On the Existence and Convergence of the Central Path for Convex Programming and Some Duality Results

Renato D.C. Monteiro* and Fangjun Zhou†

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Abstract

This paper gives several equivalent conditions which guarantee the existence of the weighted central paths for a given convex programming problem satisfying some mild conditions. When the objective and constraint functions of the problem are analytic, we also characterize the limiting behavior of these paths as they approach the set of optimal solutions. A duality relationship between a certain pair of logarithmic barrier problems is also discussed.

1 Introduction

The purpose of this work is to provide several conditions which guarantee the existence of a family of weighted central paths associated with a given convex programming problem and to study the limiting behavior of these paths as they approach the set of optimal solutions of the problem.

There are several papers in the literature which study these issues in the context of linear and convex quadratic programs. These include Adler and Monteiro [1], Güler [4], Kojima, Mizuno and Noma [7], Megiddo [9], Megiddo and Shub [10], Monteiro [11], Monteiro and Tsuchiya [13], Witzgall, Boggs and Domich [15]. For linear and convex quadratic programs, the properties of the weighted central paths are quite well understood under very mild assumptions, namely (a) existence of an interior feasible solution and, (b) boundedness of the set of optimal solutions. The paper by Güler et al. [5] gives a simplified and elegant treatment in the context of linear programs of several results treated in the forementioned papers.

Conditions which guarantee the existence of the weighted central paths have been given by Kojima, Mizuno and Noma [7] and McLinden [8] for special classes of convex programs. Namely, [7] deals with the monotone nonlinear complementarity problem which is known to include certain types of convex programs as special case and [8] deals with the pair of dual convex programs

$$\inf\{h(x) \mid x \geq 0\}, \qquad \inf\{h^*(\xi) \mid \xi \geq 0\}, \tag{1}$$

where $h(\cdot)$ is an extended proper convex function and $h^*(\cdot)$ is the conjugate function of $h(\cdot)$. We develop corresponding results for the general convex program $(\mathcal{P})$ described below. Major

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†School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, 30332.

differences between problem $(\mathcal{P})$ and the one considered by McLinden are: (i) the feasible region of problem (1) is contained in the nonnegative orthant while problem $(\mathcal{P})$ is not required to satisfy this condition; (ii) the objective and constraint functions of problem $(\mathcal{P})$ assume only real values while the objective function $h(\cdot)$ of problem (1) can assume the value $+\infty$. It might be possible to extend some of our results to the more general setting of extended convex functions but we made no attempt in this direction since such extension would needless complicate our notation and development.

McLinden in his remarkable paper [8] gives some special results about the limiting behavior of the weighted central paths with respect to the pair of convex programs (1). Specifically, he analyzes the convergence behavior of these paths assuming, in addition to conditions (a) and (b) above, the existence of a pair of primal and dual optimal solutions satisfying strict complementarity.

In this paper (Section 5), we provide convergence results for the weighted central paths assuming that the objective function and the constraint functions are analytic. As opposed to McLinden [8], we do not assume the existence of any pair of primal and dual optimal solutions satisfying strict complementarity.

Throughout this paper $\mathbf{R}^l$ denotes the l-dimensional Euclidean space; also, $\mathbf{R}^l_+$ and $\mathbf{R}^l_{++}$ denote the nonnegative and the positive orthants of $\mathbf{R}^l$, respectively. Given convex functions $f : \mathbf{R}^n \to \mathbf{R}$ and $g_j : \mathbf{R}^n \to \mathbf{R}$, $j = 1, \ldots, p$, throughout this paper we consider the following convex program

$$(\mathcal{P}) \qquad \begin{array}{ll} \inf & f(x) \\ \text{s.t.} & x \in P \equiv \{x \in \mathbf{R}^n \mid Ax = b,\ g(x) \le 0\}, \end{array}$$

where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$ and $g : \mathbf{R}^n \to \mathbf{R}^p$ denotes the function defined for every $x \in \mathbf{R}^n$ by $g(x) = (g_1(x), \ldots, g_p(x))^T$.

Given a fixed weight vector $w \in \mathbf{R}^p_{++}$, the w-central path of $(\mathcal{P})$ arises by considering the following parametrized family of logarithmic barrier problems

$$(\mathcal{P}(t)) \qquad \inf\{f(x) - t\sum_{j=1}^{p} w_j \log|g_j(x)| \ \mid \ x \in P^0\}, \tag{2}$$

where $t \in \mathbf{R}_{++}$ is the parameter of the family and

$$P^0 \equiv \{x \in \mathbf{R}^n \mid Ax = b,\ g(x) < 0\}$$

is the set of interior feasible solutions of $(\mathcal{P})$. If each problem $(\mathcal{P}(t))$ has exactly one solution $x(t)$ then the path $t > 0 \mapsto x(t) \in P^0$ is called the w-central path associated with $(\mathcal{P})$. Conditions for the existence of this path are therefore conditions which guarantee that $(\mathcal{P}(t))$ has exactly one solution for every $t > 0$.

Our paper is organized as follows. In Section 2 we introduce some notation and terminology that are used throughout the paper. We also introduce some mild assumptions that are frequently used in our results and discuss several situations in which these assumptions are satisfied. It is hoped that this discussion will convince the reader of the mildness of our assumptions.

In Section 3, several equivalent conditions which guarantee the existence of at least one solution of problem $(\mathcal{P}(t))$ are presented (see Theorem 3.1 and Theorem 3.2). A duality theory between a certain pair of logarithmic barrier problems (that is, problems (4) and (5)) is also developed in this section. A similar duality theory has been developed by Megiddo [9] for the same pair of logarithmic barrier problems in the context of linear programs.

In Section 4, we show that the several conditions developed in Section 3 are equivalent to conditions requiring boundedness of the optimal solution set of problem $(\mathcal{P})$ and/or its dual problem (see Theorem 4.1). Both the Wolfe dual and the Lagrangean dual are considered in our analysis.

Results that guarantee the uniqueness of the solution of problem $(\mathcal{P}(t))$ are stated in Section 5 and require analyticity of the functions $f(\cdot)$ and $g_j(\cdot)$, $j = 1, \ldots, p$. We observe that while most of the results of Section 3 and Section 4 do not require any analyticity condition, Section 5 deals exclusively with convex programs whose objective and constraint functions are analytic. In Section 5 the analyticity assumption plays a major role in the study of the limiting behavior of the w-central path $t \mapsto x(t)$ as $t \to 0$. Convergence of certain dual weighted central paths are also proved in Section 5 under the assumption that all constraint functions are affine.

The following notation is used throughout the paper. If $x \in \mathbf{R}^l$ and $B \subset \{1, \ldots, l\}$ then x_B denotes the subvector $(x_i)_{i \in B}$; if $h(\cdot)$ is a function taking values in $\mathbf{R}^l$ then $h_B(y)$ denotes the subvector $[h(y)]_B$ for every y in the domain of $h(\cdot)$. The vector $(1, \cdots, 1)^T$, regardless of its dimension, is denoted by $\mathbf{1}$. For any vector x, the notation $|x|$ is used for the vector whose i-th component is $|x_i|$. Throughout, we use terminology and facts from finite-dimensional convex analysis as presented by Rockafellar [14]. In particular, the symbols *, ∂ and 0^+ applied to a function signify the conjugate function, subdifferential mapping, and the recession function, respectively. Also, the symbol 0^+ applied to a convex set signify the recession cone of the set. We denote the number of elements of a finite set J by $|J|$.

2 Notation and Assumptions

In this section we introduce some notation and terminology that are used throughout the paper. We also introduce some assumptions that are frequently used in our results and also discuss several situations in which these assumptions are satisfied; it is our hope that this discussion will show that these assumptions are mild ones.

The problem we consider in this paper is the convex program $(\mathcal{P})$ stated in Section 1. The Lagrangean function $\mathcal{L} : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$ associated with $(\mathcal{P})$ is defined for every $(x, y, s) \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p$ by

$$\mathcal{L}(x, y, s) \equiv f(x) + (b - Ax)^T y + s^T g(x),$$

and the Lagrangean dual problem associated with $(\mathcal{P})$ is

$$(\mathcal{D}_L) \qquad \begin{array}{ll} \sup & L(y, s) \\ \text{s.t.} & (y, s) \in D_L \equiv \{(y, s) \mid s \geq 0, \; L(y, s) > -\infty\}, \end{array}$$

where $L : \mathbf{R}^m \times \mathbf{R}^p \to [-\infty, \infty)$ is the dual function defined by

$$L(y, s) \equiv \inf\{\mathcal{L}(x, y, s) \mid x \in \mathbf{R}^n\}, \qquad \forall (y, s) \in \mathbf{R}^m \times \mathbf{R}^p. \tag{3}$$

It is well known that L is a concave function since it is the pointwise infimum of the affine (and hence concave) functions $\mathcal{L}(x, \cdot, \cdot)$, $x \in \mathbf{R}^n$. Another dual problem associated with $(\mathcal{P})$ is the Wolfe dual given by

$$(\mathcal{D}_W) \qquad \begin{array}{ll} \sup & L(y, s) \\ \text{s.t.} & (y, s) \in D_W \equiv \{(y, s) \in D_L \mid L(y, s) = \mathcal{L}(x, y, s) \text{ for some } x \in \mathbf{R}^n \}. \end{array}$$

Hence D_W is the set of all points (y, s) in D_L for which the infimum in (3) is achieved. It is well known that the set of points (y, s) for which the infimum in (3) is achieved is given by $\{(y, s) \mid 0 \in \partial\mathcal{L}(x, y, s) \text{ for some } x \in \mathbf{R}^n\}$.

We denote the set of optimal solutions of a program $(\cdot)$ as $\mathrm{opt}(\cdot)$ and its value as $\mathrm{val}(\cdot)$. So, for instance, $\mathrm{opt}(\mathcal{P})$ denotes the set of optimal solutions of $(\mathcal{P})$ and $\mathrm{val}(\mathcal{P})$ denotes its value. By definition, the value of a program is the infimum (or the supremum) of the set of all values that the objective function can assume over the set of all feasible solutions of the problem. By convention, we assume that the infimum (supremum) of the empty set is equal to $+\infty$ $(-\infty)$ and that the infimum (supremum) of a set unbounded from below (above) is equal to $-\infty$ $(+\infty)$. Let

$$\begin{aligned} D_L^0 &\equiv \{(y, s) \in D_L \mid s > 0\}, \\ D_W^0 &\equiv \{(y, s) \in D_W \mid s > 0\}. \end{aligned}$$

We refer to D_L^0 and D_W^0 as the set of interior feasible solutions of problem $(\mathcal{D}_L)$ and $(\mathcal{D}_W)$, respectively. The following logarithmic barrier problems plays a fundamental role in our presentation. Given $w \in \mathbf{R}^p_{++}$, consider the problems

$$(\mathcal{P}^w) \qquad \inf\{p_w(x) \equiv f(x) - \sum_{j=1}^{p} w_j \log|g_j(x)| \;\mid\; x \in P^0\}, \tag{4}$$

$$(\mathcal{D}_L^w) \qquad \sup\{d_w(y, s) \equiv L(y, s) + \sum_{j=1}^{p} w_j \log s_j \mid (y, s) \in D_L^0\}, \tag{5}$$

$$(\mathcal{D}_W^w) \qquad \sup\{d_w(y, s) \mid (y, s) \in D_W^0\}. \tag{6}$$

An equivalent way to formulate problem $(\mathcal{D}_W^w)$ is as the problem

$$\sup\{\mathcal{L}(x, y, s) + \sum_{j=1}^{p} w_j \log s_j \mid s > 0,\ 0 \in \partial\mathcal{L}(x, y, s)\},$$

which, for differentiable functions $f(\cdot)$ and $g(\cdot)$, takes the form

$$\sup\{\mathcal{L}(x, y, s) + \sum_{j=1}^{p} w_j \log s_j \mid s > 0,\ \nabla f(x) - A^T y + \nabla g(x) s = 0\}.$$

Here are adopting the convention that the gradient of a scalar function is a column vector and $\nabla g(x) = [\nabla g_1(x) \cdots \nabla g_p(x)]$. We also let

$$PD^0 = \{(x, y, s) : x \in P^0,\ s > 0,\ \mathcal{L}(x, y, s) = L(y, s)\},$$

and, for $w \in \mathbf{R}^p_{++}$, we define the set

$$S_w \equiv \{(x, y, s) \in PD^0 \mid -g(x) \circ s = w\},$$

where the notation $\circ$ denotes the Hadamard product of two vectors, that is if u and v are two vectors then $u \circ v$ denotes the vector whose i-th component is equal to $u_i v_i$ for every i. When $f(\cdot)$

and $g(\cdot)$ are differentiable, the points in S_w satisfy the following "centering" conditions:

$$\begin{aligned}
&Ax = b, \quad g(x) < 0, \\
&\nabla f(x) + \nabla g(x)s - A^T y = 0, \quad s > 0, \\
&-g(x) \circ s = w.
\end{aligned}$$

We next introduce some assumptions which are frequently used in our presentation and subsequently we make some comments about the assumptions. Consider the following two assumptions:

Assumptions:
(A) $\text{rank}(A) = m$;
(B) For any $\alpha > 0$ and $\beta \in \mathbf{R}$, the set $\{x \in P \mid g(x) \geq -\alpha\mathbf{1},\ f(x) \leq \beta\}$ is bounded.

Assumption (A) is quite standard and considerably simplifies our development. We next discuss Assumption (B). First note that Assumption (B) obviously holds when $P = \emptyset$, that is, when problem $(\mathcal{P})$ is infeasible. Assumption (B) also holds when $\text{opt}(\mathcal{P})$ is nonempty and bounded. Indeed, it is easy to see that $\text{opt}(\mathcal{P})$ is nonempty and bounded if and only if there exists a constant β such that the set $\{x \in P \mid f(x) \leq \beta\}$ is bounded. It follows from Lemma 2.1 below that a necessary and sufficient condition for $\text{opt}(\mathcal{P})$ to be nonempty and bounded is that the set $\{x \in P \mid f(x) \leq \beta\}$ be bounded for *any* $\beta \in \mathbf{R}$. Since the set defined in Assumption (B) is a subset of $\{x \in P \mid f(x) \leq \beta\}$, we conclude that Assumption (B) holds when $\text{opt}(\mathcal{P})$ is nonempty and bounded.

A proof of the following well known result can be found for example in Fiacco and McCormick [2], page 93.

Lemma 2.1 *Assume that $h_j : \mathbf{R}^n \to \mathbf{R}$, $j = 1, \ldots, l$, are convex functions and that, for some scalars $\alpha_1, \ldots, \alpha_l$, the set $\{x \in \mathbf{R}^n \mid h_j(x) \leq \alpha_j,\ j = 1, \ldots, l\}$ is nonempty and bounded. Then, for any given scalars $\beta_1, \ldots, \beta_l$, the set $\{x \in \mathbf{R}^n \mid h_j(x) \leq \beta_j,\ j = 1, \ldots, l\}$ is bounded (and possibly empty).*

We should note that Assumption (B) does not imply that $\text{opt}(\mathcal{P})$ is nonempty or that $\text{opt}(\mathcal{P})$ is bounded when it is nonempty. Indeed, the problem $\inf\{x \mid x \leq 0\}$ satisfies Assumption (B) but it has no optimal solution. Moreover, the problem $\inf\{-x_1 \mid x_1 \leq 0, x_2 \leq 0\}$ satisfies Assumption (B) and has a nonempty optimal solution set which is unbounded.

An example of a problem which does not satisfy Assumption (B) is

$$\inf\{f(x) \equiv 0 \mid e^x - 1 \leq 0\}.$$

The next result gives a sufficient condition for Assumption (B) to hold when each constraint function $g_j(\cdot)$, $j = 1, \ldots, p$, is affine.

Lemma 2.2 *Assume that $g(x) = Cx - h$, where C is a $p \times n$ matrix and $h \in \mathbf{R}^p$, and that $P = \{x \in \mathbf{R}^n \mid Ax = b,\ Cx \leq h\}$ is nonempty. Then a sufficient condition for Assumption (B) to hold is that P be a pointed polyhedron (that is, P has a vertex). In addition, if $f(\cdot)$ is an affine function then this condition is also necessary.*

Proof. Assume that P is pointed and let $\alpha > 0$ and $\beta \in \mathbf{R}$ be given. Note that P is pointed if and only if the lineality space of P is equal to $\{0\}$, that is, $\{d \in \mathbf{R}^n \mid Ad = 0,\ Cd = 0\} = \{0\}$. Hence the set

$$\{x \in P \mid g(x) = Cx - h \geq -\alpha\mathbf{1}\} = \{x \mid Ax = b,\ h - \alpha\mathbf{1} \leq Cx \leq h\} \tag{7}$$

is bounded since its recession cone is equal to $\{d \in \mathbf{R}^n \mid Ad = 0, Cd = 0\} = \{0\}$. Thus the set defined in Assumption (B) is also bounded since it is a subset of the set in (7). We have thus shown that Assumption (B) holds.

To show the second part of the lemma, assume that $f(\cdot)$ is an affine function, say, $f(x) = c^T x + \gamma$ where $c \in \mathbf{R}^n$ and $\gamma \in \mathbf{R}$. The set of Assumption (B) is bounded if and only if its recession cone $\{d \mid Ad = 0, Cd = 0, c^T d \leq 0\}$ is equal to $\{0\}$. However, it is easy to see that $\{d \mid Ad = 0, Cd = 0, c^T d \leq 0\} = \{0\}$ if and only if $\{d \mid Ad = 0, Cd = 0\} = \{0\}$, that is, if and only if P is pointed. ∎

3 Conditions for the Existence of the Central Path

In this section we give several equivalent conditions which ensure that problem $(\mathcal{P}(t))$, $t > 0$, defined in (2) has at least one solution. With this goal in mind, we will consider the more general question of existence of a solution of the convex program $(\mathcal{P}^w)$ for an arbitrary $w \in \mathbf{R}^p_{++}$. We also discuss the duality relationship that exists between the pair of problems $(\mathcal{P}^w)$ and $(\mathcal{D}^w_L)$ (or $(\mathcal{D}^w_W)$). (See Megiddo [9] for a discussion of this duality relationship in the context of linear programs.)

We start by stating one of the main results of this section. Theorem 3.2, which is the other main result of this section, complements Theorem 3.1.

Theorem 3.1 *Suppose that both Assumption (A) and Assumption (B) hold. Then the following statements are all equivalent:*

(a) $P^0 \neq \emptyset$ *and* $D^0_W \neq \emptyset$;

(b) $P^0 \neq \emptyset$ *and* $D^0_L \neq \emptyset$;

(c) $opt(\mathcal{P}^w) \neq \emptyset$;

(d) $S_w \neq \emptyset$;

(e) $PD^0 \neq \emptyset$;

(f) $opt(\mathcal{D}^w_L) \neq \emptyset$.

Moreover, any of the above statements imply:

(1) $opt(\mathcal{D}^w_L) = opt(\mathcal{D}^w_W)$ *and the set* $\{s \mid (y, s) \in opt(\mathcal{D}^w_L)\}$ *is a singleton; if in addition Assumption (A) holds and the functions* $f(\cdot)$ *and* $g(\cdot)$ *are differentiable then* $opt(\mathcal{D}^w_L)$ *is also a singleton;*

(2) $S_w = \{(x, y, s) \mid x \in opt(\mathcal{P}^w), (y, s) \in opt(\mathcal{D}^w_L)\}$;

(3) for any fixed $(\bar{y}, \bar{s}) \in opt(\mathcal{D}^w_L)$,

$$opt(\mathcal{P}^w) = \{x \mid x \in P, |g(x)| \circ \bar{s} = w\} \cap \text{Argmin}\{\mathcal{L}(x, \bar{y}, \bar{s}) \mid x \in \mathbf{R}^n\};$$

(4) $val(\mathcal{P}^w) - val(\mathcal{D}^w_L) = \sum_{j=1}^p w_j(1 - \log w_j)$ *and* $val(\mathcal{D}^w_L) = val(\mathcal{D}^w_W)$.

The proof of Theorem 3.1 will be given only after we prove several preliminary results, some of which are interesting in their own right. The first two lemmas are well known and are stated without proofs.

Lemma 3.1 *Assume that X is a metric space and that $h : X \to \mathbf{R} \cup \{\infty\}$ is a proper lower semi-continuous function. Let E be a nonempty subset of X. If there exists a point $x^0 \in E$ such that $h(x^0) < \infty$ and the set $\{x \in E \mid h(x) \le h(x^0)\}$ is compact then the set of minimizers of the problem* $\inf\{h(x) \mid x \in E\}$ *is nonempty and compact (and hence bounded).*

Lemma 3.2 *Let α and β be given positive scalars and consider the function $h : (0, \infty) \to \mathbf{R}$ defined by $h(t) = \alpha t - \beta \log t$ for all $t \in (0, \infty)$. Then,*

(a) h is strictly convex;

(b) $h(t) \ge \beta[1 - \log(\beta/\alpha)]$ for all $t > 0$ with equality holding if and only if $t = \beta/\alpha$;

(c) $h(t) \to \infty$ as $t \to 0$ or $t \to \infty$.

The following simple lemma is invoked more than once in our development.

Lemma 3.3 *Suppose that $D_L^0 \neq \emptyset$. Then there exist constants $\tau_0 \in \mathbf{R}$ and $\tau_1 > 0$ such that*

$$f(x) \ge \tau_0 + \tau_1 \sum_{j=1}^{p} |g_j(x)|, \qquad \forall x \in P. \tag{8}$$

Proof. Let (y^0, s^0) be a fixed point in D_L^0. By the definition of D_L^0, there exists $\tau_0 \in \mathbf{R}$ such that

$$\mathcal{L}(x, y^0, s^0) \ge \tau_0, \qquad \forall x \in \mathbf{R}^n. \tag{9}$$

Let $\tau_1 \equiv \min\{s_j^0 \mid j = 1, \ldots, p\} > 0$. Rearranging (9), we obtain that for every $x \in P$,

$$f(x) \ge \tau_0 - (s^0)^T g(x) - (y^0)^T (b - Ax) = \tau_0 + (s^0)^T |g(x)| \ge \tau_0 + \tau_1 \sum_{j=1}^{p} |g_j(x)|.$$

■

The next result shows that the existence of interior feasible solutions for both $(\mathcal{P})$ and $(\mathcal{D}_L)$ implies that $(\mathcal{P}^w)$ has an optimal solution.

Proposition 3.1 *Suppose that Assumption (B) holds and let $w \in \mathbf{R}^p_{++}$ be given. If $P^0 \neq \emptyset$ and $D_L^0 \neq \emptyset$ then opt$(\mathcal{P}^w) \neq \emptyset$.*

Proof. Take a point $x^0 \in P^0$. In view of Lemma 3.1, the result follows once we show that the set $\Omega^0 \equiv \{x \in P^0 \mid p_w(x) \le p_w(x^0)\}$ is compact, where $p_w(\cdot)$ is defined in (4). Indeed, since $D_L^0 \neq \emptyset$, Lemma 3.3 implies that there exist constants $\tau_0 \in \mathbf{R}$ and $\tau_1 > 0$ such that (8) holds. Hence, we have

$$\begin{aligned} \sum_{j=1}^{p} (\tau_1 |g_j(x)| - w_j \log |g_j(x)|) \ &\le\ f(x) - \tau_0 - \sum_{j=1}^{p} w_j \log |g_j(x)| \\ &=\ p_w(x) - \tau_0 \ \le\ p_w(x^0) - \tau_0, \qquad \forall x \in \Omega^0. \end{aligned}$$

Using Lemma 3.2 and the fact that $\tau_1 > 0$ and $w > 0$, it is easy to verify that the above relation implies the existence of a constant $\epsilon > 0$ such that

$$\epsilon \le |g_j(x)| \le \epsilon^{-1}, \qquad \forall x \in \Omega^0 \text{ and } \forall j = 1, \ldots, p. \tag{10}$$

Relation (10) and the fact that $p_w(x)$ is bounded above on Ω^0 imply that $f(x)$ is also bounded above on Ω^0. In view of Assumption (B), it follows that Ω^0 is bounded. Relation (10) also implies that

$$\Omega^0 \equiv \{x \in P^\epsilon \mid p_w(x) \leq p_w(x^0)\}$$

where $P^\epsilon \equiv \{x \in \mathbf{R}^n \mid Ax = b,\, g(x) \leq -\epsilon \mathbf{1}\}$. Since P^ϵ is a closed set, it follows that Ω^0 is also a closed set. We have thus proved that Ω^0 is a compact set. ■

The set opt($\mathcal{P}^w$) is not necessarily a singleton. However, we have the following result whose proof is left to the reader.

Lemma 3.4 *If opt($\mathcal{P}^w$) $\neq \emptyset$ then the set $\{(f(x), g(x)) \mid x \in opt(\mathcal{P}^w)\}$ is a singleton, that is, there exist $\bar{f} \in \mathbf{R}$ and $\bar{g} \in \mathbf{R}^p$ such that $f(x) = \bar{f}$ and $g(x) = \bar{g}$ for every $x \in opt(\mathcal{P}^w)$.*

We now turn our efforts to obtaining conditions which ensure that *opt*($\mathcal{D}_L^w$) $\neq \emptyset$. We start with the following preliminary result.

Lemma 3.5 *Suppose that Assumption (A) holds and that $P^0 \neq \emptyset$. Then the following statements hold:*

(a) there exist constants $\gamma_0 \in \mathbf{R}$ and $\gamma_1 > 0$ such that

$$L(y, s) \leq \gamma_0 - \gamma_1 \sum_{j=1}^{p} s_j, \;\; \forall (y, s) \in \mathbf{R}^m \times \mathbf{R}^p_+; \tag{11}$$

(b) for any constant $\gamma \in \mathbf{R}$, the set $\Omega_\gamma \equiv \{(y, s) \in \mathbf{R}^m \times \mathbf{R}^p \mid L(y, s) \geq \gamma, \;\; s \geq 0\}$ is compact (possibly empty).

Proof. We first show (a). Since $P^0 \neq \emptyset$, let x^0 be a fixed point in P^0. Defining $\gamma_0 \equiv f(x^0)$ and $\gamma_1 \equiv \min_{j=1,\ldots,p}\{|g_j(x^0)|\}$ and using the definition of L and the fact that $Ax^0 = b$ and $g(x^0) < 0$, we obtain

$$\begin{aligned} L(y, s) \;\; &\leq \;\; \mathcal{L}(x^0, y, s) \;\; = \;\; f(x^0) + y^T(b - Ax^0) + s^T g(x^0) \\ &= \;\; f(x^0) - s^T |g(x^0)| \;\; \leq \;\; \gamma_0 - \gamma_1 \sum_{j=1}^{p} s_j, \qquad \forall (y, s) \in \mathbf{R}^m \times \mathbf{R}^p_+. \end{aligned}$$

We now show (b). The result is trivial when $\Omega_\gamma = \emptyset$, and hence we assume from now on that $\Omega_\gamma \neq \emptyset$ and that $(\bar{y}, \bar{s})$ is a fixed point in Ω_γ. Since the function $(y, s) \in \mathbf{R}^m \times \mathbf{R}^p \mapsto -L(y, s) \in (-\infty, +\infty]$ is lower semi-continuous and convex, it follows that $\{(y, s) \mid L(y, s) \geq \gamma\}$ is a closed convex set. Hence, Ω_γ is also a closed convex set. To show that Ω_γ is bounded, it is sufficient to prove that $0^+\Omega_\gamma = \{0\}$. Indeed, let $(\Delta y, \Delta s)$ be an arbitrary vector in $0^+\Omega_\gamma$. We will show that $(\Delta y, \Delta s) = 0$. By the definition of $0^+\Omega_\gamma$, we have that $(\bar{y}, \bar{s}) + \lambda(\Delta y, \Delta s) \in \Omega_\gamma$ for all $\lambda \geq 0$. More specifically, we have:

$$\bar{s} + \lambda \Delta s \geq 0, \;\; \forall \lambda \geq 0,$$

and

$$\gamma \leq L(\bar{y} + \lambda \Delta y, \bar{s} + \lambda \Delta s) \leq \gamma_0 - \gamma_1 \sum_{j=1}^{p} (\bar{s}_j + \lambda \Delta s_j), \;\; \forall \lambda \geq 0, \tag{12}$$

where the last inequality is due to (11). Since $\Delta s \geq 0$, relation (12) holds only if $\Delta s = 0$. Next, assume for contradiction that $\Delta y \neq 0$. Then, using the fact that $\text{rank}(A) = m$, it is easy to show the existence of a point $\bar{x} \in \mathbf{R}^n$ such that $\Delta y^T(b - A\bar{x}) < 0$. Using the definition of L, the fact that $\Delta s = 0$ and relation (12), we obtain

$$\begin{aligned} \gamma &\leq L(\bar{y} + \lambda\Delta y, \bar{s} + \lambda\Delta s) \\ &\leq \mathcal{L}(\bar{x}, \bar{y} + \lambda\Delta y, \bar{s} + \lambda\Delta s) \\ &= \mathcal{L}(\bar{x}, \bar{y}, \bar{s}) + \lambda(\Delta y)^T(b - A\bar{x}), \qquad \forall\lambda \geq 0. \end{aligned}$$

But this relation holds only if $\Delta y^T(b - A\bar{x}) \geq 0$, contradicting the fact that $\Delta y^T(b - A\bar{x}) < 0$. Hence, we must have $\Delta y = 0$ and the result follows. ■

The following well known result is used in the proof of the next proposition. Its proof can be found for example in Rockafellar [14], theorem 21.2.

Lemma 3.6 *A necessary and sufficient condition for P^0 to be empty is that there exists a point $(\tilde{y}, \tilde{s}) \in \mathbf{R}^m \times \mathbf{R}^p$ such that $0 \neq \tilde{s} \geq 0$ and*

$$\tilde{y}^T(b - Ax) + \tilde{s}^T g(x) \geq 0, \qquad \forall x \in \mathbf{R}^n. \tag{13}$$

Proposition 3.2 *Suppose that Assumption (A) holds and that $D_L^0 \neq \emptyset$. Let $w \in \mathbf{R}^p_{++}$ be given. Then the following implications hold:*
(a) if $P^0 = \emptyset$ then $(\mathcal{D}_L^w)$ is unbounded;
(b) if $P^0 \neq \emptyset$ then opt$(\mathcal{D}_L^w) \neq \emptyset$.

Proof. We first show implication (a). Assume that $P^0 = \emptyset$ and let $(\bar{y}, \bar{s})$ be a fixed point in D_L^0. Lemma 3.6 and the fact that $P^0 = \emptyset$ imply the existence of a point $(\tilde{y}, \tilde{s}) \in \mathbf{R}^m \times \mathbf{R}^p$ satisfying relation (13) and $0 \neq \tilde{s} \geq 0$. We next show that $(\bar{y}(\lambda), \bar{s}(\lambda)) \equiv (\bar{y}, \bar{s}) + \lambda(\tilde{y}, \tilde{s}) \in D_L^0$ for all $\lambda \geq 0$. Indeed, the fact that $\bar{s} > 0$ and $\tilde{s} \geq 0$ implies that $\bar{s}(\lambda) > 0$ for all $\lambda \geq 0$. Moreover, using relation (13) and the definition of L, we obtain

$$\begin{aligned} L(\bar{y}(\lambda), \bar{s}(\lambda)) &= \inf_x \left\{ f(x) + \bar{y}(\lambda)^T(b - Ax) + \bar{s}(\lambda)^T g(x) \right\} \\ &\geq \inf_x \left\{ f(x) + \bar{y}^T(b - Ax) + \bar{s}^T g(x) \right\} + \lambda \inf_x \left\{ \tilde{y}^T(b - Ax) + \tilde{s}^T g(x) \right\} \\ &= L(\bar{y}, \bar{s}) + \lambda \inf_x \left\{ \tilde{y}^T(b - Ax) + \tilde{s}^T g(x) \right\} \\ &\geq L(\bar{y}, \bar{s}), \qquad \forall\lambda \geq 0. \end{aligned} \tag{14}$$

Hence, $(\bar{y}(\lambda), \bar{s}(\lambda)) \in D_L^0$ for all $\lambda \geq 0$. Moreover, using relation (14) and the fact that $0 \neq \tilde{s} \geq 0$, we can easily verify that $d_w(\bar{y}(\lambda), \bar{s}(\lambda)) \equiv L(\bar{y}(\lambda), \bar{s}(\lambda)) + \sum_{j=1}^p \log w_j \bar{s}_j(\lambda) \to \infty$ as $\lambda \to \infty$. Hence, problem $(\mathcal{D}_L^w)$ is unbounded and implication (a) follows.

We next show implication (b). Assume that $P^0 \neq \emptyset$. First observe that opt$(\mathcal{D}_L^w)$ is exactly the set of minimizers of the problem $\inf\{-d_w(y, s) \mid (y, s) \in \mathbf{R}^m \times \mathbf{R}^p_{++}\}$, where $d_w(\cdot, \cdot)$ is defined in (5). In view of Lemma 3.1, it is sufficient to show that the set

$$\Gamma \equiv \{(y, s) \mid -d_w(y, s) \leq \tau, \ s > 0\}$$

is compact, where $\tau \equiv -d_w(\bar{y}, \bar{s}) < \infty$ and $(\bar{y}, \bar{s})$ is the point considered in the proof of (a). Indeed, since $P^0 \neq \emptyset$ and Assumption (A) holds, it follows that the assumptions of Lemma 3.5 are satisfied.

Hence, by Lemma 3.5(a), there exist constants $\gamma_0 \in \mathbf{R}$ and $\gamma_1 > 0$ such that (11) holds. Thus, if $(y,s) \in \Gamma$ we have,

$$\begin{aligned}
-d_w(\bar{y},\bar{s}) + \gamma_0 &\geq -d_w(y,s) + \gamma_0 \\
&= -L(y,s) + \gamma_0 - \sum_{j=1}^{p} w_j \log s_j \\
&\geq \gamma_1 \sum_{j=1}^{p} s_j - \sum_{j=1}^{p} w_j \log s_j \\
&= \sum_{j=1}^{p} \{\gamma_1 s_j - w_j \log s_j\}.
\end{aligned}$$

Using Lemma 3.2 and the above relation, it is easy to show the existence of a constant $\delta > 0$ such that

$$\delta \mathbf{1} \leq s \leq \delta^{-1}\mathbf{1}, \quad \forall (y,s) \in \Gamma. \tag{15}$$

Relation (15) and the definition of Γ then imply

$$\Gamma \subseteq \{(y,s) \mid s \geq 0,\ L(y,s) \geq -\tau - \sum_{j=1}^{p} w_j \log \delta^{-1}\},$$

which, in view of Lemma 3.5(b), yields the conclusion that Γ is bounded. Also, (15) implies that

$$\Gamma = \{(y,s) \mid -d_w(y,s) \leq \tau,\ s \geq \delta\mathbf{1}\},$$

which in turn implies that Γ is a closed set. Hence, Γ is a compact set and implication (b) follows. ■

As an immediate consequence of the previous result, we obtain the following corollary.

Corollary 3.1 *Suppose that Assumption (A) holds. Then opt($\mathcal{D}_L^w$) $\neq \emptyset$ if and only if $P^0 \neq \emptyset$ and $D_L^0 \neq \emptyset$.*

Note that Proposition 3.2 is a result about the Lagrangean dual. It is natural to ask if a similar result holds with respect to the Wolfe dual, that is, whether the two implications:
(a') if $P^0 = \emptyset$ then $(\mathcal{D}_W^w)$ is unbounded;
(b') if $P^0 \neq \emptyset$ then opt$(\mathcal{D}_W^w) \neq \emptyset$,
hold under Assumption (A) and the assumption that $D_W^0 \neq \emptyset$. It turns out that none of the two implications hold as the following example illustrates.

Example 3.1 Consider the convex set $C = \{(x_1,x_2) \in \mathbf{R}^2 \mid x_2 \geq 1/x_1, x_1 > 0\}$ and the functions $f, g : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x) = -2x_2 + \text{dist}(x,C)$ and $g(x) = x_2^2 + 2x_2 + 1 - \delta$ for all $x \in \mathbf{R}^2$, where δ is a nonnegative constant. Clearly, both $f(\cdot)$ and $g(\cdot)$ are convex functions. It is easy to verify that the dual function L restricted to $\mathbf{R}_+$ is given by

$$L(s) \equiv \inf \mathcal{L}(x,s) = \begin{cases} -\infty & \text{if } s = 0, \\ -\delta s + 2 - s^{-1} & \text{if } 0 < s < 1, \\ -\delta s + s & \text{if } 1 \leq s \leq 3/2, \\ -\delta s + 3 - (9/4)s^{-1} & \text{if } 3/2 \leq s, \end{cases}$$

and that the infimum is achieved only when $0 < s < 1$. Hence, we have $D_L = (0,\infty)$ and $D_W = (0,1)$. If $\delta = 0$, it is easy to verify that $P^0 = \emptyset$ and that problem $(\mathcal{D}_W^w)$ is bounded above with $\mathrm{opt}(\mathcal{D}_W^w) = \emptyset$. This case shows that implication (a') does not hold. Now, if $\delta > 0$ then $P^0 \neq \emptyset$ and, in addition, if $\delta \leq 1$ then problem $(\mathcal{D}_W^w)$ is also bounded above with $\mathrm{opt}(\mathcal{D}_W^w) = \emptyset$. This last case shows that implication (b') does not hold.

We can also ask whether the equivalent version of Corollary 3.1 in terms of the Wolfe dual, that is the one obtained by replacing the subscript $_L$ by $_W$ in its statement, hold. Clearly, Example 3.1 with $\delta > 0$ is a counterexample for the "if" part of this modified version of Corollary 3.1. The following example provides a counterexample for the "only if" part.

Example 3.2 Consider the convex set C as in Example 3.1 and the functions $f, g : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x) = x_1 - x_2$ and $g(x) = |x_2| + \mathrm{dist}(x, C)$ for all $x \in \mathbf{R}^2$. It is easy to verify that the dual function $L(s)$ for every $s \in \mathbf{R}_+$ is given by

$$L(s) \equiv \inf_x \mathcal{L}(x,s) = \begin{cases} -\infty & \text{if } 0 \leq s < 1, \\ 0 & \text{if } 1 \leq s, \end{cases}$$

and that the infimum is achieved and finite only when $s = 1$. Hence, we have $D_L = [1,\infty)$ and $D_W = \{1\}$. Moreover, we can easily verify that $P = \emptyset$. Clearly, we have $\mathrm{opt}(\mathcal{D}_W^w) = \{1\} \neq \emptyset$ but $P^0 = \emptyset$, and hence the "only if" part of Corollary 3.1 does not hold in the context of the Wolfe dual. Note that this example satisfies Assumption (A) since $m = 0$ and Assumption (B) since $P = \emptyset$.

Note that Theorem 3.1 guarantees that implication (b') holds if, in addition to assuming $D_W^0 \neq \emptyset$ and Assumption (A), we further impose Assumption (B). Note also that Example 3.1 with $\delta > 0$ does not satisfy Assumption (B).

We next describe the duality relationship that exists between problems $(\mathcal{P}^w)$ and $(\mathcal{D}_L^w)$.

Lemma 3.7 *If $x \in P^0$ and $(y,s) \in D_L^0$ then*

$$p_w(x) - d_w(y,s) \geq \sum_{j=1}^p w_j(1 - \log w_j). \tag{16}$$

where $p_w(\cdot)$ and $d_w(\cdot,\cdot)$ are defined in (4) and (5). Moreover, equality holds in (16) if and only if $(x,y,s) \in S_w$.

Proof. Let $x \in P^0$ and $(y,s) \in D_L^0$ be given. Using the fact that $Ax = b$, $g(x) < 0$ and $\mathcal{L}(x,y,s) \geq L(y,s)$, we obtain

$$\begin{aligned} p_w(x) - d_w(y,s) &= f(x) - \sum_{j=1}^p w_j \log|g_j(x)| - L(y,s) - \sum_{j=1}^p w_j \log s_j \\ &\geq f(x) - \mathcal{L}(x,y,s) - \sum_{j=1}^p w_j \log(s_j|g_j(x)|) \\ &= y^T(Ax-b) - s^T g(x) - \sum_{j=1}^p w_j \log(s_j|g_j(x)|) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{p} (s_j|g_j(x)| - w_j \log s_j|g_j(x)|) \\
&\geq \sum_{j=1}^{p} w_j(1 - \log w_j),
\end{aligned} \tag{17}
$$

where the last inequality is due to Lemma 3.2(b). This shows (16). Using Lemma 3.2(b) again and expression (17), it is easy to see that equality holds in (16) if and only if

$$
\mathcal{L}(x, y, s) = L(y, s), \qquad -s \circ g(x) = w. \tag{18}
$$

By definition of S_w, we immediately conclude that (18) holds if and only if $(x, y, s) \in S_w$. ■

Lemma 3.8 *Let $w \in \mathbf{R}^p_{++}$ be given and assume that $S_w \neq \emptyset$. Then, $(\bar{x}, \bar{y}, \bar{s}) \in S_w$ if and only if $\bar{x} \in opt(\mathcal{P}^w)$ and $(\bar{y}, \bar{s}) \in opt(\mathcal{D}^w_L)$, in which case $(\bar{y}, \bar{s}) \in opt(\mathcal{D}^w_W)$.*

Proof. We first show the "only if" part of the equivalence and the the fact that $(\bar{x}, \bar{y}, \bar{s}) \in S_w$ implies $(\bar{y}, \bar{s}) \in opt(\mathcal{D}^w_W)$. Indeed, assume that $(\bar{x}, \bar{y}, \bar{s}) \in S_w$. By the definition of S_w, we have $\bar{x} \in P^0$ and $(\bar{y}, \bar{s}) \in D^0_W \subseteq D^0_L$, and by the "if and only if" statement of Lemma 3.7, we conclude that

$$
p_w(\bar{x}) - d_w(\bar{y}, \bar{s}) = \sum_{j=1}^{p} w_j(1 - \log w_j).
$$

This relation together with relation (16) of Lemma 3.7 implies

$$
p_w(x) - d_w(y, s) \geq p_w(\bar{x}) - d_w(\bar{y}, \bar{s}), \ \forall x \in P^0, \ \forall (y, s) \in D^0_L. \tag{19}
$$

Fixing $(y, s) = (\bar{y}, \bar{s})$ in (19), we obtain the conclusion that $\bar{x} \in \mathrm{opt}(\mathcal{P}^w)$. Similarly, fixing $x = \bar{x}$ in (19), we obtain that $(\bar{y}, \bar{s}) \in \mathrm{opt}(\mathcal{D}^w_L)$. Since $(\bar{y}, \bar{s}) \in D^0_W$, this also implies that $(\bar{y}, \bar{s}) \in \mathrm{opt}(\mathcal{D}^w_W)$.

We next show the "if" part of the equivalence. Assume that $\bar{x} \in \mathrm{opt}(\mathcal{P}^w)$ and $(\bar{y}, \bar{s}) \in \mathrm{opt}(\mathcal{D}^w_L)$. Then it is easy to see that inequality (19) holds. By assumption, $S_w \neq \emptyset$ and so let (x^0, y^0, s^0) be a fixed point in S_w. As in the proof of the "only if" part, we have

$$
p_w(x^0) - d_w(y^0, s^0) = \sum_{j=1}^{p} w_j(1 - \log w_j), \tag{20}
$$

Combining inequality (19) with $x = x^0$ and $(y, s) = (y^0, s^0)$ and relation (20), we obtain

$$
\sum_{j=1}^{p} w_j(1 - \log w_j) \geq p_w(\bar{x}) - d_w(\bar{y}, \bar{s}). \tag{21}
$$

By inequality (16), we conclude that (21) must hold as equality. Hence, by the "if and only if" statement of Lemma 3.7, it follows that $(\bar{x}, \bar{y}, \bar{s}) \in S_w$. ■

As an immediate consequence of the above two lemmas, we obtain the following result.

Proposition 3.3 *Let $w \in \mathbf{R}^p_{++}$ be given and assume that $S_w \neq \emptyset$. Then the following statements hold:*

(a) *$opt(\mathcal{D}_L^w) = opt(\mathcal{D}_W^w)$ and the set $\{s \mid (y,s) \in opt(\mathcal{D}_L^w)\}$ is a singleton; if in addition Assumption (A) holds and the functions $f(\cdot)$ and $g(\cdot)$ are differentiable then $opt(\mathcal{D}_L^w)$ is a singleton;*

(b) *$S_w = \{(x,y,s) \mid x \in opt(\mathcal{P}^w),\ (y,s) \in opt(\mathcal{D}_L^w)\}$;*

(c) *for any fixed $(\bar{y}, \bar{s}) \in opt(\mathcal{D}_L^w)$,*

$$opt(\mathcal{P}^w) = \{x \mid x \in P,\ |g(x)| \circ \bar{s} = w\} \cap \text{Argmin}\{\mathcal{L}(x, \bar{y}, \bar{s}) \mid x \in \mathbf{R}^n\};$$

(d) *$val(\mathcal{P}^w) - val(\mathcal{D}_L^w) = \sum_{j=1}^p w_j(1 - \log w_j)$ and $val(\mathcal{D}_L^w) = val(\mathcal{D}_W^w)$.*

Proof. The assertion that $opt(\mathcal{D}_L^w) = opt(\mathcal{D}_W^w)$ follows from Lemma 3.8. Using the fact that the objective function $d_w(y,s)$ of problem $(\mathcal{D}_L^w)$ is strictly concave with respect to s, we can easily see that the set $\{s \mid (y,s) \in opt(\mathcal{D}_L^w)\}$ is a singleton. Assume now that Assumption (A) holds and the functions $f(\cdot)$ and $g(\cdot)$ are differentiable. Fix a point $\bar{x} \in opt(\mathcal{P}^w)$. (This point exists since $S_w \neq \emptyset$.) By Lemma 3.8, we know that $(y,s) \in opt(\mathcal{D}_L^w)$ if and only if $\nabla f(\bar{x}) - A^T y + \nabla g(\bar{x}) s = 0$ and $|g(\bar{x})| \circ s = w$. Using the fact that $rank(A) = m$, we can easily see that there exists a unique point (y,s) satisfying these two last equations. We have thus shown that $opt(\mathcal{D}_L^w)$ is a singleton. Statements (b), (c) and (d) follow immediately from (a), Lemma 3.7 and Lemma 3.8. ∎

It is worth mentioning that Lemmas 3.7, 3.8 and Proposition 3.3 hold even if we do not assume that the functions $f(\cdot)$ and $g_j(\cdot)$, $j = 1, \ldots, p$, are convex.

We are now in a position to give the proof of Theorem 3.1. The proof has already been given in the several results stated above and all we have to do is to put the pieces together.

Proof of Theorem 3.1: We will show that the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a)$, the implication $(d) \Rightarrow$ [(1), (2), (3) and (4)] and the equivalence $(b) \Leftrightarrow (f)$ hold, from which the result follows.
[$(a) \Rightarrow (b)$] This is obvious since $D_W^0 \subseteq D_L^0$.
[$(b) \Rightarrow (c)$] This follows from Proposition 3.1.
[$(c) \Rightarrow (d)$] Let $\bar{x} \in opt(\mathcal{P}^w)$. The KKT conditions for $\mathcal{P}^w$ imply the existence of $y \in \mathbf{R}^m$ such that $0 \in \partial f(\bar{x}) + \sum_{j=1}^p (w_j / |g_j(\bar{x})|) - A^T y$. Setting $s_j = w_j / |g_j(\bar{x})|$ for $j = 1, \ldots, p$, we have $(\bar{x}, y, s) \in S_w$.
[$(d) \Rightarrow (e)$] This is obvious since $S_w \subseteq PD^0$.
[$(e) \Rightarrow (a)$] This is obvious since $PD^0 \subseteq P^0 \times D_W^0$.
[$(d) \Rightarrow$ (1), (2), (3) and (4)] This follows from Proposition 3.3.
[$(b) \Leftrightarrow (f)$] This follows from Corollary 3.1. ∎

Theorem 3.1 is not completely symmetrical in the sense that the condition $\text{opt}(\mathcal{D}_W^w) \neq \emptyset$ is not equivalent to conditions (a), (b), (c), (d), (e) and (f) (see Example 3.2). However, if (i) the objective function and the constraints functions are analytic, or (ii) the constraints functions $g_j(\cdot)$, $j = 1, \ldots, p$, are affine functions, then the next result shows that $\text{opt}(\mathcal{D}_W^w) \neq \emptyset$ is equivalent to conditions (a), (b), (c), (d), (e) and (f) of Theorem 3.1.

Theorem 3.2 *Suppose that Assumption (A) and Assumption (B) hold and assume that either one of the following conditions hold:*
(a) the constraints functions $g_j(\cdot)$, $j = 1, \ldots, p$, are affine, or;
(b) the functions $f(\cdot)$ and $g_j(\cdot)$, $j = 1, \ldots, p$, are analytic.
Then, the condition $opt(\mathcal{D}_W^w) \neq \emptyset$ is equivalent to any one of the conditons (a), (b), (c), (d), (e) and (f) of Theorem 3.1.

The proof of Theorem 3.2 will be given at the end of this section after we state and prove some preliminary results. The main property that we use about an analytic convex function is that it satisfies the following flatness condition.

Definition 3.1 (Flatness condition) A function $h : \mathbf{R}^n \to \mathbf{R}$ is said to be *flat* if given any points $x, y \in \mathbf{R}^n$ such that $x \neq y$ the following implication holds: if $h(x)$ is constant on the segment $[x, y] \equiv \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}$ then $h(x)$ is constant on the entire line containing $[x, y]$.

Lemma 3.9 *Assume that* $h : \mathbf{R}^n \to \mathbf{R}$ *is a flat convex function such that* $\inf\{h(x) \mid x \in \mathbf{R}^n\}$ *is finite but it is not achieved. Then, there exists a direction* $d \in \mathbf{R}^n$ *such that the function* $\lambda \in \mathbf{R} \mapsto h(x + \lambda d) \in \mathbf{R}$ *is (strictly) decreasing for every* $x \in \mathbf{R}^n$.

Proof. Since $\inf\{h(x) \mid x \in \mathbf{R}^n\}$ is finite, we have $h0^+(d) \geq 0$ for every $d \in \mathbf{R}^n$. Moreover, since this infimum is not achieved, it follows from Theorem 27.1(d) of Rockafellar [14] that the set $\mathcal{R} = \{d \in \mathbf{R}^n \mid h0^+(d) = 0\}$ is nonempty ($\mathcal{R}$ is the set of all directions of recession of h). We will show that some $d \in \mathcal{R}$ satisfies the conclusion of the lemma. Indeed, assume for contradiction that for every $d \in \mathcal{R}$, there exists $x_d \in \mathbf{R}^n$ such that $\lambda \mapsto h(x_d + \lambda d)$ is not a strictly decreasing function. For the remaining of the proof, let d be an arbitrary direction in $\mathcal{R}$. By Theorem 8.6 of Rockafellar [14], it follows that $\lambda \mapsto h(x_d + \lambda d)$ is a non-increasing function. Hence, there exists a closed interval $[\lambda^-, \lambda^+]$ of positive length such that $\lambda \mapsto h(x_d + \lambda d)$ is constant on $[\lambda^-, \lambda^+]$. Since h is a flat function, it follows that $h(x_d + \lambda d)$ is constant on the whole line $\mathbf{R}$. By Corollary 8.6.1 of Rockafellar [14], it follows that $\lambda \mapsto h(x + \lambda d)$ is a constant function for every $x \in \mathbf{R}^n$. Since $d \in \mathcal{R}$ is arbitrary, we have thus shown that every direction of recession of $h(\cdot)$ is a direction in which $h(\cdot)$ is constant. By Theorem 27.1(b) of Rockafellar [14], it follows that $\inf\{h(x) \mid x \in \mathbf{R}^n\}$ is achieved contradicting the assumptions of the lemma. Hence, the conclusion of the lemma follows. ■

As a consequence of the previous lemma, we obtain the following result.

Lemma 3.10 *Assume that* $h, k : \mathbf{R}^n \to \mathbf{R}$ *are analytic convex functions satisfying the following properties:*

(a) $\inf\{h(x) \mid x \in \mathbf{R}^n\}$ *is finite and achieved;*

(b) $\inf\{k(x) \mid x \in \mathbf{R}^n\}$ *is finite, and;*

(c) $\inf\{h(x) + k(x) \mid x \in \mathbf{R}^n\}$ *is not achieved.*

Then the function $h - \theta k$ *is not convex for any* $\theta > 0$.

Proof. Since $\inf\{h(x) \mid x \in \mathbf{R}^n\}$ and $\inf\{k(x) \mid x \in \mathbf{R}^n\}$ are finite, we have $h0^+(d) \geq 0$ and $k0^+(d) \geq 0$ for every $d \in \mathbf{R}^n$. By Theorem 9.3 of Rockafellar [14], we have $(h+k)0^+(d) = h0^+(d) + k0^+(d)$ for every $d \in \mathbf{R}^n$. Hence, we have

$$\Gamma \equiv \{d \in \mathbf{R}^n \mid (h+k)0^+(d) = 0\} = \{d \in \mathbf{R}^n \mid h0^+(d) = 0,\ k0^+(d) = 0\}.$$

By Lemma 3.9, there exists $\bar{d} \in \Gamma$ such that $\lambda \mapsto (h+k)(x + \lambda \bar{d})$ is a decreasing function for every $x \in \mathbf{R}^n$. By (a), there exists $\bar{x} \in \mathbf{R}^n$ such that $h(\bar{x}) \leq h(x)$ for all $x \in \mathbf{R}^n$. Since $h0^+(\bar{d}) = 0$, it follows from Theorem 8.6 of Rockafellar [14] that $\lambda \mapsto h(\bar{x} + \lambda \bar{d})$ is a non-increasing function. Hence, it follows that $h(\bar{x} + \lambda \bar{d}) = h(\bar{x})$ for every $\lambda \geq 0$. Therefore, using the fact that $\lambda \mapsto (h+k)(\bar{x} + \lambda \bar{d})$ is a decreasing function, we conclude that $\lambda \in [0, \infty) \mapsto k(\bar{x} + \lambda \bar{d})$ is a decreasing function, and hence, together with (b), a strictly convex function. This implies that $\lambda \in [0, \infty) \mapsto (h - \theta k)(\bar{x} + \lambda \bar{d})$

is a strictly concave function. We have thus shown that the function $h - \theta k$ is not convex for any $\theta > 0$. ■

Theorem 3.2 is an immediate consequence of the following proposition.

Proposition 3.4 *Assume that $D^0_W \neq \emptyset$ and that either one of the following conditions hold:*
(a) the constraints functions $g_j(\cdot)$, $j = 1, \ldots, p$, are affine, or;
(b) the functions $f(\cdot)$ and $g_j(\cdot)$, $j = 1, \ldots, p$, are analytic.
Then, $P^0 = \emptyset$ implies that problem $(\mathcal{D}^w_W)$ is unbounded.

Proof. Let $(\bar{y}, \bar{s})$ be a fixed point in D^0_W. By Lemma 3.6 and the fact that $P^0 = \emptyset$, there exists $(\tilde{y}, \tilde{s}) \in \mathbf{R}^m \times \mathbf{R}^p$ such that $0 \neq \tilde{s} \geq 0$ and relation (13) is satisfied. As in the proof of Proposition 3.2, it follows that $(\bar{y}(\lambda), \bar{s}(\lambda)) \equiv (\bar{y}, \bar{s}) + \lambda(\tilde{y}, \tilde{s}) \in D^0_L$ for all $\lambda \geq 0$ and that $d_w(\bar{y}(\lambda), \bar{s}(\lambda)) \equiv L(\bar{y}(\lambda), \bar{s}(\lambda)) + \sum_{j=1}^p \log \bar{s}_j(\lambda) \to \infty$ as $\lambda \to \infty$. We will next show that $(\bar{y}(\lambda), \bar{s}(\lambda)) \in D^0_W$ for all $\lambda \geq 0$, which together with the above observations imply that $(\mathcal{D}^w_W)$ is unbounded.

We first assume that condition (a) holds. In this case, it follows that the left hand side of (13) is an affine function which is nonnegative. Then $x \in \mathbf{R}^n \mapsto \tilde{y}^T(b - Ax) + \tilde{s}^T g(x)$ is a constant function and hence, for every $\lambda > 0$, the function $x \mapsto \mathcal{L}(x, \bar{y}(\lambda), \bar{s}(\lambda))$ differs from the function $x \mapsto \mathcal{L}(x, \bar{y}, \bar{s})$ by a constant. Since $(\bar{y}, \bar{s}) \in D^0_W$, this implies that, for every $\lambda > 0$, $\inf\{\mathcal{L}(x, \bar{y}(\lambda), \bar{s}(\lambda)) \mid x \in \mathbf{R}^n\}$ is achieved, or equivalently, that $(\bar{y}(\lambda), \bar{s}(\lambda)) \in D^0_W$.

We now assume that condition (b) holds. Assume for contradiction that $(\bar{y}(\bar{\lambda}), \bar{s}(\bar{\lambda})) \notin D^0_W$ for some $\bar{\lambda} > 0$. Let $h, k : \mathbf{R}^n \to \mathbf{R}$ denote the functions defined by $h(x) = \mathcal{L}(x, \bar{y}, \bar{s})$ and $k(x) = \bar{\lambda}[\tilde{y}^T(b - Ax) + \tilde{s}^T g(x)]$ for every $x \in \mathbf{R}^n$. It is easy to see that h and k satisfy all the assumptions of Lemma 3.10. Hence, from the conclusion of this lemma it follows that the function $h - \theta k$ is not convex for any $\theta > 0$. But since $h - \theta k = \mathcal{L}(\cdot, \bar{y} - \theta\bar{\lambda}\tilde{y}, \bar{s} - \theta\bar{\lambda}\tilde{s})$ and $\bar{s} - \theta\bar{\lambda}\tilde{s} \geq 0$ for any $\theta > 0$ sufficiently small, we obtain that $h - \theta k$ is convex for any $\theta > 0$ sufficiently small, contradicting the above conclusion. Hence, it follows that $(\bar{y}(\lambda), \bar{s}(\lambda)) \in D^0_W$ for all $\lambda \geq 0$.

■

4 Other Existence Conditions for the Central Path

In this section, we derive other conditions which are equivalent to the conditions of Theorem 3.1 and/or Theorem 3.2. The conditions discussed in this section impose boundedness on the optimal solution set of $(\mathcal{P})$ and/or its dual (Lagrangean or Wolfe) problem. The main result of this section is Theorem 4.1.

The first result essentially says that boundedness of the set $opt(\mathcal{P})$ is equivalent to the existence of an interior feasible solution for the (Lagrangean or Wolfe) dual problem.

Proposition 4.1 *Suppose that Assumption (B) holds. Then, the following statements are equivalent:*
(a) $opt(\mathcal{P})$ is nonempty and bounded;
(b) $P \neq \emptyset$ and $D^0_W \neq \emptyset$;
(c) $P \neq \emptyset$ and $D^0_L \neq \emptyset$.

Proof. We first show the implication $(a) \Rightarrow (b)$. Assume that $\mathrm{opt}(\mathcal{P})$ is nonempty and bounded. Fix a point $x^0 \in P \neq \emptyset$ and let $\epsilon > 0$ be such that $f(x^0) < \epsilon^{-1}$. Consider the following convex program

$$
\begin{array}{ll}
\inf & f(x) - \sum_{j=1}^{p} w_j \log(\epsilon - g_j(x)) \\
\text{s.t.} & f(x) \leq \epsilon^{-1}, \\
& Ax = b, \\
& g(x) \leq (\epsilon/2)\mathbf{1},
\end{array}
\tag{22}
$$

Observe that x^0 is a feasible point of (22) and that all the inequality constraints of (22) are strictly satisfied by x^0. Moreover, using the fact that *opt*($\mathcal{P}$) is nonempty and bounded and Lemma 2.1, we can easily see that the feasible region of (22) is compact. Hence, the problem has a minimizer $\bar{x}$ which satisfy the KKT conditions:

$$
0 \in \partial f(\bar{x}) + \sum_{j=1}^{p} \left(\frac{w_j}{\epsilon - g_j(\bar{x})} \right) \partial g_j(\bar{x}) + \bar{\lambda} \partial f(\bar{x}) - A^T \bar{y} + \partial g(\bar{x}) \bar{s}, \tag{23}
$$

$$
\bar{\lambda} \geq 0, \qquad \bar{s} \geq 0, \qquad \bar{\lambda}[\epsilon^{-1} - f(\bar{x})] = 0, \qquad \bar{s}^T[(\epsilon/2)\mathbf{1} - g(\bar{x})] = 0. \tag{24}
$$

Rearranging (23), we obtain the condition that $0 \in \partial \mathcal{L}(\bar{x}, \hat{y}, \hat{s})$, where

$$
\hat{y} \equiv \frac{1}{1+\bar{\lambda}} \bar{y}, \qquad \hat{s}_j \equiv (1+\bar{\lambda})^{-1} \left(\frac{w_j}{\epsilon - g_j(\bar{x})} + \bar{s}_j \right) > 0, \qquad \forall j = 1, \ldots, p.
$$

Hence, $(\hat{y}, \hat{s}) \in D_W^0$ and therefore $D_W^0 \neq \emptyset$.

The implication $(b) \Rightarrow (c)$ is straightforward since $D_W^0 \subseteq D_L^0$.

We now prove the implication $(c) \Rightarrow (a)$. Assume that $P \neq \emptyset$ and $D_L^0 \neq \emptyset$. Take a point $x^0 \in P$. In view of Lemma 3.1, the conclusion that opt($\mathcal{P}$) is nonempty and bounded will follow if we show that the set $\Omega \equiv \{x \in P \mid f(x) \leq f(x^0)\}$ is compact. This set is clearly closed since P is closed and $f(x)$ is continuous. It remains to show that Ω is bounded. Indeed, since $D_L^0 \neq \emptyset$, it follows from Lemma 3.3 that there exist constants $\tau_0 \in \mathbf{R}$ and $\tau_1 > 0$ such that relation (8) holds. This implies that

$$
\Omega = \{x \in P \mid f(x) \leq f(x^0),\ g_j(x) \geq -(f(x^0) - \tau_0)/\tau_1,\ \forall j = 1, \ldots, p\}.
$$

By Assumption (B), the set in the right hand side of the above expression is bounded. Hence, Ω is bounded and the result follows. ■

The proof of the implication (a) $\Rightarrow$ (b) is based on ideas used in Lemma 13 of Monteiro and Pang [12]. Note that the implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) hold regardless of the validity of Assumption (B). On the other hand, Assumption (B) is needed to guarantee the reverse implications. Indeed, consider Example 3.1 with $\delta > 0$. Clearly, it does not satisfy Assumption (B), $P^0 \neq \emptyset$, $D_L^0 \neq \emptyset$ and $D_W^0 \neq \emptyset$. But it is easy to see that opt($\mathcal{P}$) $= \emptyset$ if $\delta \leq 1$ and that opt($\mathcal{P}$) is nonempty and unbounded if $\delta > 1$.

We next turn our efforts to show that, under certain mild assumptions, the existence of an interior feasible solution for problem ($\mathcal{P}$) is essentially equivalent to boundedness of the set of optimal solutions of the (Lagrangean or Wolfe) dual problem. With this goal in mind, it is useful to recall the notion of a Kuhn-Tucker vector as defined in Rockafellar [14], pages 274-275. A point $(y, s) \in \mathbf{R}^m \times \mathbf{R}^p_+$ is called a *Kuhn-Tucker vector* for ($\mathcal{P}$) if $L(y, s) = \text{val}(\mathcal{P}) \in \mathbf{R}$. We denote the set of all Kuhn-Tucker vectors for ($\mathcal{P}$) by $\mathcal{KT}$.

By Theorem 28.1 and Theorem 28.3 of Rockafellar [14], we have that a necessary and sufficient condition for $x \in \mathrm{opt}(\mathcal{P})$ and $(y,s) \in \mathcal{KT}$ is that the following relations hold:

$$x \in P, \quad s \geq 0, \quad s^T g(x) = 0 \text{ and } \mathcal{L}(x,y,s) = L(y,s). \tag{25}$$

(The last relation in (25) is also equivalent to $0 \in \partial f(x) - A^T y + s_1 \partial g_1(x) + \ldots + s_p \partial g_p(x)$.) Hence, when $\mathrm{opt}(\mathcal{P}) \neq \emptyset$, we have $\mathcal{KT} \subset D_W$.

The next two results give the relationship between the set $\mathcal{KT}$ and the sets $\mathrm{opt}(\mathcal{D}_L)$ and $\mathrm{opt}(\mathcal{D}_W)$.

Lemma 4.1 *$\mathcal{KT} \neq \emptyset$ if and only if $opt(\mathcal{D}_L) \neq \emptyset$ and $val(\mathcal{P}) = val(\mathcal{D}_L)$, in which case $\mathcal{KT} = opt(\mathcal{D}_L)$.*

Proof. The proof of this lemma follows straightforwardly from the definition of $\mathcal{KT}$ and from the weak duality result, namely: $L(y,s) \leq f(x)$ for every $x \in P$ and $(y,s) \in \mathbf{R}^m \times \mathbf{R}^p_+$. ∎

Lemma 4.2 *Assume that* $\mathrm{opt}(\mathcal{P}) \neq \emptyset$*. Then, $\mathcal{KT} \neq \emptyset$ if and only if* $\mathrm{opt}(\mathcal{D}_W) \neq \emptyset$ *and* $\mathrm{val}(\mathcal{P}) = \mathrm{val}(\mathcal{D}_W)$*, in which case $\mathcal{KT} = opt(\mathcal{D}_W)$.*

Proof. Assume that $\mathrm{opt}(\mathcal{P}) \neq \emptyset$. First we observe that $\mathcal{KT} \subseteq \mathrm{opt}(\mathcal{D}_W)$. This inclusion follows from the definition of $\mathcal{KT}$, the weak duality result and the fact that $\mathcal{KT} \subseteq D_W$ which holds under the assumption that $\mathrm{opt}(\mathcal{P}) \neq \emptyset$ (see the observation preceding Lemma 4.1). Under the assumption that $opt(\mathcal{D}_W) \neq \emptyset$ and $val(\mathcal{P}) = val(\mathcal{D}_W)$, the reverse inclusion $\mathcal{KT} \supseteq \mathrm{opt}(\mathcal{D}_W)$ is immediate. ∎

Lemma 4.3 *Suppose that Assumption (A) holds. Then, $P^0 \neq \emptyset$ and $val(\mathcal{P}) > -\infty$ imply that $\mathcal{KT}$ is nonempty and bounded.*

Proof. Assume that $P^0 \neq \emptyset$ and $\mathrm{val}(\mathcal{P}) > -\infty$. Then Theorem 28.2 of Rockafellar [14] implies that $\mathcal{KT} \neq \emptyset$. The boundedness of $\mathcal{KT}$ follows from Lemma 3.5 and the fact that

$$\mathcal{KT} = \{(y,s) \in \mathbf{R}^m \times \mathbf{R}^p_+ \mid L(y,s) \geq \mathrm{val}(\mathcal{P})\}.$$

∎

Lemma 4.4 *If $opt(\mathcal{D}_L)$ is nonempty and bounded then $P^0 \neq \emptyset$.*

Proof. Assume that $\mathrm{opt}(\mathcal{D}_L)$ is nonempty and bounded and let $(\bar{y},\bar{s})$ be a fixed point in $\mathrm{opt}(\mathcal{D}_L)$. Assume for contradiction that $P^0 = \emptyset$. Lemma 3.6 then implies the existence of a point $(\tilde{y},\tilde{s}) \in \mathbf{R}^m \times \mathbf{R}^p$ satisfying relation (13) and $0 \neq \tilde{s} \geq 0$. We next show that $(\bar{y}(\lambda),\bar{s}(\lambda)) \equiv (\bar{y},\bar{s}) + \lambda(\tilde{y},\tilde{s}) \in \mathrm{opt}(\mathcal{D}_L)$ for all $\lambda \geq 0$, a fact that contradicts the boundedness of $\mathrm{opt}(\mathcal{D}_L)$. Indeed, the fact that $\bar{s} \geq 0$ and $\tilde{s} \geq 0$ implies that $\bar{s}(\lambda) \geq 0$ for all $\lambda \geq 0$. Moreover, using (13) and (14) we obtain,

$$L(\bar{y}(\lambda),\bar{s}(\lambda)) \geq L(\bar{y},\bar{s}), \qquad \forall \lambda \geq 0. \tag{26}$$

Since $(\bar{y},\bar{s}) \in \mathrm{opt}(\mathcal{D}_L)$, relation (26) clearly implies that $(\bar{y}(\lambda),\bar{s}(\lambda)) \in \mathrm{opt}(\mathcal{D}_L)$ for all $\lambda \geq 0$. ∎

As a consequence of the lemmas stated above we have the following result.

Proposition 4.2 *Assume that Assumption (A) holds. If val*$(\mathcal{P}) > -\infty$ *then the following statements are equivalent:*

(a) $P^0 \neq \emptyset$*;*

(b) $\mathcal{KT}$ *is nonempty and bounded;*

(c) opt$(\mathcal{D}_L)$ *is nonempty and bounded,*

in which case $\mathcal{KT} =$ *opt*$(\mathcal{D}_L)$ *and val*$(\mathcal{P}) =$ *val*$(\mathcal{D}_L)$*. If instead the stronger condition that opt*$(\mathcal{P}) \neq \emptyset$ *is assumed then (a), (b) and (c) above are also equivalent to the following statement:*

(d) opt$(\mathcal{D}_W)$ *is nonempty and bounded and val*$(\mathcal{P}) =$ *val*$(\mathcal{D}_W)$*,*

in which case $\mathcal{KT} =$ *opt*$(\mathcal{D}_W)$*.*

Proof. Assume that val$(\mathcal{P}) > -\infty$. The implication $(a) \Rightarrow (b)$ follows from Lemma 4.3. By Lemma 4.1, we conclude that (b) implies (c) and the fact that $\mathcal{KT} = \text{opt}(\mathcal{D}_L)$ and $\text{val}(\mathcal{P}) = \text{val}(\mathcal{D}_L)$. Lemma 4.4 yields the implication $(c) \Rightarrow (a)$. This shows the first part of the proposition. Assume now that $\text{opt}(\mathcal{P}) \neq \emptyset$. In this case, Lemma 4.2 yields the equivalence $(b) \Leftrightarrow (d)$ and that, in this case, $\mathcal{KT} = \text{opt}(\mathcal{D}_W)$. ■

A natural question to ask is whether the condition that $\text{val}(\mathcal{P}) = \text{val}(\mathcal{D}_W)$ can be omitted from statement (d) of Proposition 4.2. The following example shows that this condition can not be omitted.

Example 4.1 Consider the functions $f, g : \mathbf{R}^2 \to \mathbf{R}$ defined for every $x = (x_1, x_2) \in \mathbf{R}^2$ by

$$f(x) = \begin{cases} -1 & \text{if } x_1 \le -1; \\ x_1 & \text{if } x_1 \ge -1 \end{cases}$$

and $g(x) = \|x\| - x_2$, where $\| \cdot \|$ denotes the two-norm. It is easy to verify that the dual function $L(s) = \inf_x f(x) + sg(x)$ is given by $L(s) = -1$ for every $s \in \mathbf{R}_+$ and that the infimum is achieved only for $s = 0$. Hence, we have $D_L = [0, \infty)$ and $D_W = \{0\}$. Moreover, we can easily verify that $P^0 = \emptyset$, $\text{opt}(\mathcal{P}) = \{(0, x_2) \mid x_2 \ge 0\}$ and $\text{opt}(\mathcal{D}_W) = \{0\}$. Note that $\text{val}(\mathcal{P}) = 0$ and $\text{val}(\mathcal{D}_W) = -1$. Note also that this example satisfies Assumption (A) since $m = 0$.

Before stating the next result, we note that the equivalence of statements (a) and (b) of Proposition 4.2 is well known under the assumption *opt*$(\mathcal{P}) \neq \emptyset$ (see for example Hiriart–Urruty and Lemaréchal [6], theorem 2.3.2, chapter VII).

Under the stronger assumption that opt$(\mathcal{P})$ is nonempty and bounded, the next result shows that the condition $\text{val}(\mathcal{P}) = \text{val}(\mathcal{D}_W)$ is not needed in statement (d) of Proposition 4.2.

Proposition 4.3 *Assume that Assumption (A) holds and that opt*$(\mathcal{P})$ *is nonempty and bounded. Then any of the statements (a), (b) and (c) of Proposition 4.2 is equivalent to the condition that opt*$(\mathcal{D}_W)$ *is nonempty and bounded. In this case, we have* $\mathcal{KT} =$ *opt*$(\mathcal{D}_L) =$ *opt*$(\mathcal{D}_W)$ *and val*$(\mathcal{P}) =$ *val*$(\mathcal{D}_L) =$ *val*$(\mathcal{D}_W)$*.*

The proof of Proposition 4.3 will be given below after we state a preliminary result. Consider the following perturbed problem

$$(\mathcal{P}(d)) \qquad v(d) \equiv \inf\{f(x) \mid Ax = b,\, g(x) \le d\},$$

where $d \in \mathbf{R}^p$ is a given perturbation vector. It is well known that the function $v(\cdot)$ is convex; $v(\cdot)$ is usually referred to as the perturbation function associated with problem $(\mathcal{P})$. The following result due to Geoffrion (see [3], Theorem 8) is needed in the proof of Proposition 4.3.

Lemma 4.5 (Geoffrion) *Assume that opt($\mathcal{P}$) is nonempty and bounded. Then $v(\cdot)$ is a lower semi-continuous function at $d = 0$.*

Note that the dual function $L_d : \mathbf{R}^m \times \mathbf{R}^p \to [-\infty, \infty)$ associated with problem $(\mathcal{P}(d))$ is given by

$$L_d(y, s) = L(y, s) - d^T s, \quad \forall (y, s) \in \mathbf{R}^m \times \mathbf{R}^p. \tag{27}$$

Hence, it follows that, for every $d \in \mathbf{R}^p$, D_L and D_W are the sets of feasible solutions of the Lagrangean dual and the Wolfe dual associated with problem $(\mathcal{P}(d))$, respectively. We are now in a position to give the proof of Proposition 4.3. The arguments used in the proof are based on the proof of Theorem 7 of Geoffrion [3].

Proof of Proposition 4.3: Assume that Assumption (A) holds and opt($\mathcal{P}$) is nonempty and bounded. In view of Proposition (4.2), it remains to show that $val(\mathcal{P}) = val(\mathcal{D}_W)$ holds when opt($\mathcal{D}_W$) is nonempty and bounded. Indeed, let $\{d^k\}$ be a sequence of strictly positive vectors converging to 0. Clearly, the set of interior feasible solutions of $(\mathcal{P}(d^k))$ is nonempty. Moreover, using the fact that opt($\mathcal{P}$) is nonempty and bounded, we can show that opt($\mathcal{P}(d^k)$) is also nonempty and bounded by Lemma 2.1. Hence, in view of the equivalence of statements (a) and (d) of Proposition (4.2), there exists a sequence $\{(y^k, s^k)\} \subseteq D_W$ such that

$$v(d^k) = L_d(y^k, s^k), \quad \forall k.$$

Using this relation, relation (27), the weak duality result and the fact that d^k, $s^k \geq 0$, we obtain

$$v(0) \geq L(y^k, s^k) = L_d(y^k, s^k) + (d^k)^T s^k \geq v(d^k), \quad \forall k. \tag{28}$$

Since $v(.)$ is lower semi-continuous at $d = 0$ and $v(0) \geq v(d^k)$ for all k, it follows that $\lim_{k\to\infty} v(d^k) = v(0)$. Hence, relation (28) implies that $\lim_{k\to\infty} L(y^k, s^k) = v(0)$. Clearly, this shows that $val(\mathcal{P}) = val(\mathcal{D}_W)$. ∎

We end this section by giving other conditions which are equivalent to conditions (a), (b), (c), (d), (e) and (f) of Theorem 3.1 and the condition of Theorem 3.2. The main result given below is a consequence of the results already stated in this section. Consider the conditions:

(1) $P^0 \neq \emptyset$;
(2) opt($\mathcal{D}_L$) is nonempty and bounded;
(3) opt($\mathcal{D}_W$) is nonempty and bounded,

and the conditions:

(a) $D_L^0 \neq \emptyset$;
(b) $D_W^0 \neq \emptyset$;
(c) opt($\mathcal{P}$) is nonempty and bounded.

By combining one condition from conditions (1), (2) and (3) with one condition from conditions (a), (b) and (c), we obtain a total of nine conditions which we refer to as (1a), (1b), (1c), (2a), (2b), (2c), (3a), (3b) and (3c). The following result gives the relationship between these nine conditions.

Theorem 4.1 *Suppose that both Assumption (A) and Assumption (B) hold. Then, conditions (1a), (1b), (1c), (2a), (2b), (2c) and (3c) are all equivalent. Moreover, any of these conditions implies (3b), which in turn implies (3a). In addition, if all the constraints functions $g_j(\cdot)$, $j = 1, \ldots, p$, are affine then the nine conditions are equivalent.*

Proof. The equivalence of the conditions (1a), (1b) and (1c) follows from Proposition 4.1. Note that any of the conditions (a), (b) or (c) implies that $val(\mathcal{P}) > -\infty$. Hence, it follows from Proposition 4.2 that (1) and (2) are equivalent under any of the conditions (a), (b) or (c). Moreover, it follows from Proposition 4.3 that (3) is also equivalent to both (1) and (2) when condition (c) holds. We have thus shown the equivalences $(1a) \Leftrightarrow (2a)$, $(1b) \Leftrightarrow (2b)$ and $(1c) \Leftrightarrow (2c) \Leftrightarrow (3c)$. By Proposition 4.1 and the fact that $D_W \subset D_L$, we know that $(c) \Rightarrow (b) \Rightarrow (a)$. These two implications obviously yield the implications $(3c) \Rightarrow (3b) \Rightarrow (3a)$. We have thus shown the first part of the proposition. The second part of the result follows trivially from the lemma stated below. ∎

Lemma 4.6 *Assume that each constraint function $g_j : \mathbf{R}^n \to \mathbf{R}$ $(j = 1, \ldots, p)$ is an affine function. If opt($\mathcal{D}_W$) is nonempty and bounded then $P^0 \neq \emptyset$.*

Proof. The proof of this result uses arguments similar to the ones used in the proofs of Lemma 4.4 and Proposition 3.4(a). We leave the details to the reader. ∎

In general the implications $(3a) \Rightarrow (3b)$ and $(3b) \Rightarrow (3c)$ do not hold. Indeed, the problem stated in Example 3.2 satisfies (3b) but not (3c) since it has no feasible solution. This shows that $(3b) \Rightarrow (3c)$ does not hold. The following simple example shows that $(3a) \Rightarrow (3b)$ does not hold either.

Example 4.2 Consider the functions $f, g : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 0$ and $g(x) = e^x$ for every $x \in \mathbf{R}$. It is easy to verify that $L(s) \equiv \inf_x f(x) + sg(x) = 0$ for every $s \geq 0$ and that the infimum is achieved only for $s = 0$. Hence, we have $D_L = [0, \infty)$ and $D_W = \{0\}$. Thus (3a) is satisfied but not (3b). Note that $P = \emptyset$ and $m = 0$ and so both Assumptions (A) and (B) are satisfied.

5 Limiting Behavior of the Weighted Central Path

In this section we analyze the limiting behavior of the path of solutions of the parametrized family of logarithmic barrier problems (2). This path is referred in this paper to as the w-central path (or the weighted central path when the reference to w is not relevant). When $w = \mathbf{1}$, the w-central path is usually referred to as the central path. As opposed to McLinden [8], we do not assume the existence of a pair of primal and dual optimal solutions satisfying strict complementarity. However, we assume throughout this section that the functions $f(\cdot)$ and $g_j(\cdot)$, $j = 1, \ldots, p$, are analytic. Two main results are proved. The first one (Theorem 5.1) states that the weighted central path converges to a well characterized optimal solution of $(\mathcal{P})$. The second result (Theorem 5.2) shows that a certain dual weighted central path also converges to a well characterized dual optimal solution under the assumption that all constraint functions are affine.

We begin by explicitly stating the assumptions used throughout this section. In addition to Assumptions (A) and (B) of Section 3, we impose throughout this section the following two assumptions:

Assumption (C): $PD^0 \neq \emptyset$.

Assumption (D): The functions $f(x)$ and $g_j(x)$, $j = 1, \ldots, p$, are analytic.

In view of the equivalence of statements (c) and (e) of Theorem 3.1, we know that problem $(\mathcal{P}^w)$ has at least one optimal solution. This problem can however have more than one optimal solution. The following result shows that under the assumptions above, this possibility can not occur.

Lemma 5.1 *For any fixed $w \in \mathbf{R}^p_{++}$, problem $(\mathcal{P}^w)$ has exactly one optimal solution.*

Proof. Existence of at least one optimal solution has already been established. To prove that there is at most one optimal solution, assume for contradiction that $\hat{x}$ and $\bar{x}$ are two distinct optimal solutions of problem $(\mathcal{P}^w)$. Hence, all points in the segment $[\bar{x}, \hat{x}] \equiv \{\lambda\bar{x} + (1-\lambda)\hat{x} \mid \lambda \in [0,1]\}$ are also optimal solutions of $(\mathcal{P}^w)$. It then follows from Lemma 3.4 that the functions $f(\cdot)$ and $g_j(\cdot)$, $j = 1, \ldots, p$, are constant over $[\bar{x}, \hat{x}]$. Since these functions are analytic in view of Assumption (D), we conclude that they are constant over the whole straight line L containing $[\bar{x}, \hat{x}]$. Since any point x in the line L satisfies $Ax = b$, the set $\{x \mid f(x) = f(\bar{x}),\, Ax = b,\, g(x) = g(\bar{x})\}$ contains L, and hence it is unbounded. However, one can easily see that this set must be bounded due to Assumption (B). We have thus obtained a contradiction and the result follows. ∎

It follows from Lemma 5.1 that problem $(\mathcal{P}(t))$ has a unique optimal solution which we denote by $x(t)$. In what follows we are interested in analyzing the limiting behavior of the w-central path $t \mapsto x(t)$, as $t > 0$ tends to 0. We show in Theorem 5.1 below that this path converges to a specific optimal solution of $(\mathcal{P})$, namely the w-center of $\text{opt}(\mathcal{P})$, which we define next.

If $\text{opt}(\mathcal{P})$ consists of a single point x^* then the w-center of $\text{opt}(\mathcal{P})$ is defined to be x^*. Consider now the case in which $\text{opt}(\mathcal{P})$ consists of more than one point and define

$$B \equiv \{j \mid g_j(x) < 0 \text{ for some } x \in \text{opt}(\mathcal{P})\}.$$

It can be shown using arguments similar to the ones used in the proof of Lemma 5.1 that $B \neq \emptyset$ when $\text{opt}(\mathcal{P})$ has more than one point. The w-center of $\text{opt}(\mathcal{P})$ is then defined to be the unique optimal solution of the following convex program:

$$(\mathcal{C}) \qquad \begin{array}{ll} \max & \sum_{j \in B} w_j \log |g_j(x)| \\ \text{s.t.} & x \in \text{opt}(\mathcal{P}), \quad g_B(x) < 0. \end{array} \tag{29}$$

It remains to verify that the above definition is meaningful, that is, that problem $(\mathcal{C})$ has a unique optimal solution. We start by showing that the set of feasible solutions of $(\mathcal{C})$ is nonempty.

Lemma 5.2 *The set $O_B \equiv \{x \mid x \in opt(\mathcal{P}),\ g_B(x) < 0\}$ is nonempty.*

Proof. It follows from the definition of B that, for every $j \in B$, there exists $x^j \in \text{opt}(\mathcal{P})$ such that $g_j(x^j) < 0$. Define $\bar{x} = (1/|B|)\sum_{j \in B} x^j$. Clearly, $\bar{x} \in \text{opt}(\mathcal{P})$ since $\text{opt}(\mathcal{P})$ is a convex set. Moreover, the convexity of $g_j(\cdot)$ implies that

$$g_j(\bar{x}) \leq \frac{1}{|B|} \sum_{j \in B} g_j(x^j) < 0, \quad \forall j \in B.$$

Hence the set $\{x \mid x \in \text{opt}(\mathcal{P}),\ g_B(x) < 0\}$ is nonempty. ∎

Lemma 5.3 *Problem $(\mathcal{C})$ has a unique optimal solution.*

Proof. Fix a point $\bar{x} \in O_B$. In view of Lemma 3.1, the existence of an optimal solution of $(\mathcal{C})$ follows once we show that the set $\Gamma_B \equiv \{x \in O_B \mid \phi_B(x) \geq \phi_B(\bar{x})\}$ is compact, where $\phi_B(x) \equiv \sum_{j \in B} w_j \log |g_j(x)|$ for every $x \in O_B$. Indeed, first observe that Assumption (C) and Proposition

4.1 imply that opt($\mathcal{P}$) is a compact set. This implies that the sets $g_j(\text{opt}(\mathcal{P}))$, $j = 1, \ldots, p$, are bounded. Using this observation, we can easily show the existence of a constant $\delta > 0$ such that $g_B(x) \leq -\delta \mathbf{1}$ for all $x \in \Gamma_B$. Hence,

$$\Gamma_B = \{x \in \text{opt}(\mathcal{P}) \mid g_B(x) \leq -\delta \mathbf{1},\ \phi_B(x) \geq \phi_B(\bar{x})\},$$

from which it follows that the set Γ_B is both bounded and closed, and hence compact.

We now show that problem ($\mathcal{C}$) has at most one optimal solution. Assume by contradiction that x^1 and x^2 are two distinct optimal solutions of problem ($\mathcal{C}$). Then every point in the segment $[x^1, x^2]$ is also an optimal solution. Now, it is easy to see that $g_B(x)$ is constant over the set of optimal solutions of problem ($\mathcal{C}$). Moreover, we also know that $f(x)$ and $g_j(x)$ with $j \notin B$ are constant over opt($\mathcal{P}$). Therefore, we conclude that $f(x)$ and $g(x)$ are constant over the segment $[x^1, x^2]$, and hence, in view of Assumption (D), over the whole straight line containing $[x^1, x^2]$. But one can easily verify that this conclusion contradicts Assumption (B). ∎

We now state and prove one of the main results of this section.

Theorem 5.1 *Suppose that Assumptions (A), (B), (C) and (D) hold and let $w \in \mathbf{R}^p_{++}$ be given. Then, the w-central path $t \mapsto x(t)$ converges to the w-center of opt($\mathcal{P}$) as t tends to 0.*

Proof. Let x^* denote the w-center of opt($\mathcal{P}$), that is the optimal solution of problem ($\mathcal{C}$), and let $\bar{x}$ denote an arbitrary accumulation point of $x(t)$ as t tends to 0, that is $\bar{x} = \lim_{k\to\infty} x(t^k)$, where $\{t^k\}$ is a sequence of positive scalars converging to 0. The theorem follows once we show that $x^* = \bar{x}$. Assume for contradiction that $x^* \neq \bar{x}$ and let $\Delta x = x^* - \bar{x}$. Consider the sequence of points $\{x^k\}$ defined by $x^k \equiv x(t^k) + \Delta x$ for every k. Clearly, we have $\lim_{k\to\infty} x^k = x^*$. We next show that $x^k \in P^0$ for every k sufficiently large. Using the definition of x^k and the fact that $A\Delta x = 0$, we obtain that $Ax^k = b$ for every k. Since $g(\cdot)$ is a continuous function, $g_B(x^*) < 0$ and $\lim_{k\to\infty} x^k = x^*$, we conclude that $g_B(x^k) < 0$ for every k sufficiently large. Now, it is easy to see that $\bar{x} \in \text{opt}(\mathcal{P})$. Hence, due to the convexity of opt($\mathcal{P}$), we have $[\bar{x}, x^*] \subseteq \text{opt}(\mathcal{P})$. This implies that $g_j(x) = 0$ with $j \notin B$ for every $x \in [\bar{x}, x^*]$. Since $g_j(\cdot)$ with $j \notin B$ is analytic, it follows that $g_j(x) = 0$ with $j \notin B$ over the whole straight line containing $[\bar{x}, x^*]$, that is $g_j(\bar{x} + \lambda\Delta x) = 0$ with $j \notin B$ for every $\lambda \in \mathbf{R}$. By Corollary 8.6.1 of Rockafellar [14], it follows that $\lambda \mapsto g_j(x(t^k) + \lambda\Delta x)$ with $j \notin B$ is a constant function for every k. In particular, it follows that $g_j(x^k) = g_j(x(t^k)) < 0$ with $j \notin B$ for every k. We have thus shown that $x^k \in P^0$ for every k sufficiently large. Since $x(t)$ is by definition the optimal solution of problem (2), we conclude that for every k sufficiently large,

$$f(x^k) - t^k \sum_{j=1}^{p} w_j \log|g_j(x^k)| \geq f(x(t^k)) - t^k \sum_{j=1}^{p} w_j \log|g_j(x(t^k))|. \tag{30}$$

The same arguments used to prove that $g_j(x^k) = g_j(x(t^k))$ with $j \notin B$ can also be used to show that $f(x^k) = f(x(t^k))$. Using these two equalities into relation (30), we obtain

$$\sum_{j\in B} w_j \log|g_j(x^k)| \leq \sum_{j\in B} w_j \log|g_j(x(t^k))|,$$

for all k sufficiently large. Letting k go to ∞ in the last relation, we obtain

$$\sum_{j\in B} w_j \log|g_j(x^*)| \leq \sum_{j\in B} w_j \log|g_j(\bar{x})| \tag{31}$$

if $g_B(\bar{x}) < 0$, or that

$$\sum_{j \in B} w_j \log |g_j(x^*)| \leq -\infty \tag{32}$$

if $g_j(\bar{x}) = 0$ for some $j \in B$. Relation (31) is not possible since x^* is the only optimal solution of (29). Obviously, (32) is not possible either. ∎

Associated with problem $\mathcal{P}$, we can also define a dual w-central path as the path of solutions of the following parametrized family of dual logarithmic barrier problems

$$(\mathcal{D}(t)) \qquad \max\{L(y,s) + t\sum_{j=1}^{p} w_j \log s_j \mid (y,s) \in D_L^0\}, \tag{33}$$

where again $t > 0$ represents the parameter of the family. By Theorem 3.1, we know that Assumptions (A), (C) and (D) imply that, for each $t > 0$, problem $(\mathcal{D}(t))$ has a unique optimal solution which we denote by $(y(t), s(t))$. The path $t > 0 \mapsto (y(t), s(t))$ is then called the *dual w-central path* associated with $(\mathcal{P})$. In what follows we characterize the limit of the path $t \mapsto (y(t), s(t))$ as $t > 0$ tends to 0 for the case in which the constraints functions $g_j(\cdot)$, $j = 1, \ldots, p$, are affine. The corresponding characterization for the more general case in which the functions $g_j(\cdot)$, $j = 1, \ldots, p$, are allowed to be nonlinear remains open.

Before stating the above characterization, we first define the w-center of $\text{opt}(\mathcal{D}_L) = \text{opt}(\mathcal{D}_W)$. Let

$$N \equiv \{j \mid s_j > 0 \text{ for some } (y,s) \in \text{opt}(\mathcal{D}_L)\}.$$

The w-center of $\text{opt}(\mathcal{D}_L)$ is defined to be the unique optimal solution of the following convex program:

$$\begin{aligned} (\mathcal{DC}) \qquad & \max \quad \sum_{j \in N} w_j \log s_j \\ & \text{s.t.} \quad (y,s) \in \text{opt}(\mathcal{D}_L), \quad s_N > 0. \end{aligned}$$

It can be easily verified that the above problem has a unique optimal solution, and hence that the above definition is meaningful.

Theorem 5.2 *Assume that the functions $g_j(\cdot)$, $j = 1, \ldots, p$, are affine and let $w \in \mathbf{R}^p_{++}$ be given. Then, the dual w-central path $t \mapsto (y(t), s(t))$ converges to the w-center of* $\text{opt}(\mathcal{D}_L)$ *as t tends to* 0.

Proof. This proof closely follows the proof of Theorem 5.1. Assume that $g(x) = Cx - h$, $\forall x \in \mathbf{R}^n$, where C is a $p \times n$ matrix and $h \in \mathbf{R}^p$. Let (y^*, s^*) denote the w-center of $\text{opt}(\mathcal{D}_L)$. Let $(\bar{y}, \bar{s})$ be an arbitrary accumulation point of $(y(t), s(t))$ as t tends to 0, that is, $(\bar{y}, \bar{s}) = \lim_{k \to \infty}(y(t^k), s(t^k))$, where $\{t^k\}$ is a sequence of positive scalars converging to 0. The result follows once we show that $(\bar{y}, \bar{s}) = (y^*, s^*)$. Assume for contradiction that $(\bar{y}, \bar{s}) \neq (y^*, s^*)$ and define $(\Delta y, \Delta s) \equiv (y^* - \bar{y}, s^* - \bar{s})$. It is easy to verify that $(\bar{y}, \bar{s}) \in \text{opt}(\mathcal{D}_L)$. Consider the sequence of points $\{(y^k, s^k)\}$ defined by $(y^k, s^k) \equiv (y(t^k), s(t^k)) + (\Delta y, \Delta s)$ for all k. We claim that there exists k_0 such that

$$(y^k, s^k) \in D_L^0, \quad \forall k \geq k_0, \tag{34}$$

$$L(y^k, s^k) = L(y(t^k), s(t^k)), \quad \forall k \geq 0, \tag{35}$$

$$s_j^k = s_j(t^k), \quad \forall j \notin N, \ \forall k \geq 0. \tag{36}$$

We now prove the theorem assuming for the moment that the above claim is true. Indeed, using (34) and the fact that $(y(t^k), s(t^k))$ is the optimal solution of $(\mathcal{D}(t^k))$, we obtain

$$L(y(t^k), s(t^k)) + t^k \sum_{j=1}^{p} w_j \log s_j(t^k) \geq L(y^k, s^k) + t^k \sum_{j=1}^{p} w_j \log s_j^k, \qquad \forall k \geq k_0.$$

Combining (35) and (36) with the above relation yield

$$\sum_{j \in N} w_j \log s_j(t^k) \geq \sum_{j \in N} w_j \log s_j^k, \quad \forall k \geq k_0.$$

Making k go to ∞ in the above relation and using the fact that (y^*, s^*) is the unique optimal solution of $(\mathcal{DC})$, we can easily obtain a contradiction.

It remains to show that the claim holds. By the definition of N, we have $s_j^* = \bar{s}_j = 0$ for every $j \notin N$. Hence, $\Delta s_j = 0$ for every $j \notin N$, and this implies (36). Clearly, we have $\lim_{k\to\infty}(y^k, s^k) = (y^*, s^*)$, and since $s_N^* > 0$, we conclude that there exists k_0 such that $s_N^k > 0$ for all $k \geq k_0$. This observation together with relation (36) imply that $s^k > 0$ for every $k \geq k_0$. It is now immediate that (34) holds once we show the validity of (35). Observe that by the definition of the function $L(\cdot, \cdot)$, (35) follows immediately once we show that

$$(\Delta y)^T (b - Ax) + (\Delta s)^T (Cx - h) = 0, \quad \forall x \in \mathbf{R}^n. \tag{37}$$

To show this last relation, fix a point $\bar{x} \in \text{opt}(\mathcal{P})$. Using the fact that $(\bar{y}, \bar{s})$ and (y^*, s^*) are in $\text{opt}(\mathcal{D}_L) = \mathcal{KT}$ and the observation preceding (25), we conclude that

$$\begin{aligned} \nabla f(\bar{x}) - A^T y^* + C^T s^* = 0, &\quad (s^*)^T (C\bar{x} - h) = 0, \\ \nabla f(\bar{x}) - A^T \bar{y} + C^T \bar{s} = 0, &\quad (\bar{s})^T (C\bar{x} - h) = 0, \end{aligned}$$

which in turn imply that

$$-A^T \Delta y + C^T \Delta s = 0, \quad (\Delta s)^T (C\bar{x} - h) = 0.$$

These two relations and the fact that $A\bar{x} = b$ then imply

$$\begin{aligned} (\Delta y)^T (b - Ax) + (\Delta s)^T (Cx - h) &= b^T \Delta y - h^T \Delta s \\ &= \bar{x}^T A^T \Delta y - \bar{x}^T C^T \Delta s = 0, \end{aligned}$$

for every $x \in \mathbf{R}^n$. ■

It is worth noting that in the proof of Theorem 5.2 we used only the fact that $f(\cdot)$ is differentiable, and hence it is not necessary to assume that $f(\cdot)$ is analytic.

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References

[1] I. Adler and R. D. C. Monteiro. Limiting behavior of the affine scaling continuous trajectories for linear programming problems. *Mathematical Programming*, 50:29–51, 1991.

[2] A. V. Fiacco and G. P. McCormick. *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*. John Wiley & Sons, New York, 1968. Reprint : Volume 4 of *SIAM Classics in Applied Mathematics*, SIAM Publications, Philadelphia, PA 19104–2688, USA, 1990.

[3] A. M. Geoffrion. Duality in nonlinear programming: a simplified applications-oriented development. *SIAM Review*, 13(1):1–37, 1971.

[4] O. Güler. Limiting behavior of the weighted central paths in linear programming. *Mathematical Programming*, 65:347–363, 1994.

[5] O. Güler, C. Roos, T. Terlaky, and J. P. Vial. Interior point approach to the theory of linear programming. Cahiers de Recherche 1992.3, Faculte des Sciences Economique et Sociales, Universite de Geneve, Geneve, Switzerland, 1992.

[6] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms I*, volume 305 of *Comprehensive Study in Mathematics*. Springer-Verlag, New York, 1993.

[7] M. Kojima, S. Mizuno, and T. Noma. Limiting behavior of trajectories by a continuation method for monotone complementarity problems. *Mathematics of Operations Research*, 15(4):662–675, 1990.

[8] L. McLinden. An analogue of Moreau's proximation theorem, with application to the nonlinear complementarity problem. *Pacific Journal of Mathematics*, 88:101–161, 1980.

[9] N. Megiddo. Pathways to the optimal set in linear programming. In N. Megiddo, editor, *Progress in Mathematical Programming: Interior Point and Related Methods*, pages 131–158. Springer Verlag, New York, 1989. Identical version in: *Proceedings of the 6th Mathematical Programming Symposium of Japan, Nagoya, Japan*, pages 1–35, 1986.

[10] N. Megiddo and M. Shub. Boundary behavior of interior point algorithms in linear programming. *Mathematics of Operations Research*, 14:97–114, 1989.

[11] R. D. C. Monteiro. Convergence and boundary behavior of the projective scaling trajectories for linear programming. *Mathematics of Operations Research*, 16(4):842–858, 1991.

[12] R. D. C. Monteiro and J.-S. Pang. Properties of an interior-point mapping for mixed complementarity problems. *Mathematics of Operations Research*, 21:629–654, 1996.

[13] R. D. C. Monteiro and T. Tsuchiya. Limiting behavior of the derivatives of certain trajectories associated with a monotone horizontal linear complementarity problem. *Mathematics of Operations Research*, 21:793–814, 1996.

[14] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, NJ, 1970.

[15] C. Witzgall, P. T. Boggs, and P. D. Domich. On the convergence behavior of trajectories for linear programming. In J. C. Lagarias and M. J. Todd, editors, *Mathematical Developments Arising from Linear Programming: Proceedings of a Joint Summer Research Conference held at Bowdoin College, Brunswick, Maine, USA, June/July 1988*, volume 114 of *Contemporary Mathematics*, pages 161–187. American Mathematical Society, Providence, Rhode Island, USA, 1990.