Principal-Agent Problems

ISyE 6230 Spring 2007

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Principal-agent examples

- Restaurant owner – waiter
- Software company – salesman
- Auto manufacturer – customer leasing a car
- Insurance company – insured
- Others?

- How can the manager ensure the agent performs as desired?

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General Principal Agent Problem

- Principal designs contract and offers it to agent
- Agent decides to accept or not
  - If accepts, then the agent decides on the level of effort
- The firm's revenue is observed
- Principal pays the agent according to contract terms

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PA Example

- Effort level of a waiter (agent):
  - e = 0 if shirks
  - e = 2 if works hard
- Utility of agent
  - U = w – e if devotes e
  - U = 10 if rejects contract
- Revenue of restaurant owner (principal)
  - R(e) = H if e = 2
  - R(e) = L if e = 0
- Owner's profit is Π = R(e) – w
- Owner will choose wages (w^H, w^L) to offer agent based on the effort level (high, low)
  - (We assume H – L is sufficiently large to desire high effort)

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PA Example

- The agent won’t accept the offer unless his utility exceeds his other (outside) options
  - $w^H - 2$ ¸ $10 \rightarrow w^H$ ¸ 12
- The principal wants the agent to work hard, so must give an incentive for the agent to want to work hard
  - $w^H - 2$ ¸ w^L

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PA Example

- What is the resulting contract?
- w^L?
- w^H?
- $\Pi^H = H - w^H$
- $\Pi^L = L - w^L$
- (For this contract to be optimal for principal, we need Π^H ¸ Π^L)

- What’s wrong with the contract?

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Moral hazard

In reality, the action taken by the agent is not usually observable.

"Moral hazard": the agent takes a decision or action that affects his or her utility as well as the principal's, the principal only observes the "outcome" (as an imperfect signal of the action taken), and the agent does not necessarily choose the action in the interest of the principal.

<u>Alternative</u>: Offer a contract where the wage depends on the outcome.

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PA Example with Unknown Effort

- Suppose that the revenues H and L are uncertain and partly determined by effort level: R(e)
  - R(2) = H with probability 0.8, L with probability 0.2
  - R(0) = H with probability 0.4, L with probability 0.6
- Agent's utility depends on effort:
  - U = E[w] – e   if agent devotes e
  - U = 10   if rejects contract
- Wage is expected value
  - E[w] = $0.8w^H + 0.2w^L$ when e = 2
  - E[w] = $0.4w^H + 0.6w^L$ when e = 0

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PA Contract with Unknown Effort

- Participation constraint
  - $0.8\ w^H + 0.2\ w^L – 2$ ¸ 10
    - → w^L ¸ $60 – 4\ w^H$
- Incentive Compatability constraint
  - $0.8\ w^H + 0.2\ w^L – 2$ ¸ $0.4\ w^H + 0.6\ w^L$
    - → w^L · $w^H – 5$
- Optimal contract
  - $w^L = 8$ → $w^H = 13$
- Implications?

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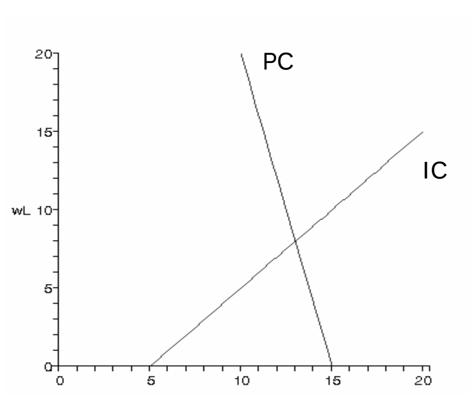
PA Constraints



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PA Contract Implications

- Manager can achieve goals without monitoring
- But how costly is it to implement?
  - Perfect monitoring:
    - $w^H = 12$ and $w^L = 10$
  - No-monitoring under uncertainty:
    - $w^H = 13$ and $w^L = 8$
    - $E[w] = 0.8(13) + 0.2\,(8) = 12$!

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Risk aversion

- What if the agent is risk-averse?
  - A person who prefers to get the expected value of a gamble for sure instead of taking the risky gamble is risk averse
  - E.g.: getting $25 for sure vs. getting $0 with probability 0.75 and $100 with probability 0.25
- The agent and the principal may have different “beliefs” about the probabilities of different outcomes under different effort levels

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PA Example – Risk averse agent

- Owner believes:
  - $R^P(2)$ = H with prob 0.8, L with prob 0.2
  - $R^P(0)$ = H with prob 0.4, L with prob 0.6
- Agent believes:
  - $R^A(2)$ = H with prob 0.7, L with prob 0.3
  - $R^A(0) = R^P(0)$

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PA Example – Risk averse agent

- Participation constraint

  $0.3w^L + 0.7w^H - 2 \geq 10$

  $\rightarrow w^H = (12 - 0.3w^L)/(0.7)$
- Incentive constraint

  $0.3w^L + 0.7w^H - 2 \geq 0.6w^L + 0.4w^H$

  $\rightarrow w^H = 2/0.3 + w^L$
- Principal's objective

  $\Pi = \min_{wH,\, wL} E^P[w] = 0.8w^H + 0.2w^L$

Solution: $w^L = 22/3$ $w^H = 14$ $\Pi = 12.66 > 10 + 2$

Recall if the agent also shares the principal's belief

$w^L = 8$ $w^H = 13$ $\Pi = 12$

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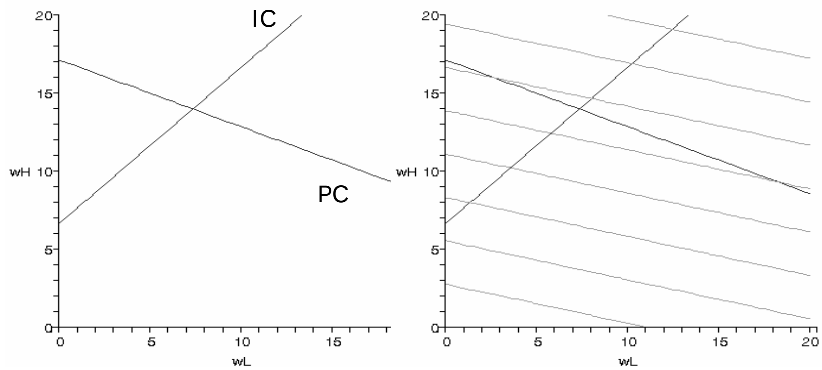
PA Example – Risk averse agent



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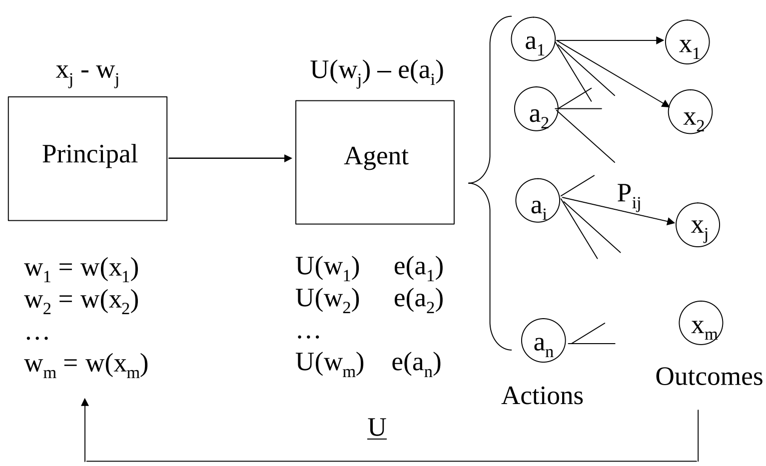
Principal-Agent Problem



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Principal-agent problem

- If the agent doesn't accept the contract, his payoff is his reservation utility, $\underline{U}$
- If the agent accepts the contract, he chooses between n possible actions: $a_1, \ldots, a_n$.
- These actions produce m possible outcomes: $x_1, \ldots, x_m$.
- There is a stochastic relationship between actions and outcomes (called "technology"). When the action is a_i, the principal observes outcome x_j with probability P_{ij}.
- If the principal observes outcome x_j, she pays the agent w_j.
- The agent's payoff is $U(w)-e(a_i)$, where $U(w)$ is the utility of wage w to the agent and $e(a_i)$ is the cost of action a_i to the agent.
  - U is increasing, differentiable and concave.
- Assuming the principal is risk-neutral, her payoff is $x_j - w_j$.

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What if the agent's actions can be observed?

- The principal can design a contract where the wages are conditioned on the actions, i.e., $w(a_i)$

$$\max \sum_{j=1}^{m} P_{ij} x_j - w_i \ \equiv \ \min w_i$$

$$\text{Participation constraint}: U(w_i) - e(a_i) \geq \underline{U}$$

Incentive constraint:

$$U(w_i) - e(a_i) \geq U(w_k) - e(a_k) \quad \forall k \neq i$$

To induce the agent to choose action a_i, set w_i such that $U(w_i) = \underline{U} + e(a_i)$ and set all other w_k sufficiently low.

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Unobservable actions - Principal's problem: Step 1

- Given an action a_i, how to set the wages such that the agent chooses a_i and the principal's payoff is maximized?

Participation constraint: $\sum_{j=1}^{m} P_{ij} U(w_j) - e(a_i) \geq \underline{U}$

Incentive constraint:

$$\sum_{j=1}^{m} P_{ij} U(w_j) - e(a_i) \geq \sum_{j=1}^{m} P_{kj} U(w_j) - e(a_k) \quad \forall k \neq i$$

Principal's objective (minimize the expected cost of inducing the agent to choose a_i):

$$\max \sum_{j=1}^{m} P_{ij}(x_j - w_j) \equiv \min \sum_{j=1}^{m} P_{ij} w_j = C(a_i)$$

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Principal's problem: Step 1

- $C(a_i)$ is the minimal cost (to the principal) of inducing the agent to take action a_i.
- $C(a_i)$ is convex → the original maximization objective is concave
- Well-behaved mathematical program with a concave objective function (maximization) and linear constraints.

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Principal's problem: Step 1

For a given action a_i :

$$\max \sum_{j=1}^{m} P_{ij}(x_j - w_j)$$

$$\sum_{j=1}^{m} P_{ij}U(w_j) - e(a_i) \geq \underline{U} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad m$$

$$\sum_{j=1}^{m} P_{ij}U(w_j) - e(a_i) \geq \sum_{j=1}^{m} P_{kj}U(w_j) - e(a_k) \;\; \forall k \neq i \;\ldots. \quad l_k$$

$$L(w, l, m) = \sum_{j=1}^{m} P_{ij}(x_j - w_j) + m(\sum_{j=1}^{m} P_{ij}U(w_j) - e(a_i) - \underline{U}) +$$

$$\sum_{k \neq i} l_k (\sum_{j=1}^{m} P_{ij}U(w_j) - e(a_i) - \sum_{j=1}^{m} P_{kj}U(w_j) + e(a_k))$$

Principal's problem: Step 1

$$L(w, l, m) = \sum_{j=1}^{m} P_{ij}(x_j - w_j) + m(\sum_{j=1}^{m} P_{ij}U(w_j) - e(a_i) - \underline{U}) +$$

$$\sum_{k \neq i} l_k \left(\sum_{j=1}^{m} P_{ij}U(w_j) - e(a_i) - \sum_{j=1}^{m} P_{kj}U(w_j) + e(a_k) \right)$$

$$\frac{\partial L(w, l, m)}{\partial w_j} = -P_{ij} + mP_{ij} \frac{\partial U(w_j)}{\partial w_j} + \sum_{k \neq i} l_k \left(P_{ij} \frac{\partial U(w_j)}{\partial w_j} - P_{kj} \frac{\partial U(w_j)}{\partial w_j} \right) = 0 \rightarrow$$

$$-P_{ij} + \frac{\partial U(w_j)}{\partial w_j} \left(mP_{ij} + \sum_{k \neq i} l_k (P_{ij} - P_{kj}) \right) = 0$$

$$-1 + \frac{\partial U(w_j)}{\partial w_j} \left(m + \sum_{k \neq i} l_k \left(1 - \frac{P_{kj}}{P_{ij}} \right) \right) = 0 \rightarrow \frac{1}{\frac{\partial U(w_j)}{\partial w_j}} = m + \sum_{k \neq i} l_k \left(1 - \frac{P_{kj}}{P_{ij}} \right)$$

Principal's problem: Step 1

$$\frac{1}{U'(w_j)} = m + \sum_{k \neq i} l_k \left(1 - \frac{P_{kj}}{P_{ij}}\right)$$

$\frac{P_{kj}}{P_{ij}}$: likelihood ratio

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