

Competitive Analysis for Dynamic Multi-Period Uncapacitated Routing Problems

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Abstract

We study a dynamic multi-period routing problem where, at the beginning of each time period, a set of orders arrive that have to be fulfilled either that time period or the next. Thus, in each time period there are customers which have to be served and customers whose service may be postponed. Once it has been decided which customers to serve, an optimal route is constructed and executed. The objective of the problem is to minimize the total distance traveled during the planning horizon. Deciding which customers to serve in a time period is done on the basis of incomplete information, analyzing simultaneously customers in two consecutive periods. No knowledge is available about customers requiring service in future time periods. We introduce simple algorithms, ones which naturally arise in practice, and analyze these algorithms by studying their competitive ratio.

1 Introduction

Dynamic routing problems are concerned with situations in which information about the customers to be visited is not revealed all at once, but rather is revealed dynamically over time. The importance of dynamic routing problems has always been recognized (see [15] for an early discussion on the topic), but they are becoming increasingly important due to the technological advances that are transforming the transportation sector. In the near future, it will be commonplace that planners will know the exact position of every

vehicle in their fleet at every moment in time, will have the capability to communicate with each of the vehicles at all times, and thus will be able to re-route vehicles, if needed and desirable, when new information about customers to be visited becomes available. Unfortunately, the decision technology to effectively exploit the opportunities provided by these technological advances for dynamic routing is still in its infancy. Relatively little research has been done on dynamic routing problems compared to the huge amount of time and effort invested in analyzing and solving static routing problems.

As mentioned earlier, in dynamic routing problems customer demands are not known in advance but become available incrementally over time. A typical approach is to construct a routing plan based on the known demands and modify this plan when new demands become available (e.g., [7], [8], [9], [13], [14], and [18]). The performance of dynamic routing algorithms is often analyzed by studying their competitive ratio (e.g., [1], [5], [10], [11], and [12]), as introduced in [16], for the analysis of online algorithms. The competitive ratio measures the price that has to be paid for the lack of information. As such it provides insight in the sensitivity of algorithms to the availability of information. In stochastic routing problems, which are related but quite different, the goal is to find, given a set of customers with stochastic demand, an a priori routing plan that minimizes the expected delivery costs. This approach often incorporates a recourse function to correct the routing plan when vehicle capacity constraints are violated (e.g., [6]). Routing environments with both dynamic and stochastic aspects have also been studied (e.g., [2], [3], [4] and [17]).

We study and analyze algorithms for a dynamic multi-period routing problem motivated by a routing environment we encountered in practice, and which is often found at companies involved in service delivery management. The situation arises due to contracts guaranteeing a certain service level (often referred to as service level agreements or SLAs), measured in terms of the number of days within which the service has to be provided. In our motivating application, the company receives orders each day, but has the flexibility to fulfill the orders within the next two days, i.e., on the day after the order was received or on the day after that. Consequently, each day the distribution planner at the company has to decide which orders to fulfill on the basis of partial information. The distribution planner has no knowledge of or insight in the orders that will arrive in the next few days. Obviously, orders that were received two days before and have not yet been served, have to be served today. The other orders, however, can be served today but can also be postponed.

We introduce the dynamic multi-period routing problem to capture the key characteristic of situations such as the one described above. More specifically, at the beginning of each time period a set of customers requires a service to be fulfilled either that time period or the next. Thus, in each time period there are customers which need to be served, namely the customers that have been postponed the previous time period, and customers whose service may be postponed, namely the customers that have arrived this time period. The objective of the problem is to minimize the total distance traveled during the planning

horizon. To be able to focus first and foremost on the dynamic nature of demand, i.e., orders being revealed over time, we keep the routing component of the problem as simple as possible by not including capacity considerations, i.e., by assuming a single vehicle with unlimited capacity.

We present a few simple algorithms, ones which naturally arise in practice, and analyze these algorithms by studying their competitive ratio. We study these algorithms in different settings by varying the length of the planning horizon (two periods, more than two periods) and the locations of the customers (on the real line, in the Euclidean plane). More specifically, we analyze the performance of Algorithm *IMMEDIATE*, which always serves customers as soon as possible, of Algorithm *DELAY*, which always serves customers as late as possible, and of Algorithm *SMART*, which in each time period decides whether to serve customers immediately or whether to postpone service based on the routing cost of serving the set of customers that have to be served that time period and the routing cost of serving all the customers. Both Algorithm *IMMEDIATE* and Algorithm *DELAY* have a competitive ratio equal to 2. When we restrict the planning horizon to two time periods, Algorithm *SMART* is shown to be optimal with a competitive ratio of $\sqrt{2}$ for instances with customers located on the real line and with a competitive ratio of $\frac{3}{2}$ for instances with customers located in the Euclidean plane. The competitive ratio of *SMART* increases and the algorithm is no longer optimal when the number of time periods in the planning horizon is greater than or equal to three.

As mentioned above, the competitive ratio measures the price that has to be paid for the lack of information, and therefore provides insight in the sensitivity of algorithms to the availability of information. Even though the algorithms studied in this paper may not be used directly in practical situations, the insights provided by their analysis may help in the design and implementation of effective practical heuristics.

The remainder of the paper is organized as follows. In Section 2 we formally introduce the dynamic multi-period routing problem and discuss its key characteristics. In Section 3 we analyze the performance of Algorithms *IMMEDIATE*, *DELAY*, and *SMART*, for the special case in which the planning horizon consists of only two time periods, and in Section 4 we present results for the general case in which the planning horizon has more than two time periods. Finally, in Section 5 we summarize our key findings and discuss future research directions.

2 A Dynamic Multi-Period Routing Problem

We will now define the dynamic multi-period routing problem (DMPRP) studied in this paper.

The planning horizon is divided into time periods of equal length. Orders from customers arrive at the beginning of each time period. Each order has to be fulfilled within

two time periods after arrival. Consequently, at the beginning of each time period t , the distribution planner learns about a set of customers $C_{t|t+1}$ that can be served either in the current time period or in the next time period. A single vehicle is available at a central depot to make deliveries and fulfill orders in each time period. The vehicle has to return to the depot at the end of each time period. The objective is to minimize the total travel cost, where we assume that the travel costs depend linearly on the distance.

In case of a finite planning horizon, i.e., consisting of a finite number of time periods $t = 1, 2, \dots, T$, we assume that every order has to be fulfilled within the planning horizon, i.e., orders that arrive in the last time period have to be served in that time period, and that the system did not necessarily start empty. We denote the set of customers already in the system at the start of the planning horizon, which have to be served in the first time period, by C_1 and we denote the set of customers arriving in the last time period, which have to be served in the last period, by C_T .

When making decisions at the beginning of the first time period, the decision maker knows the customers in C_1 and in $C_{1|2}$. At the beginning of time period t , $2 \leq t \leq T - 1$, the decision maker knows the customers in $C_{t-1|t}$ and $C_{t|t+1}$ and knows which customers of the set $C_{t-1|t}$ have been served at time $t - 1$, but has no knowledge of the customers in $C_{s|s+1}$ for $s = t + 1, \dots, T - 1$ and in C_T . Obviously, the decision maker has also knowledge of the customers in $C_{s-1|s}$, for $s < t$, but these customers have already been served and do not influence the decision at time t . At time T the customers in $C_{T-1|T}$ which have not been served at time $T - 1$ need to be served together with the customers in C_T .

In general, the decision maker has to make decisions based on partial information. Consider the situation at the beginning of time period t . A set of customers which have to be served in time period t (because they were postponed in time period $t - 1$) and a set of customers which may be served at time period t or postponed to time period $t + 1$ (because they just arrived) are known. Clearly, if the newly arrived customers are located close to the customers that have to be served, it may be advantageous to serve these customers in time period t too. On the other hand, if the newly arrived customers are located far from the customer that have to be served, it may be advantageous to postpone their service in the hope that in the next time period their service can be combined with the customers that arrive in that time period.

It is common to analyze the performance of algorithms for optimization problems with incomplete information by studying their competitive ratio (see [16]). Let $z(A, I)$ denote the value produced by algorithm A on instance I , and let $z^*(I)$ denote the value produced by an optimal off-line algorithm, i.e., an algorithm with access to perfect information, on instance I . The *competitive ratio* r_A of algorithm A is defined as

$$r_A = \max_I \frac{z(A, I)}{z^*(I)}.$$

An on-line algorithm A is *optimal* if no other algorithm A' has a competitive ratio $r_{A'} < r_A$.

Throughout the paper we will adopt the same approach for studying the performance of algorithms for the DMPRP and we calculate the competitive ratio of various algorithms bounding from above the ratio $r_A(I) = z(A, I)/z^*(I)$ for all instances I .

Even with perfect information, or perfect hindsight, the DMPRP is NP-hard as it contains the TSP as a subproblem.

3 Dynamic Multi-Period Routing Problem - A Special Case

We start our analysis of the DMPRP by considering the special case in which the planning horizon has only two time periods, i.e., $T = 2$. We believe it is of interest to study this special case, because it already allows us to concentrate on the need to make decisions with only partial information.

Since the planning horizon consists of only two time periods ($t = 1$ and $t = 2$), there are three types of customers, customers C_1 that are known at the beginning of the first time period and have to be served in that time period, customers $C_{1|2}$ that are known at the beginning of the first time period and can be served either in the first or the second time period, and customers C_2 that are unknown at the beginning of the first time period but become known at the beginning of the second time period and have to be served in the second time period. In this setting, there is only one decision that needs to be made at the beginning of the first time period: which customers in $C_{1|2}$ to serve and which customers to postpone. Once this decision has been made, all that remains is to solve a traveling salesman problem for the customers that are served in the first time period and for the customers that are served in the second time period. The objective is to choose the set of customers to be served in the first time period in such a way that the total cost, i.e., the cost of the traveling salesman tour in the first time period plus the cost of the traveling salesman tour in the second time period, is minimum.

Let us denote by L_1 , $L_{1|2}$, L_2 the length of an optimal tour starting and ending at the depot and visiting all customers in C_1 , $C_{1|2}$, C_2 , respectively, by $L_{1,1|2}$ and $L_{1|2,2}$ the length of an optimal tour visiting all customers in $C_1 \cup C_{1|2}$, and in $C_{1|2} \cup C_2$, respectively. Moreover, we denote by L_{all} the length of an optimal tour starting and ending at the depot and visiting all customers, i.e., the customers in $C_1 \cup C_{1|2} \cup C_2$.

In our analysis, we restrict the set of possible customer locations to the real line and the Euclidean plane. Since the triangle inequality holds in both cases, we have that $\max(L_1, L_{1|2}) \leq L_{1,1|2} \leq L_1 + L_{1|2}$. Similarly, $\max(L_{1|2}, L_2) \leq L_{1|2,2} \leq L_{1|2} + L_2$.

3.1 The Real Line

To gain an understanding of the complexity of DMPRP, we initially restrict ourselves to instances where customers are located on the non-negative real line (R^+) and the depot is located at the origin. It is useful to observe that, in this setting, the length of an optimal

tour visiting a set of customers is twice the distance to the farthest (rightmost) customer. Accordingly, the length of an optimal tour visiting two sets of customers together equals the maximum length of the tours that visit each set of customers independently. Namely, $L_{A,B} = \max(L_A, L_B)$.

The next theorem provides a lower bound on the competitive ratio for any on-line algorithm.

Theorem 1. *The competitive ratio of any algorithm for DMPRP with customer locations on the non-negative real line is greater than or equal to $\sqrt{2}$.*

Proof. Consider the following instance. The customer set C_1 contains a single customer at distance 1 from the depot (located at the origin) and the customer set $C_{1|2}$ contains a single customer on the non-negative real line at distance $a > 1$ from the depot.

First, we consider algorithm A' of the class of algorithms that at the beginning of the first time period decide to visit both customers. Now assume that the customer set C_2 , i.e., the set of customers that place an order at the beginning of the second time period, contains a single customer at distance a from the depot. Then, $z(A', I) = 2a + 2a$, while the off-line optimum is $z^*(I) = 2 + 2a$. Thus, $r_{A'} \geq \frac{2a}{1+a}$ for all $a > 1$.

Next, we consider algorithm A'' of the class of algorithms that at the beginning of the first time period decide to visit only the customer in C_1 . Now, assume that the customer set C_2 is empty, i.e., no orders arrive at the beginning of the second time period. Then, $z(A'', I) = 2 + 2a$, while $z^*(I) = 2a$. Thus, $r_{A''} \geq \frac{1+a}{a}$ for all $a > 1$.

Consequently, for any algorithm A , we have

$$r_A \geq \min\left(\frac{2a}{1+a}, \frac{1+a}{a}\right) \text{ for all } a > 1.$$

Since $\max_{a>1}[\min(\frac{2a}{1+a}, \frac{1+a}{a})] = \sqrt{2}$ for $a = 1 + \sqrt{2}$, we obtain

$$r_A \geq \sqrt{2}.$$

□

There are two natural algorithms to consider and analyze as a first step towards constructing an effective algorithm for DMPRP. Namely, the algorithm which serves all customers in $C_{1|2}$ in the first time period, which we will call *IMMEDIATE*, and the algorithm which postpones all customers in $C_{1|2}$ to the second time period, which we will call *DELAY*.

Our analysis of these two simple algorithms will show that their competitive ratio is quite far from the lower bound $\sqrt{2}$ shown in Theorem 1.

Algorithm IMMEDIATE: *Visit all customers in C_1 and $C_{1|2}$ in the first time period and visit all customers in C_2 in the second time period.*

Theorem 2. *The competitive ratio of Algorithm IMMEDIATE for instances with customer locations on the non-negative real line is 2.*

Proof. It is easy to see that $z^*(I) \geq L_{all} \geq \max(L_{1,1|2}, L_2)$ and $z(IMMEDIATE, I) = L_{1,1|2} + L_2$. Therefore,

$$r_{IMMEDIATE}(I) \leq \frac{L_{1,1|2} + L_2}{L_{all}} \leq \frac{2L_{all}}{L_{all}} = 2 \text{ for all instances } I. \text{ Thus}$$

$$r_{IMMEDIATE} \leq 2.$$

To see that the bound is tight, consider any instance I where $C_1 = \emptyset$ and the sets $C_{1|2}$ and C_2 are identical and non-empty. Then,

$$\frac{z(IMMEDIATE, I)}{z^*(I)} = \frac{2L_{1|2}}{L_{1|2}} = 2.$$

□

Algorithm DELAY: *Visit all customers in C_1 in the first time period and visit all customers in $C_{1|2}$ and C_2 in the second time period.*

Theorem 3. *The competitive ratio of Algorithm DELAY for instances with customer locations on the non-negative real line is 2.*

Proof. Since $z^*(I) \geq L_{all} \geq \max(L_1, L_{1|2,2})$ and $z(DELAY, I) = L_1 + L_{1|2,2}$, we obtain

$$r_{DELAY}(I) \leq \frac{L_1 + L_{1|2,2}}{L_{all}} \leq \frac{2L_{all}}{L_{all}} \leq 2 \text{ for all instances } I. \text{ Thus}$$

$$r_{DELAY} \leq 2.$$

To see that the bound is tight, consider any instance I where the sets C_1 and $C_{1|2}$ are identical and non-empty and $C_2 = \emptyset$. Then,

$$\frac{z(DELAY, I)}{z^*(I)} = \frac{L_1 + L_1}{L_1} = 2.$$

□

We now consider a parameterized algorithm that applies either *IMMEDIATE* or *DELAY*, based on the information available at the beginning of the first period.

Algorithm SMART(p): *If $L_1 > 0$ and $L_{1,1|2} \leq pL_1$, then apply IMMEDIATE, else apply DELAY.*

Algorithm $SMART(p)$ visits all customers in $C_{1|2}$ in the first time period if C_1 is non-empty and $L_{1,1|2} \leq pL_1$, i.e. if the customers in $C_{1|2}$ are not “too far” from the customers in C_1 , while it postpones all customers in $C_{1|2}$ to the second time period otherwise. Throughout the paper we will assume $p > 1$.

Next, we will show that the proposed algorithm, for an appropriate parameter choice, is optimal in the sense that no algorithm exists with a better competitive ratio.

Theorem 4. *The competitive ratio of Algorithm $SMART(1 + \sqrt{2})$ for instances with customer locations on the non-negative real line is $\sqrt{2}$. The algorithm is optimal.*

Proof. The cases $L_1 = 0$ and $L_{1|2} \leq L_1$ are trivial. In the first case Algorithm $DELAY$ is applied and the off-line optimal solution is found. In the second case Algorithm $IMMEDIATE$ is applied and the off-line optimal solution is found. In both cases the ratio $r_{SMART}(I)$ equals 1. Now, let us assume $0 < L_1 < L_{1|2}$ and observe that $L_{1,1|2} = L_{1|2}$.

When $L_1 < L_2$, the optimal strategy is to serve the customers in $C_{1|2}$ in the second time period, resulting in a total cost $z^*(I) = L_1 + L_{1|2,2}$.

If $L_{1,1|2} > (1 + \sqrt{2})L_1$, then Algorithm $DELAY$ is applied and it finds the optimal solution: $r_{SMART}(I) = 1$.

If $L_{1,1|2} \leq (1 + \sqrt{2})L_1$, then Algorithm $IMMEDIATE$ is applied. We derive

$$\begin{aligned} r_{SMART}(I) &= \frac{z(IMMEDIATE, I)}{z^*(I)} = \frac{L_{1,1|2} + L_2}{L_1 + L_{1|2,2}} \\ &= \frac{L_{1|2} + L_2}{L_1 + L_{1|2,2}} \leq \frac{L_{1|2} + L_{1|2,2}}{L_1 + L_{1|2,2}} \leq \frac{L_{1|2} + L_{1|2}}{L_1 + L_{1|2}}, \end{aligned}$$

where the latter inequality is due to the fact that the ratio $\frac{L_{1|2}+x}{L_1+x}$ is a decreasing function of x and $L_{1|2,2} \geq L_{1|2}$. Thus,

$$r_{SMART}(I) \leq \frac{2L_{1|2}}{L_1 + L_{1|2}} \leq \frac{2(1 + \sqrt{2})L_1}{L_1 + (1 + \sqrt{2})L_1} = \sqrt{2}.$$

The latter equality is due to the fact that the ratio $\frac{2x}{L_1+x}$ is an increasing function of x and $L_{1|2} = L_{1,1|2} \leq (1 + \sqrt{2})L_1$.

When $L_2 \leq L_1$, the optimal strategy is to serve the customers in $C_{1|2}$ in the first time period, resulting in a total cost $z^*(I) = L_{1,1|2} + L_2 = L_{1|2} + L_2$.

If $L_{1,1|2} \leq (1 + \sqrt{2})L_1$, then Algorithm $IMMEDIATE$ is applied and it finds the optimal solution: $r_{SMART}(I) = 1$.

If $L_{1,1|2} > (1 + \sqrt{2})L_1$, then Algorithm *DELAY* is applied. We derive

$$\begin{aligned} r_{SMART}(I) &= \frac{z(DELAY, I)}{z^*(I)} = \frac{L_1 + L_{1|2,2}}{L_{1|2} + L_2} = \frac{L_1 + L_{1|2}}{L_{1|2} + L_2} \\ &\leq \frac{L_1 + L_{1|2}}{L_{1|2}} < \frac{L_1 + (1 + \sqrt{2})L_1}{(1 + \sqrt{2})L_1} = \sqrt{2}. \end{aligned}$$

In conclusion, we have $r_{SMART}(I) \leq \sqrt{2}$ for all instances I . Thus, $r_{SMART} \leq \sqrt{2}$ when $p = 1 + \sqrt{2}$. The tightness of the bound and the optimality of the algorithm come from Theorem 1. \square

Next, we consider what happens when we restrict ourselves to instances where customer locations are on the entire real line. Obviously, the lower bound provided in Theorem 1 is still valid.

Theorem 5. *An optimal algorithm with competitive ratio $\sqrt{2}$ exists for instances with customer locations on the real line ($R^- \cup R^+$).*

Proof. Observe that the vehicle has to pass the depot when going from a customer located on R^- to a customer located on R^+ and vice versa. Therefore, the length of any tour is the sum of the distances traveled on R^- and R^+ . It follows that it is optimal to partition each customers set C_1 , $C_{1|2}$ and C_2 into a subset of customers located on R^- and a subset of customers located on R^+ (customers located at the origin may be included in either one of the sets), to compute the length of an optimal tour starting and ending at the depot and visiting all customers in each subset, and to apply $SMART(1 + \sqrt{2})$ on R^- and $SMART(1 + \sqrt{2})$ on R^+ . \square

3.2 The Euclidean Plane

A lower bound on the competitive ratio of any algorithm for DMPRP with customer locations in the Euclidean plane (R^2) is given in the next theorem.

Theorem 6. *The competitive ratio of any algorithm for DMPRP with customer locations in the Euclidean plane is greater than or equal to $\frac{3}{2}$.*

Proof. Consider the following instance. The depot is located at the origin O . Customers are located on three line segments emanating from O . The endpoints A, B and C of the three line segments lie on the circle with radius $\frac{1}{2}$. The distance between A and B as well as the distance between B and C is ϵ (distance between A and C is approximately 2ϵ for ϵ small). The customers in C_1 , with $|C_1| = n$, are distributed uniformly on the line segment OA . Half of the customers in $C_{1|2}$, with $|C_{1|2}| = 2n$, are distributed uniformly on the line segment OB and the other half of the customers are distributed uniformly over

the line segment OC . For large enough n (depends on ϵ), the customers on a line segment are always visited consecutively in an optimal vehicle route, i.e., it never pays to zig-zag between customers on different line segments.

Consequently, any algorithm for DMPRP has three choices. The first choice is to serve only the customers in C_1 in the first time period. In that case, let the customers in C_2 , with $|C_2| = n$, be distributed uniformly on the line segment OA . The algorithm produces a solution with cost $1 + (2 + \epsilon)$ ($OA + AO$ plus $OA + \epsilon + BO$ and $OC + CO$). The optimal off-line solution has cost $2 + 3\epsilon$ ($OA + \epsilon + BO$ plus $OA + 2\epsilon + CO$). The ratio tends to $\frac{3}{2}$ when ϵ tends to 0. The second choice is to serve the customers in $C_1 \cup C_{1|2}$ in the first time period. In that case, let the customers in C_2 , with $|C_2| = n$, be distributed uniformly on the line segment OC . The algorithm produces a solution with cost $(2 + \epsilon) + 1$ ($OA + AO$ and $OB + \epsilon + CO$ plus $OC + CO$). The optimal off-line solution has cost $2 + \epsilon$ ($OA + \epsilon + BO$ plus $OC + CO$). The ratio tends to $3/2$ when ϵ tends to 0. The final choice is to serve the customer in C_1 and the customers in $C_{1|2}$ on the line segment OB in the first time period. In that case, let half of the customers in C_2 , with $|C_2| = 2n$, be distributed uniformly on the line segment OA and the other half of the customers be distributed uniformly on the line segment OB . The algorithm produces a solution with cost $(1 + \epsilon) + (2 + \epsilon)$ ($OA + \epsilon + BO$ and $OA + \epsilon + BO$ plus $OC + CO$). The optimal off-line solution has cost $(1 + 2\epsilon) + (1 + \epsilon)$ ($OA + 2\epsilon + CO$ plus $OA + \epsilon + BO$). The ratio tends to $\frac{3}{2}$ when ϵ tends to 0.

As observed earlier, no other partition of the customer sets can lead to a better solution. Therefore, any algorithm has a competitive ratio greater than or equal to $\frac{3}{2}$. \square

Next, we analyze the performance of the algorithms *IMMEDIATE* and *DELAY* when applied to Euclidean instances.

Theorem 7. *The competitive ratio of Algorithm IMMEDIATE for Euclidean instances is 2 and the bound is tight.*

Proof. Same argument as Theorem 2. \square

Theorem 8. *The competitive ratio of Algorithm DELAY for Euclidean instances is 2 and the bound is tight.*

Proof. Same argument as Theorem 3. \square

Next, we analyze the performance of algorithm *SMART*(p) on Euclidean instances when $p = 2$.

Theorem 9. *The competitive ratio of Algorithm SMART(2) for Euclidean instances is $\frac{3}{2}$. The algorithm is optimal.*

Proof. The optimal solution for any instance partitions the customers in $C_{1|2}$ into two sets, the set of customers visited in the first time period, $C_{(1|2)a}$, and the set of customers visited in the second time period, $C_{(1|2)b}$.

As before, we denote by $L_{(1|2)a}$ and $L_{(1|2)b}$ the length of an optimal tour starting and ending at the depot and visiting all customers in $C_{(1|2)a}$ and $C_{(1|2)b}$, respectively. In a similar way, we define $L_{1,(1|2)a}$ and $L_{2,(1|2)b}$ so that $z^*(I) = L_{1,(1|2)a} + L_{2,(1|2)b}$. Furthermore, let $M = \max(L_2, L_{(1|2)b}) \leq L_{2,(1|2)b}$. Observe that from the triangle inequality we have the following inequalities

$$L_{1,1|2} \leq L_{1,(1|2)a} + L_{(1|2)b} \leq L_{1,(1|2)a} + M \quad (1)$$

$$z^*(I) \geq L_{1,(1|2)a} + M \geq L_1 + M. \quad (2)$$

If $L_{1,1|2} \leq 2L_1$, then Algorithm *IMMEDIATE* is applied and, from (1) and (2), we derive

$$\begin{aligned} r_{SMART}(I) &= \frac{L_{1,1|2} + L_2}{L_{1,(1|2)a} + L_{2,(1|2)b}} \leq \frac{L_{1,(1|2)a} + M + L_2}{L_{1,(1|2)a} + M} \leq \frac{L_{1,(1|2)a} + 2M}{L_{1,(1|2)a} + M} \\ &\leq \frac{L_1 + 2M}{L_1 + M} = \frac{1 + 2(M/L_1)}{1 + (M/L_1)}. \end{aligned}$$

Moreover, from $L_{1,1|2} \leq 2L_1$ and (2), we derive

$$\begin{aligned} r_{SMART}(I) &= \frac{L_{1,1|2} + L_2}{L_{1,(1|2)a} + L_{2,(1|2)b}} \leq \frac{L_{1,1|2} + L_2}{L_1 + M} \leq \frac{2L_1 + L_2}{L_1 + M} \\ &\leq \frac{2L_1 + M}{L_1 + M} = \frac{2 + (M/L_1)}{1 + (M/L_1)}. \end{aligned}$$

Thus, $r_{SMART}(I) \leq \min(\frac{1+2(M/L_1)}{1+(M/L_1)}, \frac{2+(M/L_1)}{1+(M/L_1)}) \leq \frac{3}{2}$ for all M and L_1 .

If $L_{1,1|2} > 2L_1$, then Algorithm *DELAY* is applied and from $z^*(I) \geq L_{all} \geq \max(L_{1,1|2}, L_{1|2,2})$, we obtain

$$r_{SMART}(I) \leq \frac{L_1 + L_{1|2,2}}{L_{all}} \leq \frac{L_1}{L_{all}} + \frac{L_{1|2,2}}{L_{all}} \leq \frac{L_1}{L_{1,1|2}} + \frac{L_{1|2,2}}{L_{1|2,2}} \leq \frac{L_1}{2L_1} + 1 < \frac{3}{2}.$$

In conclusion, we have $r_{SMART}(I) \leq \frac{3}{2}$ for all instances I . Thus, $r_{SMART} \leq \frac{3}{2}$ when $p = 2$. The tightness of the bound and the optimality of the algorithm come from Theorem 6. \square

Observe that Algorithm *SMART*(p) either serves all customers in $C_{1|2}$ in the first time period, when Algorithm *IMMEDIATE* is selected, or serves all customers in $C_{1|2}$ in the second time period, when Algorithm *DELAY* is selected. This is an algorithmic choice. The DMPRP, as defined in Section 2, does allow customers in $C_{1|2}$ to be split over the first and second time period.

4 Dynamic Multi-Period Routing Problem - General Case

The algorithms defined in Section 3 naturally extend to a planning horizon with more than two periods: apply the same decision rule at the beginning of each time period, i.e., at the beginning of time periods $1, 2, \dots, T-1$ for a planning horizon with T periods, since no decision has to be made at the beginning of the last period (all remaining orders have to be delivered).

In this section, we provide a lower bound on the competitive ratio of any online algorithm for DMPRP with a planning horizon $T > 2$, we establish the competitive ratio of Algorithm *IMMEDIATE* and Algorithm *DELAY* for the case $T > 2$, and we show that Algorithm *SMART*(p) is no longer optimal for the case $T > 2$, as a variant in which the parameter p is changed from time period to time period already outperforms *SMART*(p) for the case $T = 3$. We restrict the analysis to instances with customers on the non-negative real line.

Let L_1 denote the length of the tour that visits only customers in C_1 , let L_T denote the length of the tour that visits only customers in C_T , and let $L_{t-1|t}$ denote the length of the tour that visits only customers $C_{t-1|t}$ for $t = 2, 3, \dots, T$. Finally, let $L_{t-1|t,t|t+1}$ denote the length of the optimal merge of the tours $L_{t-1|t}$ and $L_{t|t+1}$, for $2 \leq t \leq T-1$. Let $L_{1,1|2}$ denote the optimal merge of the tours L_1 and $L_{1|2}$ and $L_{T-1|T,T}$ the optimal merge of the tours $L_{T-1|T}$ and L_T .

4.1 A lower bound

The next theorem shows that for planning horizons with more than two time periods, the lower bound on the competitive ratio of any algorithm is strictly greater than $\sqrt{2}$.

Theorem 10. *The competitive ratio of any algorithm for DMPRP with a planning horizon $T > 2$ and with customer locations on the non-negative real line is greater than or equal to 1.44.*

Proof. Consider the following instance with three time periods, $T = 3$. The customer set C_1 contains a single customer at distance 1 from the depot (located at the origin) and the customer set $C_{1|2}$ contains a single customer at distance $a > 1$ from the depot.

First, we consider algorithm A' of the class of algorithms that at the beginning of the first time period decide to visit the customers in C_1 and $C_{1|2}$. Now assume that the customer set $C_{2|3}$ contains a single customer at distance a from the depot and that $C_3 = \emptyset$. Then, $z(A', I) = 2a + 2a$, while the off-line optimum is $z^*(I) = 2 + 2a$. Thus, $r_{A'} \geq \frac{2a}{1+a}$.

Next, we consider algorithm A'' of the class of algorithms that at the beginning of the first time period decide to visit only the customers in C_1 . Now assume that the customer set $C_{2|3}$ contains a customer at distance ab , with $b > 1$, from the depot. If algorithm A'' at the beginning of the second time period decides to visit the customers in $C_{1|2}$ and $C_{2|3}$,

then assume that C_3 contains a customer at distance ab . Then, $z(A'', I) = 2 + 2ab + 2ab$, while the off-line optimum is $z^*(I) = 2a + 2ab$. Thus, $r_{A''} \geq \frac{1+2ab}{a+ab}$. If algorithm A'' at the beginning of the second time period decides to visit only the customers in $C_{1|2}$, then assume that $C_3 = \emptyset$. Then, $z(A'', I) = 2 + 2a + 2ab$, while the off-line optimum is $z^*(I) = 2 + 2ab$. Thus, $r_{A''} \geq \frac{1+a+ab}{1+ab}$.

Consequently, for any algorithm A we have

$$r_A \geq \min \left(\frac{2a}{1+a}, \frac{1+2ab}{a+ab}, \frac{1+a+ab}{1+ab} \right). \quad (3)$$

We want to establish the maximum lower bound (3). Suppose that a is fixed and focus on the last two terms of the minimum. Since the first of these is an increasing function of b while the last is a decreasing function of b , the maximum value of the minimum of these two terms is attained at equality, i.e., when

$$\frac{1+2ab}{a+ab} = \frac{1+a+ab}{1+ab},$$

which implies $b = \frac{a-1+\sqrt{a(2a-1)}}{a}$. Therefore

$$r_A \geq \min \left(\frac{2a}{1+a}, \frac{2a-1+2\sqrt{a(2a-1)}}{2a-1+\sqrt{a(2a-1)}} \right).$$

The first term is an increasing function of a and the second term is a decreasing function of a . Therefore, the maximum value of the lower bound (3) is attained at $a \simeq 2.575$ (with $b \simeq 1.881$) and its value is approximately 1.44. Thus, $r_A \geq 1.44$. \square

4.2 Analysis of Algorithms

Next, we turn our attention to the performance of algorithms *IMMEDIATE* and *DELAY*.

Theorem 11. *The competitive ratio of Algorithm IMMEDIATE for instances with customer locations on the non-negative real line is 2 for any length of the planning horizon.*

Proof. Assume T is even. (The case T is odd is handled similarly.) Let I be an instance for which the maximum ratio $\frac{z(\text{IMMEDIATE}, I)}{z^*(I)}$ is attained. It is easy to see that $z(\text{IMMEDIATE}, I) = L_{1,1|2} + L_{2|3} + \dots + L_{T-1|T} + L_T$. Next, consider an optimal solution. In an optimal solution all customers in $C_{1|2}$ have to be served in time periods 1 and 2. Therefore, the cost incurred in an optimal solution over time periods 1 and 2 is at least $L_{1|2}$. More generally, for t even, all customers in $C_{t-1|t}$ have to be served in time periods $t-1$ and t in an optimal solution. Therefore, the cost incurred in an optimal solution over time periods $t-1$ and t is at least $L_{t-1|t}$. Consequently,

$$z^*(I) \geq L_{1|2} + L_{3|4} + \dots + L_{T-1|T}.$$

Analogously, for t odd ($t \geq 3$), all customers in $C_{t-1|t}$ have to be served in time periods $t-1$ and t in an optimal solution. Since an optimal solution also incurs a cost of at least L_1 in the first time period and at least L_T in the last time period, we find that

$$z^*(I) \geq L_1 + L_{2|3} + \dots + L_{T-2|T-1} + L_T.$$

Combining these two observations we find

$$2z^*(I) \geq L_{1,1|2} + L_{2|3} + \dots + L_{T-2|T-1} + L_{T-1|T} + L_T.$$

This shows that Algorithm *IMMEDIATE* never incurs a cost that is more than twice that of an optimal solution. \square

Theorem 12. *The competitive ratio of Algorithm DELAY for instances with customer locations on the non-negative real line is 2 for any length T of the planning horizon.*

Proof. Similar to the proof of Theorem 11. \square

Finally, we turn to the performance analysis of *SMART*(p). We consider the case $T = 3$. We start by examining optimal off-line algorithms. When the planning horizon has three time periods, there are two decision points, since in the third time period we have to serve all remaining customers. Let i denote the decision to immediately serve all available customers, and let d denote the decision to delay all customers that can be postponed. Let the sequence id denote an algorithm that applies *IMMEDIATE* on the first day and *DELAY* on the second day. Similarly we define the other decision sequences ii , di and dd .

Theorem 13. *The off-line optimal solution $z^*(I)$ for instances I with customers on the non-negative real line and a planning horizon of three time periods is given either by $z(id, I)$ or $z(di, I)$. Moreover, $z(di, I) < z(id, I)$ if and only if $L_1 + L_3 < \min(L_{1|2}, L_{2|3})$.*

Proof. The decision sequences ii and dd are dominated by id , since

$$\begin{aligned} z(ii, I) &= L_{1,1|2} + L_{2|3} + L_3 \geq L_{1,1|2} + L_{2|3,3} = z(id, I) \text{ and} \\ z(dd, I) &= L_1 + L_{1|2} + L_{2|3,3} \geq L_{1,1|2} + L_{2|3,3} = z(id, I). \end{aligned}$$

In order to prove the second part of the theorem, observe that

$$\begin{aligned} z(id, I) &= L_{1,1|2} + L_{2|3,3} = \max(L_1, L_{1|2}) + \max(L_{2|3}, L_3) \text{ and} \\ z(di, I) &= L_1 + L_{1|2,2|3} + L_3 = L_1 + \max(L_{1|2}, L_{2|3}) + L_3. \end{aligned}$$

We compare the two non-dominated decision sequences by analyzing a number of cases:

- If $L_1 \geq L_{1|2}$ and $L_3 \geq L_{2|3}$, then $z(id, I) \leq z(di, I)$ because

$$z(id, I) = L_1 + L_3 \leq L_1 + \max(L_{1|2}, L_{2|3}) + L_3 = z(di, I).$$

- If $L_1 \geq L_{1|2}$ and $L_3 < L_{2|3}$, then $z(id, I) \leq z(di, I)$ because

$$z(id, I) = L_1 + L_{2|3} \leq L_1 + \max(L_{1|2}, L_{2|3}) + L_3 = z(di, I).$$

- If $L_1 < L_{1|2}$ and $L_3 \geq L_{2|3}$, then $z(id, I) \leq z(di, I)$ because

$$z(id, I) = L_{1|2} + L_3 \leq L_1 + \max(L_{1|2}, L_{2|3}) + L_3 = z(di, I).$$

- If $L_1 < L_{1|2}$ and $L_3 < L_{2|3}$, then $z(id, I) \leq z(di, I)$ if and only if $\min(L_{1|2}, L_{2|3}) \leq L_1 + L_3$ because

$$z(id, I) = L_{1|2} + L_{2|3} \leq L_1 + \max(L_{1|2}, L_{2|3}) + L_3 = z(di, I)$$

\Leftrightarrow

$$\min(L_{1|2}, L_{2|3}) \leq L_1 + L_3.$$

We conclude that $z^*(I) = z(di, I) < z(id, I)$ if and only if $L_1 + L_3 < \min(L_{1|2}, L_{2|3})$. \square

Theorem 14. *The competitive ratio of Algorithm SMART(p) for instances with customers on the non-negative real line and a planning horizon of three time periods is $\max(\frac{1+2p^2}{p+p^2}, \frac{1+p}{p})$, with a best ratio of $\frac{3}{2}$ for $p = 2$.*

Proof. The performance of SMART(p) can be analyzed by studying its three possible decision sequences. There are only three decision sequences that need to be analyzed, because the decision sequence ii will never be applied by SMART.

Case dd . SMART(p) applies dd under the following conditions:

$$\begin{cases} L_{1,1|2} > pL_1 \\ L_{1|2,2|3} > pL_{1|2} \end{cases} \text{ which imply } \begin{cases} L_{1|2} > pL_1 \\ L_{2|3} > pL_{1|2} > p^2L_1. \end{cases}$$

Since $p > 1$, the inequality $L_1 < L_{1|2} < L_{2|3}$ holds. For the case $z(id, I) \leq z(di, I)$, the ratio is

$$\begin{aligned} r_{SMART}(I) &= \frac{L_1 + L_{1|2} + L_{2|3,3}}{L_{1,1|2} + L_{2|3,3}} = \frac{L_1 + L_{1|2} + L_{2|3,3}}{L_{1|2} + L_{2|3,3}} \\ &\leq \frac{L_1 + L_{1|2}}{L_{1|2}} \leq \frac{L_1 + pL_1}{pL_1} = \frac{1+p}{p}, \end{aligned}$$

while for the case $z(di, I) < z(id, I)$, the ratio is

$$\begin{aligned} r_{SMART}(I) &= \frac{L_1 + L_{1|2} + L_{2|3,3}}{L_1 + L_{1|2,2|3} + L_3} = \frac{L_1 + L_{1|2} + L_{2|3}}{L_1 + L_{2|3} + L_3} < \frac{L_1 + L_{1|2} + L_{2|3}}{L_1 + L_{2|3}} \\ &< \frac{L_1 + L_{1|2} + pL_{1|2}}{L_1 + pL_{1|2}} < \frac{L_{1|2} + pL_{1|2}}{pL_{1|2}} < \frac{1+p}{p}. \end{aligned}$$

Therefore,

$$r_{SMART}(I) \leq \frac{1+p}{p}.$$

Case di . $SMART(p)$ applies di under the following conditions:

$$\begin{cases} L_{1,1|2} > pL_1 \\ L_{1|2,2|3} \leq pL_{1|2} \end{cases} \text{ which imply } \begin{cases} L_{1|2} > pL_1 > L_1 \\ L_{2|3} \leq pL_{1|2}. \end{cases}$$

If $z^*(I) = z(di, I)$, then $r_{SMART}(I) = 1$. Thus, we consider the case for which the off-line optimum is $z^*(I) = z(id, I)$ and the ratio is

$$r_{SMART}(I) = \frac{L_1 + L_{1|2,2|3} + L_3}{L_{1|2} + L_{2|3,3}} \leq \frac{L_1 + L_{1|2,2|3} + L_{2|3,3}}{L_{1|2} + L_{2|3,3}}.$$

If $L_{2|3} < L_{1|2}$, then

$$r_{SMART}(I) \leq \frac{L_1 + L_{1|2} + L_{2|3,3}}{L_{1|2} + L_{2|3,3}} \leq \frac{L_1 + L_{1|2}}{L_{1|2}} < \frac{1+p}{p}.$$

If $L_{1|2} < L_{2|3}$, then

$$\begin{aligned} r_{SMART}(I) &\leq \frac{L_1 + L_{2|3} + L_{2|3,3}}{L_{1|2} + L_{2|3,3}} \leq \frac{L_1 + L_{2|3} + L_{2|3}}{L_{1|2} + L_{2|3}} = \frac{L_1 + 2L_{2|3}}{L_{1|2} + L_{2|3}} \\ &\leq \frac{L_1 + 2pL_{1|2}}{L_{1|2} + pL_{1|2}} \leq \frac{L_1 + 2p^2L_1}{pL_1 + p^2L_1} \leq \frac{1 + 2p^2}{p + p^2}. \end{aligned}$$

Therefore,

$$r_{SMART}(I) \leq \max\left(\frac{1+p}{p}, \frac{1+2p^2}{p+p^2}\right).$$

Case *id*. $SMART(p)$ applies *id* under the following condition:

$$L_{1,1|2} \leq pL_1 \text{ which implies } L_{1|2} < pL_1.$$

If $z^*(I) = z(id, I)$, then $r_{SMART}(I) = 1$. Thus, we consider the case for which the off-line optimum is $z^*(I) = z(di, I)$. By Theorem 13 we assume $L_1 + L_3 < \min(L_{1|2}, L_{2|3})$ and the ratio is bounded as follows

$$\begin{aligned} r_{SMART}(I) &= \frac{L_{1,1|2} + L_{2|3,3}}{L_1 + L_{1|2,2|3} + L_3} = \frac{\max(L_1, L_{1|2}) + \max(L_{2|3}, L_3)}{L_1 + \max(L_{1|2}, L_{2|3}) + L_3} \\ &= \frac{L_{1|2} + L_{2|3}}{L_1 + \max(L_{1|2}, L_{2|3}) + L_3} \leq \frac{L_{1|2} + L_{2|3}}{L_1 + \max(L_{1|2}, L_{2|3})} \\ &\leq \frac{\min(L_{1|2}, L_{2|3}) + \max(L_{1|2}, L_{2|3})}{L_1 + \max(L_{1|2}, L_{2|3})} \leq \frac{2 \min(L_{1|2}, L_{2|3})}{L_1 + \min(L_{1|2}, L_{2|3})} \\ &< \frac{2pL_1}{L_1 + pL_1} = \frac{2p}{1+p}. \end{aligned}$$

The latter inequality comes from $\min(L_{1|2}, L_{2|3}) \leq L_{1|2} < pL_1$.

Combining these results, we obtain

$$r_{SMART} \leq \max\left(\frac{1+2p^2}{p+p^2}, \frac{1+p}{p}, \frac{2p}{1+p}\right) = \max\left(\frac{1+2p^2}{p+p^2}, \frac{1+p}{p}\right).$$

The minimum upper bound on the competitive ratio is attained for $p = 2$ for which $SMART(p)$ guarantees a worst-case ratio less or equal to $\frac{3}{2}$.

To show that the bound is tight, consider the following two instances:

1. Let C_1 contain a single customer at distance 1, $C_{1|2}$ a single customer at distance $p + \varepsilon$, and let $C_{2|3} = C_3 = \emptyset$. Then, $SMART(p)$ visits C_1 in the first time period and $C_{1|2}$ in the second time period. The ratio is $\frac{1+p+\varepsilon}{p+\varepsilon}$, which can be arbitrarily close to $\frac{1+p}{p}$. Thus, $r_{SMART(p)} \geq \frac{1+p}{p}$.
2. Let C_1 contain a single customer at distance 1, $C_{1|2}$ a single customer at distance $p + \varepsilon$, and $C_{2|3}$ and C_3 a single customer at distance p^2 . Then, $SMART(p)$ visits C_1 in the first time period, $C_{1|2}$ and $C_{2|3}$ in the second time period, and C_3 in the last time period. The ratio is $\frac{1+2p^2}{p+\varepsilon+p^2}$, which can be arbitrarily close to $\frac{1+2p^2}{p+p^2}$. Thus, $r_{SMART(p)} \geq \frac{1+2p^2}{p+p^2}$.

Therefore, for any $p > 1$, we have $r_{SMART} = \max\left(\frac{1+2p^2}{p+p^2}, \frac{1+p}{p}\right)$. Its minimum value is $\frac{3}{2}$ for $p = 2$. \square

4.3 A Time-Dependent Variant of Algorithm *SMART*

For instances with a planning horizon of three time periods, we have established a lower bound on the competitive ratio of any algorithm of 1.44 and a competitive ratio for *SMART*(2) of $\frac{3}{2}$. It may still be the case that *SMART*(2) is an optimal algorithm. Unfortunately, the following theorem shows that a version of *SMART* with time-dependent parameters achieves a better competitive ratio than *SMART*(2). Consider *SMART*(p_1, p_2), which applies *SMART*(p_1) in the first time period and *SMART*(p_2) in the second time period.

Theorem 15. *The competitive ratio of Algorithm *SMART*(p_1, p_2) for instances with customers on the non-negative real line is $\max(\frac{2p_1}{1+p_1}, \frac{1+p_1}{p_1}, \frac{1+p_2}{p_2}, \frac{1+2p_1p_2}{p_1+p_1p_2})$, with a best ratio of 1.473 for $p_1 \simeq 2.79$ and $p_2 \simeq 2.11$.*

Proof. With arguments similar to those in the proof of Theorem 14, we find that

$$r_{SMART(p_1, p_2)} \leq \beta(p_1, p_2) = \max\left(\frac{2p_1}{1+p_1}, \frac{1+p_1}{p_1}, \frac{1+p_2}{p_2}, \frac{1+2p_1p_2}{p_1+p_1p_2}\right),$$

$$r_{SMART(p_1, p_2)} \leq \tilde{\beta} = \min_{p_1, p_2 > 1} \beta(p_1, p_2).$$

We make the following observations:

- $\frac{1+p_2}{p_2}$ is decreasing with p_2 for any p_1 ;
- $\frac{1+2p_1p_2}{p_1+p_1p_2}$ is increasing with p_2 for any p_1 .

In order to minimize β , p_2 should be taken such that $\frac{1+p_2}{p_2} = \frac{1+2p_1p_2}{p_1+p_1p_2}$, that is $p_2 = \frac{2p_1 - 1 + \sqrt{(8p_1^2 - 4p_1 + 1)}}{2p_1}$. Thus, we consider

$$\beta(p_1) = \beta(p_1, p_2(p_1)) = \max\left(\frac{1+p_1}{p_1}, \frac{2p_1}{1+p_1}, \frac{4p_1 - 1 + \sqrt{8p_1^2 - 4p_1 + 1}}{2p_1 - 1 + \sqrt{8p_1^2 - 4p_1 + 1}}\right).$$

The minimum is attained at $\tilde{p}_1 \simeq 2.79$. The guaranteed performance is $r_{SMART(p_1, p_2)} \leq \tilde{\beta} \simeq 1.473$, for $p_1 \simeq 2.79$ and $p_2 \simeq 2.11$.

The tightness of this bound can be proved with an argument similar to the one used in Theorem 14. \square

In this section, we have established that algorithms *IMMEDIATE* and *DELAY* have a competitive ratio of 2 independently of the length of the planning horizon, but that the competitive ratio of *SMART*(p) increases when we increase the length of the planning horizon to $T = 3$ and that it is no longer optimal.

The lower bound obtained for the non-negative real line clearly holds for Euclidean instances as well. The extension of the analysis of *IMMEDIATE* and *DELAY* to the Euclidean plane is straightforward and gives the same competitive ratio.

5 Future Research

We have studied a dynamic multi-period routing problem and have analyzed three simple algorithms for its solution, i.e., *IMMEDIATE*, *DELAY*, and *SMART*(p). We have shown that *SMART*(p), with an appropriate parameter p , outperforms *IMMEDIATE* and *DELAY* on instances with short planning horizons.

A natural extension of this research is to investigate an asymptotic lower bound on the competitive ratio of any on-line algorithm for T tending to infinity and to analyze the performance of *SMART* and its time-dependent variant for longer planning horizons.

We have assumed that orders have to be fulfilled within the next two time periods. In the near future, we plan to investigate the impact of that restriction, i.e., whether significantly different results are obtained when more than two time periods are available to fulfill orders.

We have ignored some key aspects of the motivating application, most glaringly everything related to the size of orders and the limited capacity of the vehicles that deliver orders. As a result, the routing problem to be solved in each time period is a traveling salesman problem. Capacitated variants of the DMPRP deserve to be studied.

Finally, establishing the competitive ratio of an on-line algorithm provides only partial information on its performance. We also plan to study, theoretically as well as empirically, optimal and suboptimal policies under a probabilistic characterization of the demand.

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