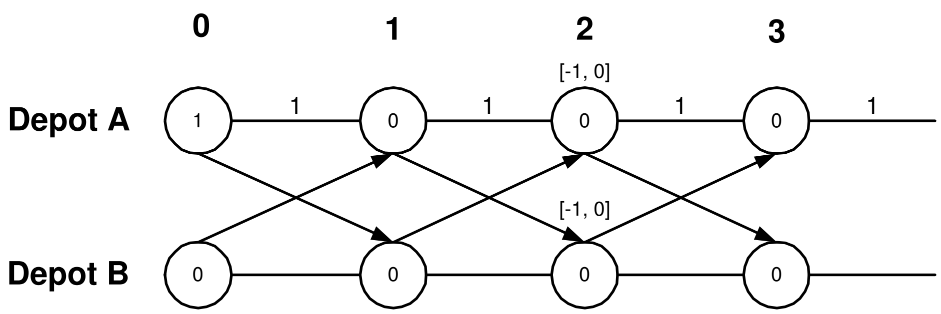

Figure 1: The solution is robust for $k = 1$ if containers in Depot A can be repositioned reactively to Depot B.

We now formally define a feasibility recovery set W_2 for the inventory-pooling scenario:

$$W_2 = \{w \in \mathbb{Z}^{|\mathcal{A}|} \mid w(a) \geq 0 \quad \forall \; a \in R, \quad w(a) = 0 \quad \forall \; a = (v_0^j, u) \in R\}.$$

The set W_2 allows any integer flow change on each inventory arc, and non-negative integer flow changes on repositioning arcs. By enforcing non-negativity, we assume that only minor changes are allowed to the initial repositioning plan. Further, we do not allow any reactive flow changes on repositioning arcs that begin in the initial time epoch, since such decisions are assumed to be fixed. If the decision-maker intends to fix decisions for multiple initial time epochs, additional constraints could be added to W_2.

Again setting $X = \mathbb{Z}_+^{|\mathcal{A}|}$, the recoverable set $H(W_2, \delta)$ for a given perturbation δ from nominal is given by

$$H(W_2, \delta) = \left\{x \mid \; \exists \; w \; : \; Aw = \delta, \; x + w \in \mathbb{Z}_+^{|\mathcal{A}|} \; w \in W_2 \cap \mathbb{R}^{|\mathcal{A}|}\right\},$$

and the general inventory-pooling robust repositioning problem is then:

$$\mathbf{ROP2} = \mathbf{ROP}(W_2, \varphi_k).$$

We can formulate Problem **ROP2** in extended form:

$$
\begin{aligned}
\mathbf{ROP2}(W_2, \varphi_k) \text{ min} \quad & c\,x \\
\text{s.t.} \quad & Ax = b && (13)\\
& x \in \mathbb{Z}_+^{|\mathcal{A}|} && (14)\\
& Aw_\delta = \delta \quad \forall\, \delta \in \varphi_k && (15)\\
& x + w_\delta \in \mathbb{Z}_+^{|\mathcal{A}|} \quad \forall\, \delta \in \varphi_k && (16)\\
& w_\delta(a) = 0 \quad \forall\, a = (v_0^j, u) \in R, \forall \delta \in \varphi_k && (17)\\
& w_\delta(a) \geq 0 \quad \forall\, a \in R, \forall \delta \in \varphi_k && (18)\\
& w_\delta \in \mathbb{Z}^{|\mathcal{A}|} \quad \forall\, \delta \in \varphi_k && (19)
\end{aligned}
$$

This direct integer programming formulation for **ROP2** may become difficult to solve as the space of feasible uncertain outcomes defined by φ_k grows large. Therefore, following the approach for the inventory robust repositioning problem, we seek methods for determining an optimal inventory-pooling robust solution that do not rely on enumerating the outcome space.

Creating a set of necessary and sufficient constraints for a nominal solution x to be robust with respect to any set of allowable recovery actions W requires ensuring that x is in the recoverable set $H(W, \delta)$ for every $\delta \in \varphi_k$. A useful methodology for testing this condition for the recovery actions considered in this paper is to use the existence conditions for a feasible flow on a properly defined *recovery network*.

Let $G_W = (N_W, A_W)$ refer to the recovery network corresponding to allowable recovery action set W. The node set N_W is the same as the node set N of G. The arc set A_W contains all inventory arcs in I, and all repositioning arcs in R on which recovery flow is permitted to be nonzero by W. In the case of recovery set W_2, G_{W_2} contains only inventory arcs and each repositioning arc departing a depot at time $t > 0$.

To determine whether a repositioning plan x is recoverable using the action set W_2 for a specific realization $\delta \in \varphi_k$, we add appropriate net supplies to the nodes N_{W_2} and search for a feasible flow on G_{W_2}. To do so, let $\mathcal{I}^{x,\delta}(v)$ at each time-space depot node represent the marginal net inventory of containers available (or needed) in the recovery problem given x and δ:

$$
\begin{aligned}
\mathcal{I}^{x,\delta}(v_0^d) &= 0 \quad \text{for all } d \in D \\
\mathcal{I}^{x,\delta}(v_1^d) &= x(v_1^d, v_2^d) + \delta(v_1^d) \quad \text{for all } d \in D \\
\mathcal{I}^{x,\delta}(v_t^d) &= x(v_t^d, v_{t+1}^d) - x(v_{t-1}^d, v_t^d) + \delta(v_t^d) \quad \text{for all } d \in D,\ 1 < t \leq \rho
\end{aligned}
$$

where we note that $\delta(v_0^d) = 0$ for all $d \in D$. To understand this definition, suppose initially that $\delta = \mathbf{0}$. Given repositioning plan x, $\mathcal{I}^{x,0}(v_1^d)$ is the initial inventory at depot d that could be repositioned to serve recovery needs elsewhere. For $t > 1$, a positive value of $\mathcal{I}^{x,0}(v_t^d)$ indicates an increase in the number of units in inventory at depot d at time t, and therefore an additional number of containers that may be repositioned if warranted, or held in inventory to serve container needs at future times. On the other hand, a negative value of $\mathcal{I}^{x,0}(v_t^d)$ indicates a reduction in container inventory at time t. Such a reduction represents a demand for containers at that time. A container shortage will occur if the reduction is not satisfied by inbound containers either from inventory or via reactive repositioning. Observe that $\sum_{s=1}^{t} \mathcal{I}^{x,0}(v_s^d)$ corresponds to the actual inventory at time t at depot d. Since x is feasible for the nominal problem, this inventory is nonnegative for all values of t.

Given a nonzero realization vector δ, the net inventory at node v_t^d is changed by $\delta(v_t^d)$. Hence, the definition of $\mathcal{I}^{x,\delta}(v)$ models the net inventory availability (or requirement) at each node after the realization. Observe that $\sum_{s=1}^{t} \mathcal{I}^{x,\delta}(v_s^d)$ is not necessarily greater than or equal to 0 for all values of t. A negative value for this expression implies the necessity to reactively reposition units into the depot by time t in order to avoid a container shortage.

Using $\mathcal{I}^{x,\delta}$, we can complete the definition of the recovery network G_{W_2}. Each arc $a \in A_{W_2}$ is given a flow lower bound $\ell(a) = 0$ and upper bound $u(a) = +\infty$. The net supplies d at each node

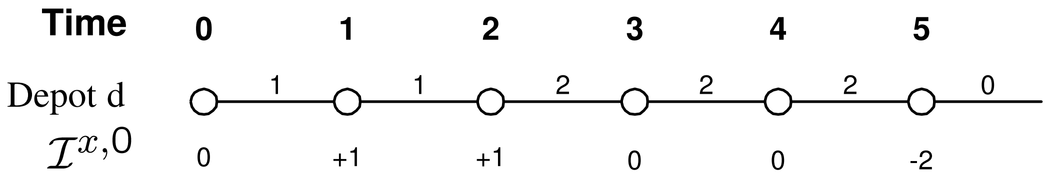

Figure 2: Given the flow on the inventory arcs of depot d, the value of $\mathcal{I}^{x,0}$ indicates that one unit at time 1 is available for reactive repositioning, and that an additional unit becomes available at time 2. The negative value of $\mathcal{I}^{x,0}$ indicates that at least 2 units must be in inventory by time 5. If any units are repositioned out of d, the same number must be repositioned back no later than time 5.

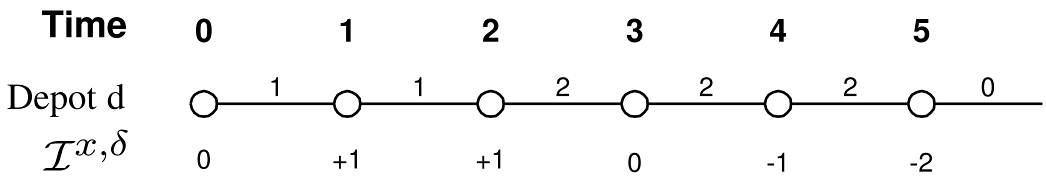

Figure 3: Value of $\mathcal{I}^{x,\delta}$ after perturbation $\delta(v_4^d) = -1$, $\delta(v_1^d) = \delta(v_2^d) = \delta(v_3^d) = \delta(v_5^d) = 0$. At least one unit must be in inventory by time 4 and at least 2 additional units by time 5 to recover feasibility.

are given by

$$\begin{aligned} d(v_t^d) &= \mathcal{I}^{x,\delta}(v_t^d) \quad \forall\, d \in D,\ t = 0, 1, 2, \cdots, \rho \\ d(s) &= -\sum_{v \in V} d(v) \end{aligned}$$

Let $G_{W_2}(x, \delta)$ refer to the recovery network with net supply vector d defined as above.

Proposition 1 *A feasible solution x for the nominal problem* (4) *belongs to the recoverable set $H(W_2, \delta)$ for a given $\delta \in \varphi_k$ if and only if there exists a feasible flow in $G_{W_2}(x, \delta)$.*

Proof: By construction of the network and its associated net supplies d, a feasible flow in $G_{W_2}(x, \delta)$ defines a set of feasible reactive repositioning decisions and inventory flow changes w restoring the feasibility of x given δ: for arcs $a \in R$, $w(a)$ is simply the flow on the corresponding arc in A_{W_2}, and for arcs $a = (v_t^d, v_{t+1}^d) \in I$ for $t > 1$, $w(a)$ is the flow on the arc in A_{W_2} minus $x(a)$. It is also not difficult to see that a w corresponding to an $x \in H(W_2, \delta)$ can be used to construct a feasible flow in $G_{W_2}(x, \delta)$: for $a \in R$, the flow on the associated arc in A_{W_2} is $w(a)$ and for $a \in I$, the flow is $x(a) + w(a)$. $\square$

We now derive necessary and sufficient conditions for the existence of a feasible flow in general recovery networks $G_W(x, \delta)$, where W allows general flow changes on all inventory arcs (v_t^d, v_{t+1}^d) for $t \geq 1$ and non-negative or zero flow changes on repositioning arcs. To do so, we first introduce two definitions.

Definition 2 (Competing Arc Set) *A set of arcs $S \subseteq I$ is competing if every directed path P in G_W has $|P \cap S| \leq 1$.*

Such arcs essentially compete for inventory to protect against uncertainty, since containers moved to satisfy a need of one arc cannot be later used to satisfy the need of any other in the set since no path for flow exists.

Definition 3 (Inbound-closed Node Set) *A set of nodes $C \subseteq N_W$ is inbound-closed if there exists no directed path P in G_W from any node $i \in N_W \setminus C$ to any node $j \in C$.*

Using these definitions, we can define a set of cuts in G_W that determine whether or not a feasible flow exists in $G_W(x, \delta)$. Let $\Delta^{\text{out}}(U) = \{(u, v) \in A_W \mid u \in U,\ v \in N_W \setminus U\}$. Define

$$\mathcal{U}_W = \{U \subseteq N \mid U \text{ is inbound-closed, } \Delta^{\text{out}}(U) \cap I \text{ is competing}\}.$$

Proposition 2 *There exists a feasible flow in $G_W(x, \delta)$ if and only if for every set of nodes $U \in \mathcal{U}_W$*

$$\sum_{v \in U} d(v) \geq 0.$$

Proof: It is known (see, *e.g.*, Cook et al., 1998) that there exists a feasible flow in $G_W(x, \delta)$ if and only if

$$\sum_{a \in \Delta^{\text{out}}(U)} \ell(a) \leq \sum_{v \in U} d(v) + \sum_{a \in \Delta^{\text{out}}(N \setminus U)} u(a) \qquad \text{for all } U \subseteq N. \tag{20}$$

Consider then any node set $U \subseteq N$ such that $U \notin \mathcal{U}_W$. By definition of $\mathcal{U}_W$, there must exist a $v_1 \in N \setminus U$ and $v_2 \in U$ where $(v_1, v_2) \in A_W$. Since $u((v_1, v_2)) = +\infty$, (20) is always satisfied for such U.

Now consider any node set $U \in \mathcal{U}_W$, and note that by definition, there are no arcs into set U. Since $\ell(a) = 0$ for all $a \in A_W$, (20) reduces to

$$0 \leq \sum_{v \in U} d(v) \quad \text{for all } U \in \mathcal{U}_W.$$

□

The necessary and sufficient conditions in Proposition 2 can be enforced through a set of constraints on the nominal flow variables given by the following theorem:

Theorem 2 *A feasible solution x of the nominal problem* (4) *is also feasible for **ROP2** if and only if for every set of nodes $U \in \mathcal{U}_{W_2}$*

$$\sum_{a \in \Delta^{out}(U) \cap I} x(a) \geq \min \left\{ k, \sum_{a \in \Delta^{out}(U) \cap I} \vartheta(a) \right\}$$

Proof: Let x be a feasible solution of the nominal problem (4) and $\delta \in \varphi_k$. By definition of $\mathcal{I}^{x,\delta}$ it is clear that

$$\sum_{s=1}^{t} \mathcal{I}^{x,\delta}(v_s^d) = x(v_t^d, v_{t+1}^d) + \sigma((v_t^d, v_{t+1}^d)) \qquad \forall\, d \in D$$

and therefore that

$$\sum_{v\in U} \mathcal{I}^{x,\delta}(v) = \sum_{a\in\Delta^{\text{out}}(U)\cap I} x(a) + \sum_{d\in D} \sigma(a^d)$$

for every $U \in \mathcal{U}_W$, where $a^d \in \Delta^{\text{out}}(U) \cap I$ is the inventory arc for depot d in cut $\Delta^{\text{out}}(U)$. Thus, by Propositions 1 and 2, solution x is feasible for **ROP2** if and only if

$$\sum_{v\in U} \mathcal{I}^{x,\delta}(v) = \sum_{a\in\Delta^{\text{out}}(U)\cap I} x(a) + \sum_{d\in D} \sigma(a^d) \geq 0 \qquad \forall\, U \in \mathcal{U}_{W_2} \tag{21}$$

holds for each $\delta \in \varphi_k$.

But when $\delta \in \varphi_k$, $-\sum_{d\in D} \sigma(a^d)$ can be bounded as $\delta \in \varphi_k$ implies that

$$-\sigma(a^d) \leq \vartheta(a^d) \quad \forall\, d \in D$$

and the total deviation will be no more than k, which yields

$$-\sum_{d\in D} \sigma(a^d) \leq \min\left\{k, \sum_{a\in\Delta^{\text{out}}(U)\cap I} \vartheta(a)\right\}.$$

We note that this bound is tight for at least one $\delta \in \varphi_k$. Thus, condition (21) simplifies to

$$\sum_{a\in\Delta^{\text{out}}(U)\cap I} x(a) \geq \min\left\{k, \sum_{a\in\Delta^{\text{out}}(U)\cap I} \vartheta(a)\right\} \quad \text{for all } U \in \mathcal{U}_{W_2}.$$

□

As an aside, we note that the conditions in Theorem 2 (and the techniques used to develop the theorem) could also apply for the inventory robust problem **ROP1** given an appropriately-defined recovery network G_{W_1} and set $\mathcal{U}_{W_1}$.

By Theorem 2, an alternative integer programming formulation for **ROP2** is:

$$\begin{aligned} \min \quad & c\,x \\ \text{s.t.} \quad & Ax = b \\ & \sum_{a\in\Delta^{\text{out}}(U)\cap I} x(a) \geq \min\left\{k, \sum_{a\in\Delta^{\text{out}}(U)\cap I} \vartheta(a)\right\} \quad \text{for all } U \in \mathcal{U}_{W_2} \\ & x \in \mathbb{Z}_+^{|\mathcal{A}|} \end{aligned}$$

Observe that the size of the constraint set specifying necessary and sufficient conditions for a feasible solution to **ROP2** is independent of the size of the uncertain outcome space characterized by φ_k. Note also that the number of variables required for the robust formulation is equal to the number required for the nominal problem. However, this formulation requires a separate constraint for each element of $\mathcal{U}_{W_2}$, which is the set of all inbound-closed subsets of the nodes N_W that satisfy the competing inventory arc condition. Although this paper does not specifically address the problem of generating this set for a given problem instance, it should be clear that this may be computationally difficult and that the resultant integer program may still be difficult to solve given the large constraint set.

4.2.3 A Restricted Inventory-Pooling Robust Repositioning Problem

A more tractable inventory-pooling robust repositioning problem with applicability to practice specifies those depots in the system that serve only as providers of reactive repositioning, and those that serve only as recipients. Suppose that D then is partitioned *a priori* into two subsets: the depots in D_s can reposition containers reactively to other depots, but do not receive such support, while the depots in D_r may receive reactive repositioning containers but do not provide them. For example, in a geographic region a container operator may have a large hub depot, and many smaller depots. The operator then might wish to include the hub in D_s, and the smaller depots in D_r.

The feasibility recovery set W_3 can be generated for this scenario by a simple modification of W_2 where we restrict outbound reactive repositioning from depots in D_r and inbound reactive repositioning to depots in D_s:

$$\begin{aligned} W_3 = \{w \in \mathbb{Z}^{|\mathcal{A}|} \mid w(a) &\geq 0 \quad \forall \quad a \in R, \\ w(a) &= 0 \quad \forall \quad a \in \{(v_0^j, u) \in R\} \;\cup\; \{(v_t^a, v_{t+h}^b) \in R \mid a \in D_r \text{ or } b \in D_s\}\} \end{aligned}$$

With $X = \mathbb{Z}_+^{|\mathcal{A}|}$, the recoverable set $H(W_3, \delta)$ for a given realization $\delta \in \varphi_k$ is then

$$H(W_3, \delta) = \left\{x \mid \; \exists\, w \;:\; Aw = \delta,\; x + w \in \mathbb{Z}_+^{|\mathcal{A}|} \; w \in W_3 \cap \mathbb{R}^{|\mathcal{A}|}\right\},$$

and a restricted inventory-pooling robust repositioning problem is

$$\mathbf{ROP3} = \mathbf{ROP}(W_3, \varphi_k).$$

In this setting, it is clear that each depot in D_s must not dispatch inventory for reactive repositioning that it will need later to cover its own needs. Recall that given a nominal solution x and a realization $\delta \in \varphi_k$, the inventory available at time t at depot $d \in D_s$ is given by

$$\sum_{s=1}^{t} \mathcal{I}^{x,\delta}(v_s^d) = x(v_t^d, v_{t+1}^d) + \sigma((v_t^d, v_{t+1}^d)) \qquad \forall\, d \in D.$$

Furthermore, any containers that are reactively repositioned from depot d at or before time t will reduce this available inventory. Since this adjusted inventory cannot fall below zero, we can define the available container *support* $\mathcal{S}^{x,\delta}(v_t^d)$ at d at time t as the maximum number of containers that can be reactively repositioned from depot d by time t given nominal flow x and uncertain outcome δ, such that no container shortage occurs at d after time t. Mathematically,

$$\mathcal{S}^{x,\delta}(v_t^d) = \begin{cases} 0 & \text{if } t = 0 \\ \min_j \{x(v_j^d, v_{j+1}^d) + \sigma((v_j^d, v_{j+1}^d)) \;\mid\; t \leq j \leq \rho\} & \text{otherwise} \end{cases}$$

Note that for a fixed d, $\mathcal{S}^{x,\delta}(v_t^d)$ is a *non-decreasing* function of t. Support at time 0 is defined again to indicate that no reactive repositioning is allowed at that time epoch.

Following the approach for **ROP2**, we define a recovery network G_{W_3} that will be used to develop conditions for the existence of a feasible recovery flow given a nominal problem solution. The network G_{W_3} is specified using the procedure in Section 4.2.2; the arc set A_{W_3} thus contains no repositioning arcs outbound from depots in D_r and none inbound to depots in D_s.

The supply vector d in this case can be specified using the definitions of $\mathcal{I}^{x,\delta}$ and $\mathcal{S}^{x,\delta}$. For nodes associated with depots in D_r, the definition is unchanged. However, for nodes associated with depots in D_s, the net supply is equal to the incremental support available at time t:

$$\begin{aligned} d(v_0^d) &= 0 \quad \forall\, d \in D \\ d(v_t^d) &= \mathcal{I}(x,\delta,v_t^d) \quad \forall\, d \in D_r,\ t = 1,\cdots,\rho \\ d(v_t^d) &= \mathcal{S}^{x,\delta}(v_t^d) - \mathcal{S}^{x,\delta}(v_{t-1}^d) \quad \forall\, d \in D_s,\ t = 1,\cdots,\rho \\ d(s) &= -\sum_{v \in N\setminus\{s\}} d(v) \end{aligned}$$

A negative net supply at some node v_t^d where $d \in D_r$ indicates a demand for containers that must be served from inventory, or via reactive repositioning. The net supplies for nodes v_t^d where $d \in D_s$ specify the maximum number of additional units that can be repositioned out of depot d at time t so that no container shortage occurs later in time. Note that by the definition of support, a negative net supply can only occur at such a node at $t = 1$; clearly in this case, there exists no feasible recovery flow.

Let $G_{W_3}(x,\delta)$ represent the recovery network G_{W_3} along with the associated net supply vector d. Again, a feasible flow in this recovery network has a one-to-one correspondence with a valid feasibility recovery vector for a given realization δ.

Proposition 3 *A feasible solution x of the nominal problem (4) is a member of the recoverable set $H(W_3,\delta)$ for a given $\delta \in \varphi_k$ if and only if there exists a feasible flow in $G_{W_3}(x,\delta)$.*

Proof: Parallel to proof of Proposition 1. □

Valid necessary and sufficient conditions for the existence of a feasible flow in $G_{W_3}(x,\delta)$ are given by Proposition 2 using set $\mathcal{U}_{W_3}$, since W_3 is a recovery set of the form required by the Proposition. However, it is possible to develop a potentially smaller set of conditions given the special network structure of **ROP3**.

To do so, we first introduce some additional notation. Given a set of inventory arcs $\alpha \subseteq I$, let $T(\alpha)$ be the set of tail nodes of arcs in α. Let $C(\alpha)$ be the set of nodes from which the tail nodes of arcs in α can be reached, *i.e.*, the set of nodes from which there exists a directed path to the tail node of an arc in α. Note that $T(\alpha) \subseteq C(\alpha)$, and that $C(\alpha)$ is an inbound-closed set. Finally, let $D(\alpha)$ be the set of depots corresponding to α.

In this case, instead of considering all subsets of nodes $U \in \mathcal{U}_{W_3}$, we can determine whether or not a feasible flow in $G_{W_3}(x,\delta)$ exists by considering only sets of nodes $C(\alpha)$ defined by sets of competing inventory arcs α at depots in D_r, where each arc $a \in \alpha$ has a shortage with respect to δ: $x(a) + \sigma(a) < 0$.

Proposition 4 *There exists a feasible flow in $G_{W_3}(x,\delta)$ if and only if*

$$d(v_1^d) = \mathcal{S}^{x,\delta}(v_1^d) \geq 0 \quad \forall\, d \in D_s \tag{22}$$

and

$$\sum_{v \in C(\alpha)} d(v) \geq 0 \tag{23}$$

for all $\alpha \subseteq I$ where α is competing, $D(\alpha) \subseteq D_r$, and $x(a) + \sigma(a) < 0$ for each $a \in \alpha$.

Proof: From Proposition 2, necessary and sufficient conditions for the existence of a feasible flow in $G_{W_3}(x, \delta)$ are

$$\sum_{v \in U} d(v) \geq 0 \quad \forall \; U \in \mathcal{U}_{W_3} \tag{24}$$

It is now shown that the conditions in the proposition are equivalent to the conditions given by (24).

The necessity of (22) and (23) is clear, since each constraint in the two sets corresponds directly to some set $U \in \mathcal{U}_{W_3}$ for which the expression in (24) must hold. For each $d \in D_s$, (22) corresponds to $U = \{v_0^d, v_1^d\}$ which is clearly inbound-closed by definition of D_s. Each $C(\alpha)$ generating a constraint of type (23) is also inbound-closed. Further, the arc set $\Delta^{\text{out}}(C(\alpha)) \cap I$ can be shown to be competing. This set is comprised of α and $\overline{\alpha}$, where each $a \in \overline{\alpha}$ is associated with a different depot $d \in D_s$. Therefore, since α is a competing arc set, and no path exists containing arcs both in α and arcs in $\overline{\alpha}$ by definition of $C(\alpha)$, and no path exists containing more than one arc $\overline{\alpha}$ by definition of D_s, $\alpha \cup \overline{\alpha}$ is competing. Thus, constraint of type (23) corresponding to $C(\alpha)$ also has a corresponding constraint in (24).

We now show the sufficiency of the conditions by showing that if they hold, conditions (24) hold for all $U \in \mathcal{U}_{W_3}$. Consider any $U \in \mathcal{U}_{W_3}$, and let $D_u \subseteq D_r$ be the set of depots d where there exists an arc $a^d \in \Delta^{\text{out}}(U) \cap I$ satisfying $x(a^d) + \sigma(a^d) \geq 0$. Note then that for each $d \in D_u$,

$$\sum_{v \in V^d \cap U} d(v) = x(a^d) + \sigma(a^d) \geq 0. \tag{25}$$

We claim first then that the conditions for U in (24) are redundant with those for $\widetilde{U} = U \setminus \bigcup_{d \in D_u}(V^d \cap U)$ in this case. Clearly this is true if $\widetilde{U} = \emptyset$. If $\widetilde{U} \neq \emptyset$, note that $\widetilde{U} \in \mathcal{U}_{W_3}$ since the subset $\Delta^{\text{out}}(\widetilde{U})$ of competing arcs $\Delta^{\text{out}}(U) \cap I$ is competing, and since no outbound reactive repositioning arcs exist in A_{W_3} from any depot $d \in D_u$ the set $\widetilde{U}$ remains inbound-closed. Suppose first that $\widetilde{U}$ contains only nodes associated with depots in D_s. In this case, conditions (22) guarantee that $\sum_{s=1}^{t} d(v_s^d) \geq 0$ for any $0 \leq t \leq \rho$ by definition of $\mathcal{S}^{x,\delta}$, and that therefore for any such $\widetilde{U}$, (24) holds. Along with (25), this in turn implies that (24) is satisfied for U.

Finally, suppose instead that $\widetilde{U}$ contains nodes for each depot in some set $\widetilde{D}_r \subseteq D_r$, in addition perhaps to nodes for some depots in D_s. Let $\alpha \subset \Delta^{\text{out}}(\widetilde{U}) \cap I$ where $D(\alpha) = \widetilde{D}_r$. Clearly, α is competing, and $x(a) + \sigma(a) < 0$ for each $a \in \alpha$ by the definition of $\widetilde{U}$. Further, $C(\alpha) \subseteq \widetilde{U}$, where any additional nodes in $\widetilde{U}$ must be associated with depots in D_s. Since $d(v_t^d) \geq 0$ for $d \in D_s$ by (22) and the definition of $\mathcal{S}^{x,\delta}$, if (23) holds for $C(\alpha)$ then (24) holds for $\widetilde{U}$, and finally (25) implies further that (24) is also satisfied for U. □

Proposition 4 can now be used to specify necessary and sufficient conditions for a feasible nominal repositioning plan x to be a feasible solution to **ROP3**. To do so, we must introduce several additional definitions. First, we identify sets of arcs, denoted *vulnerable* sets, that require a constraint of type (23) to protect against a joint container shortage that will arise for at least one uncertain realization $\delta \in \varphi_k$:

Definition 4 *Given a feasible solution x to the nominal problem* (4)*, a set $\alpha \subset I$ is vulnerable if and only if the following three conditions hold:*

1. Arc set α is competing in G_{W_3} and $D(\alpha) \subseteq D_r$,

2. Each arc $a \in \alpha$ is weak,

3. $\sum_{a\in\alpha} x(a) + |\alpha| \leq k$.

Let $\Psi(x) = \{\alpha \subset I \mid \alpha \text{ is vulnerable}\}$. For a given $\delta \in \varphi_k$, a constraint (23) is required by Proposition 4 for arc set α if $x(a) < -\sigma(a)$ for each $a \in \alpha$. Since $-\sigma(a) < \min\{k, \vartheta(a)\}$, arc a therefore must be weak. To ensure that each arc in α might simultaneously face a container shortage, we sum the condition $x(a) + \sigma(a) \leq -1$ for all $a \in \alpha$ yielding:

$$\sum_{a\in\alpha} x(a) + |\alpha| \leq -\sum_{a\in\alpha} \sigma(a) \leq k.$$

Note that a vulnerable set corresponds to a set of arcs α for which we know there exists at least one $\delta \in \varphi_k$ that would create a container shortage at all depots $D(\alpha)$ unless containers are reactively repositioned. Further, note that an arc set α which is not vulnerable will not require a constraint (23) for any $\delta \in \varphi_k$.

For a vulnerable set $\alpha \in \Psi(x)$, let $C'(\alpha) = C(\alpha) \bigcap \{\bigcup_{d\in(D\setminus D(\alpha))} V^d\}$ be the set of nodes associated with depots in $D \setminus D(\alpha)$ from which an arc in the vulnerable set can be reached. Let $D'(\alpha) \subseteq D \setminus D(\alpha)$ be the set of depots with nodes from which an arc in the vulnerable set can be reached, *i.e.*, the set of depots $d \in D\setminus D(\alpha)$ for which $C'(\alpha) \cap V^d \neq \emptyset$. Let $t^d_\alpha = \text{argmax}_t\{v^d_t \in C'(\alpha)\}$ for all $d \in D'(\alpha)$. Finally, let

$$I^d(\alpha) = \left\{(v^d_t, v^d_{t+1}) \in I \;\mid\; t \geq t^d_\alpha\right\} \quad \forall\, d \in D'(\alpha).$$

Definition 5 *A layer of a vulnerable set $\alpha \in \Psi(x)$, denoted $\theta(\alpha) \subset \bigcup_{d\in D'(\alpha)} I^d(\alpha)$, is a set of inventory arcs where*

$$|\theta(\alpha) \cap I^d(\alpha)| = 1 \quad \forall\, d \in D'(\alpha).$$

Each layer of α, therefore, contains one inventory arc from each depot in $D'(\alpha)$ leaving d at some time greater than or equal to t^d_α. Let $\Theta(\alpha) = \{\theta(\alpha) \subset I \mid \theta(\alpha) \text{ is a layer of } \alpha\}$.

Definition 6 *Given a feasible solution x to the nominal problem (4), a vulnerable set $\alpha \in \Psi(x)$, and a layer $\theta(\alpha) \in \Theta(\alpha)$, let*

$$\sum_{a\in\alpha} x(a) + \sum_{a\in\theta(\alpha)} x(a) \geq \min\left\{k, \sum_{a\in\alpha} \vartheta(a) + \sum_{a\in\theta(\alpha)} \vartheta(a)\right\}$$

be the layer $\theta(\alpha)$ inequality.

We are now ready for the main theorem in this subsection:

Theorem 3 *A feasible solution x for the nominal problem (4) is also a feasible solution for **ROP3** if and only if*

$$x(a) \geq \min\{k, \vartheta(a)\} \quad \forall\; a = (v^d_t, v^d_{t+1}) \in I \text{ where } d \in D_s$$

and for every vulnerable set $\alpha \in \Psi(x)$, the layer $\theta(\alpha)$ inequality is satisfied for each $\theta(\alpha) \in \Theta(\alpha)$.

Proof: Let x be a feasible solution of the nominal problem (4). By Propositions 3 and 4, x is also a feasible solution for **ROP3** if and only if conditions (22) and (23) are satisfied for all $\delta \in \varphi_k$.

First we consider conditions (22). By the definition of $\mathcal{S}^{x,\delta}$, these conditions become

$$\min_{1 \le j \le \rho} \left\{ x(v_j^d, v_{j+1}^d) + \sigma((v_j^d, v_{j+1}^d)) \right\} \ge 0 \quad \forall\, d \in D_s \quad \forall\, \delta \in \varphi_k,$$

which are equivalent to the following sets of constraints for each $d \in D_s$:

$$x(v_t^d, v_{t+1}^d) \ge -\sigma((v_t^d, v_{t+1}^d)) \quad \forall\, t \in \{1, \cdots, \rho\} \quad \forall\, \delta \in \varphi_k. \tag{26}$$

For any $\delta \in \varphi_k$,

$$-\sigma((v_t^d, v_{t+1}^d)) \le \min\{k, \vartheta(v_t^d, v_{t+1}^d)\}$$

where we note that this bound is tight for at least one $\delta \in \varphi_k$. Therefore (26) simplifies to

$$x(v_t^d, v_{t+1}^d) \ge \min\{k, \vartheta(v_t^d, v_{t+1}^d)\} \quad \forall\, t \in \{1, \cdots, \rho\},$$

which must hold for all $d \in D_s$. Thus,

$$x(a) \ge \min\{k, \vartheta(a)\} \quad \forall\, a = (v_t^d, v_{t+1}^d) \in I \text{ where } d \in D_s.$$

Now consider conditions (23). Using the definition of vulnerable sets, these conditions become

$$\sum_{v \in C(\alpha)} d(v) \ge 0 \quad \forall\, \alpha \in \Psi(x), \tag{27}$$

which again by Proposition 3 must hold for all $\delta \in \varphi_k$. The sum can be rewritten by partitioning the depots into those associated with the vulnerable arcs and those providing reactive repositioning support to the vulnerable arcs, and simplified as follows:

$$\begin{aligned}
\sum_{v \in C(\alpha)} d(v) &= \sum_{d \in D(\alpha)} \sum_{v \in C(\alpha) \cap V^d} d(v) + \sum_{d \in D'(\alpha)} \sum_{v \in C'(\alpha) \cap V^d} d(v) \\
&= \sum_{d \in D(\alpha)} \sum_{v \in C(\alpha) \cap V^d} \mathcal{I}^{x,\delta}(v) + \sum_{d \in D'(\alpha)} \sum_{s=1}^{t_\alpha^d} \left(\mathcal{S}^{x,\delta}(v_s^d) - \mathcal{S}^{x,\delta}(v_{s-1}^d) \right) \\
&= \sum_{a \in \alpha} x(a) + \sum_{d \in D(\alpha)} \sigma(a^d) + \sum_{d \in D'(\alpha)} \mathcal{S}^{x,\delta}(v_{t_\alpha^d}^d) \\
&= \sum_{a \in \alpha} (x(a) + \sigma(a)) + \sum_{d \in D'(\alpha)} \min_{a \in I^d(\alpha)} \{x(a) + \sigma(a)\}.
\end{aligned}$$

where a^d is the inventory arc for depot d in α. Thus, the condition for a specific α in (27) can now be replaced by a set of inequalities, one for each layer $\theta(\alpha) \in \Theta(\alpha)$:

$$\sum_{a \in \alpha} (x(a) + \sigma(a)) + \sum_{a \in \theta(\alpha)} (x(a) + \sigma(a)) \ge 0.$$

Rewriting yields

$$\sum_{a \in \alpha} x(a) + \sum_{a \in \theta(\alpha)} x(a) \ge -\sum_{a \in \alpha} \sigma(a) - \sum_{a \in \theta(\alpha)} \sigma(a). \tag{28}$$

Since $\delta \in \varphi_k$, we can bound the right-hand side of (28) using $-\sigma(a) \leq \vartheta(a)$ for all a, and $-\sum_{a\in\alpha\cup\theta(\alpha)} \sigma(a) \leq k$

$$\sum_{a\in\alpha} x(a) + \sum_{a\in\theta(\alpha)} x(a) \geq \min\left\{k, \sum_{a\in\alpha} \vartheta(a) + \sum_{a\in\theta(\alpha)} \vartheta(a)\right\},$$

which is the layer $\theta(\alpha)$ inequality. □

Theorem 3 shows that **ROP3** can be formulated as the following integer program:

$$\begin{aligned}
\min \quad & c\,x \\
\text{s.t.} \quad & Ax = b \\
& x(a) \geq \min\{k, \vartheta(a)\} \quad \forall\, a = (v_t^d, v_{t+1}^d) \in I,\ \ d \in D_s \\
& \sum_{a\in\alpha} x(a) + \sum_{a\in\theta(\alpha)} x(a) \geq \min\left\{k, \sum_{a\in\alpha} \vartheta(a) + \sum_{a\in\theta(\alpha)} \vartheta(a)\right\} \quad \forall\, \alpha \in \Psi(x),\ \ \forall \theta(\alpha) \in \Theta(\alpha) \\
& x \in \mathbb{Z}_+^{|\mathcal{A}|}
\end{aligned}$$

Note that the layer inequality constraints are specified for each vulnerable set α, and that the vulnerable sets depend on the solution x. If the layer inequality constraints were alternatively required regardless of x for each competing arc set α in G_{W_3} where $D(\alpha) \subseteq D_r$, the formulation would remain valid. However, Theorem 3 shows that a given solution x can be checked for feasibility to **ROP3** using far fewer constraints. These ideas motivate computational approaches to this problem as future research.

4.2.4 Using the Robust Repositioning Models in Practice

Although the sets of constraints required to specify the robust repositioning problems **ROP2** and **ROP3** grow large as the number of spatial locations grows, many real-world repositioning problems can be decomposed geographically such that the approaches proposed in this paper should be tractable. For example, consider an international container manager that provides service from many container depots globally. These depots may be naturally grouped into regions: *e.g.*, Southeast Asia, East Coast North America, Northern Europe, etc. While depots within a region might pool inventory to hedge against uncertainty, it might not be practical to allow reactive sharing across regional boundaries. This scenario can be modelled using a simple modification of **ROP2** or **ROP3** where inter-regional repositioning arcs are included in the nominal problem network G, but excluded from the specification of G_W. It should be clear, then, that the conditions in Theorems 2 and 3 decompose by region, leading to much smaller sets of constraints guaranteeing feasibility. Such decomposition also could allow the manager to specify certain regions where complete reactive pooling of the type specified by **ROP2** is allowed, and other regions where the restricted reactive pooling of **ROP3** is considered.

5 A Numerical Example

In this section, we present a small numerical example to illustrate the robust repositioning ideas developed in this paper. The example will provide some insight into how different levels of conservatism, measured by k, and different degrees of flexibility for performing reactive repositioning affect the cost of a robust repositioning plan, and therefore the price of robustness. Each solution to follow is obtained by formulating and solving the robust models derived in the previous sections for the example problem using an integer programming solver.

The example problem consists of three container depots and a planning horizon of three time units. The problem is depicted graphically in Figure 4, where the number inside of each time-space node v is the nominal net supply $b(v)$.

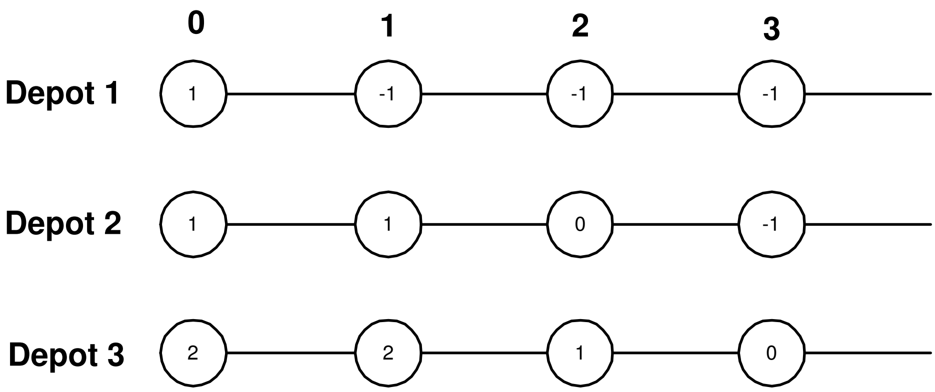

Figure 4: An example repositioning problem, where the number inside of each time-space node v is the nominal net supply $b(v)$.

In the example, transportation costs are symmetric, where the empty repositioning cost per container is \$1 between depots 1 and 2, \$1.5 between depots 1 and 3, and \$1 between depots 2 and 3. Travel times are also symmetric, where the travel time is 1 time unit between depots 1 and 2, 2 time units between depots 1 and 3, and 1 time unit between depots 2 and 3. Transportation between depots can be initiated at all times, and inventory costs are zero at all depots between all time periods. The solution to the nominal problem in this case yields a total cost of \$ 2.5, and is depicted graphically in Figure 5.

Suppose now that the uncertainty in this problem is such that in time periods 1 and 2, each depot may experience a reduction in net supply of 1 and in time period 3, each depot may experience a reduction of 2. Mathematically,

$$\underline{b}(v_s^d) = \begin{cases} 1 & \text{if } s = 1, 2 \\ 2 & \text{if } s = 3 \end{cases} \quad \forall\, d = \{1, 2, 3\},$$

and therefore that the vulnerability of the arcs in this system is

$$\vartheta((v_s^d, v_{s+1}^d)) = \begin{cases} 1 & \text{if } s = 1 \\ 2 & \text{if } s = 2 \\ 4 & \text{if } s = 3 \end{cases} \quad \forall\, d = \{1, 2, 3\}.$$

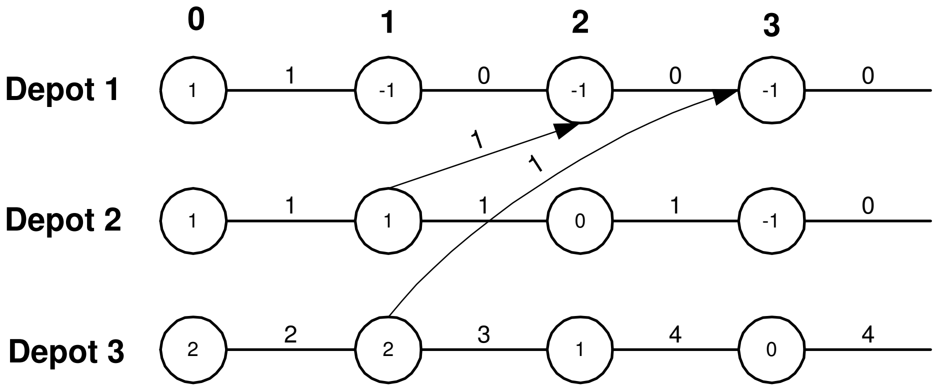

Figure 5: Optimal solution to nominal problem for the example, total cost = \$2.5

For this example, the solution of problem **ROP1** when $k = 1$ is depicted in Figure 6. This solution has total cost \$5. Note that the cost is double the cost of the nominal problem solution. There does not exist a feasible solution of **ROP1** when $k \geq 2$.

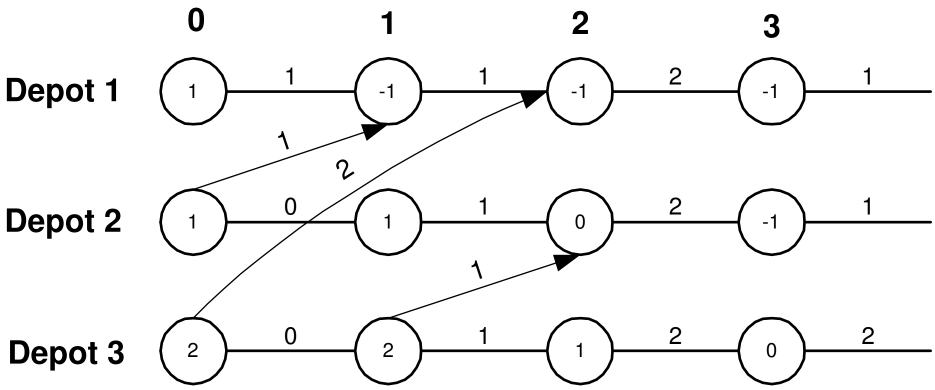

Figure 6: Optimal solution to **ROP1** when $k = 1$ for the example, total cost = \$5

Now consider problem **ROP3** for this example and let depot 1 be the only member of D_r. Depots 2 and 3 are in D_s, and therefore must hedge independently against uncertainty but may provide reactive repositioning support to depot 1. An optimal solution with cost of \$4.5 is presented in Figure 7 for **ROP3** when $k = 2$. Observe that by allowing the possibility of reactive repositioning to depot 1, a higher level of robustness can be obtained at a lower cost than in the previous case. Also, note that although there are weak inventory arcs at time 2 and 3 for depot 1, there are enough containers at depots 2 and 3 to provide support for every realization of uncertain net supplies. There does not exist a feasible solution when $k \geq 3$ for this scenario.

Consider now **ROP3** where both depots 1 and 2 are in D_r. Figure 8 shows an optimal solution when $k = 3$ with a total cost of \$4 . Again, a solution with a higher level of robustness is obtained at a lower cost in this case, since depot 2 can also benefit from reactive reposition from depot 3. It is not possible to find a solution to **ROP3** when $k \geq 4$ for this scenario.

Finally, we consider the additional reactive repositioning flexibility provided by **ROP2** for this

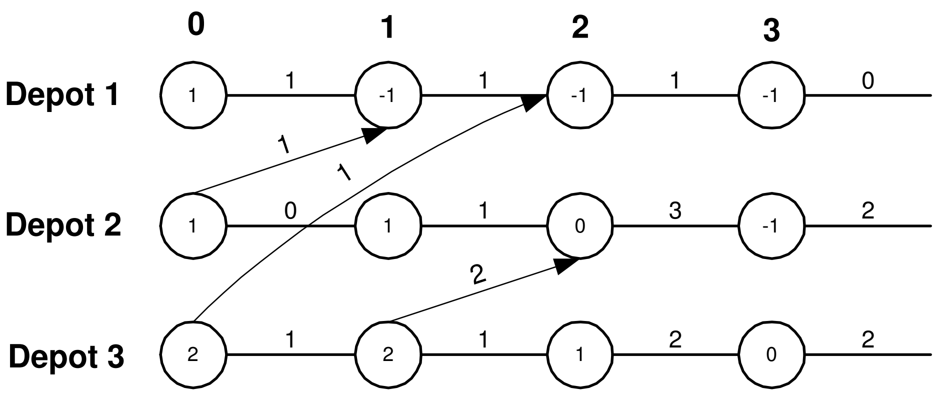

Figure 7: Optimal solution to **ROP3** when $k = 2$ for the example, where only depot 1 is in D_r, cost \$4.5

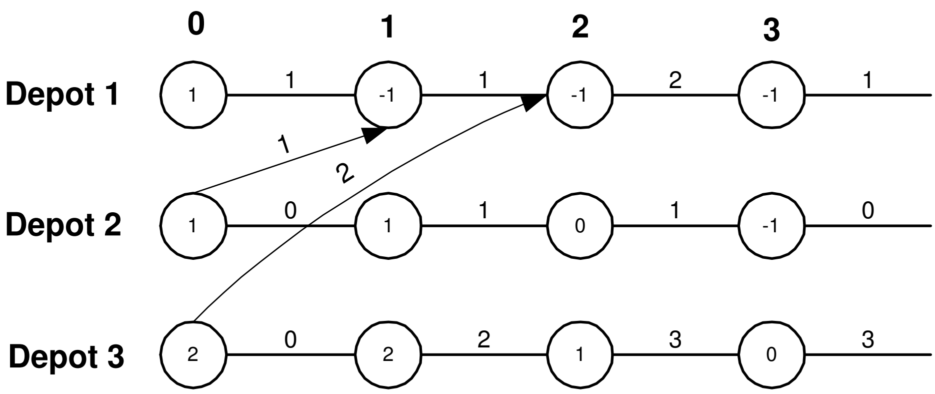

Figure 8: Optimal solution of **ROP3** when $k = 3$ for the example, where depots 1 and 2 are in D_r, cost \$4

example. Now, any depot can potentially receive or provide reactive containers using feasible repositioning arcs departing depots at time $t \geq 1$. Figure 9 shows an optimal solution in this case for $k = 4$ with a cost of \$3.5. Again, note the increase in robustness level and decrease in cost.

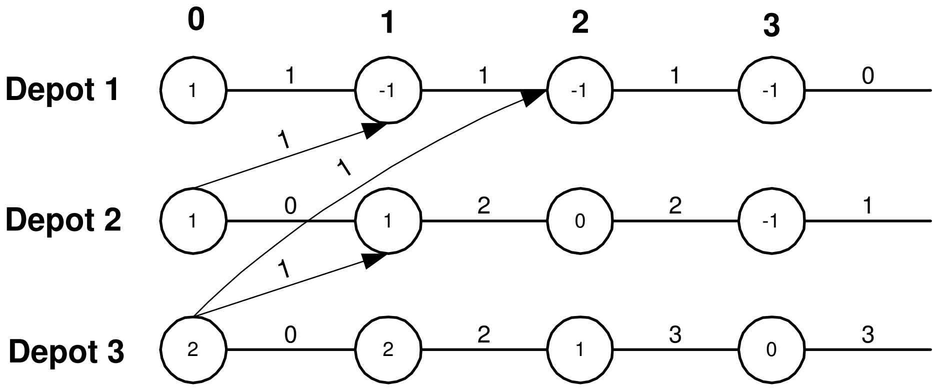

Figure 9: Optimal solution of **ROP2** when $k = 4$ for the example, cost \$3.5

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